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NEW EXISTENCE THEOREMS FOR VECTOR-VALUED FUNCTIONS WITH ESSENTIAL VECTORIAL SEMICONTINUITY AND ADJUSTABILITY CONVEXITY

WEI-SHIH DU

Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 824004, Taiwan

In memory of Professor Petr Zabreiko

Abstract. In this paper, the new concepts of essential vectorial lower semicontinuity, essential vectorial upper semicontinuity and essential vectorial continuity are introduced and their properties and auxiliary results are established. We present new existence theorems for C-adjustability convex vector-valued functions in the weakly compact setting and apply them to obtain new fixed point theorem and existence theorems for root-finding problem, eigenvector problem and minimization problem.

Keywords. Vectorial lower semicontinuity; C-adjustability convex; Fixed point theorem; Root-finding problem; Eigenvector problem; Minimization problem.

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1. INTRODUCTION AND PRELIMINARIES

During the previous more than a half-century, convexity theory has widespread and significant applications in many areas at the core of many branches of pure and applied mathematics and modern science, including functional analysis, data analysis and modeling, nonlinear ordinary and partial differential equations, nonlinear optimization, fractional calculus, physics, economics and finance, automatic control systems, estimation and signal processing, communications and networks, and many more besides. For more details, we refer the readers to [1, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] and references therein.

Let E be a vector space. A nonempty subset A of E is called *convex* if for any $x, y \in A$, $\lambda x + (1-\lambda)y \in A$ for all $\lambda \in [0,1]$. Let M be a nonempty convex subset of E. A real-valued function $f : M \to \mathbb{R}$ is called *convex* if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \tag{1.1}$$

*Corresponding author.

E-mail address: wsdu@mail.nknu.edu.tw

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for all $x,y \in M$ and $t \in [0,1]$. If the above inequality (1.1) is strict whenever $x \neq y$ and $0 < t < 1$, then f is called *strictly convex*. A function $f : M \to \mathbb{R}$ is called *concave* (resp. *strictly concave*) if $-f$ is convex (resp. strictly convex). Let X be a topological space. A real-valued function $f : X \to \mathbb{R}$ is *lower semicontinuous* (in short *l.s.c*) (resp. *upper semicontinuous*, in short *u.s.c*) if $\{x \in X : f(x) \leq r\}$ (resp. $\{x \in X : f(x) \geq r\}$) is closed for each $r \in \mathbb{R}$.

Let V be a topological vector space (t.v.s. for short) with its zero vector θ_V and A be a nonempty subset of V. We use the notations $\overline{A}$, $co(A)$, $\overline{co}(A)$, $intA$ and $\partial(A)$ to denote the closure, convex hull and closed convex hull (*i.e.* the closure of the convex hull), interior and boundary of A, respectively. A nonempty subset C of V is said to be (i) *proper* if $C \neq \emptyset$; (ii) a *cone* if $\lambda C \subseteq C$ for $\lambda \geq 0$; (iii) a *convex cone* if $C + C \subseteq C$ and $\lambda C \subseteq C$ for $\lambda \geq 0$. It is obvious that if C is a convex cone in V, then $\theta_V \in C$. A cone C in V is said to be *pointed* if $C \cap (-C) = \{\theta_V\}$. For a given proper, pointed and convex cone C in V, we can define a partial ordering $\precsim_C$with respect to C by

$$x \precsim_C y \Longleftrightarrow y - x \in C.$$

$x \prec_C y$ will stand for $x \precsim_C y$ and $x \neq y$, while $x \ll_C y$ will stand for $y - x \in intC$. As usual, we allow the use of these symbols: (i) $x \precsim_C y \Longleftrightarrow y \succsim_C x$, (ii) $x \prec_C y \Longleftrightarrow y \succ_C x$, and (iii) $x \ll_C y \Longleftrightarrow y \gg_C x$. A function $\varphi : V \to V$ is called to be $\precsim_C$*-nondecreasing* if $x,y \in V$ with $x \precsim_C y$ implies $\varphi(x) \precsim_C \varphi(y)$.

In reality, we often encounter non-convex functions or non-concave functions when solving real-world problems, so this makes known results in the literature about convex or concave functions not easily applicable to dealt with these problems. Motivated by that reason, Du [5] introduced and studied the concepts of C-adjustability convexity and strictly C-adjustability convexity as follows:

Definition 1.1 (see [5, Definition 1])**.** Let V_1 and V_2 be vector spaces, X be a nonempty convex set in V_1, C be a given convex cone in V_2 and $\mu : V_2 \to V_2$ be a mapping. A vector-valued function $f : X \to V_2$ is called

(i) *C-adjustability convex with respect to μ* (abbreviated as *(C,μ)-adjconvex*) if

$$\mu(tf(x) + (1-t)f(y)) - f(tx + (1-t)y) \in C \tag{1.2}$$

for all $x,y \in X$ and $t \in [0,1]$. In particular, f is called *C-convex* if μ is an identity mapping on V_2 and (1.2) becomes

$$tf(x) + (1-t)f(y) - f(tx + (1-t)y) \in C$$

for all $x,y \in X$ and $t \in [0,1]$.

(ii) *strictly C-adjustability convex with respect to μ* (abbreviated as *strictly (C,μ)-adjconvex*) if

$$\mu(tf(x) + (1-t)f(y)) - f(tx + (1-t)y) \in intC \tag{1.3}$$

for all $x,y \in X$ with $x \neq y$ and $t \in (0,1)$. In particular, f is called *strictly C-convex* if μ is an identity mapping on V_2 and (1.3) becomes

$$tf(x) + (1-t)f(y) - f(tx + (1-t)y) \in intC$$

for all $x,y \in X$ with $x \neq y$ and $t \in (0,1)$.

Example A (see [5, Example 1])**.** Let $V_1 = \mathbb{R}$, $V_2 = \mathbb{R}^2$, $X = [-1,1]$ and

$$C = \mathbb{R}^2_+ := \left\{(x_1,x_2) \in \mathbb{R}^2 : x_i \geq 0, i = 1,2\right\}.$$

Then X is a nonempty convex subset of V_1 and C is a convex cone in V_2. Let $f : X \to V_2$ be defined by

$$f(x) = \begin{cases} (-x,0), & x \in [0,1], \\ (0,x), & x \in [-1,0). \end{cases}$$

Take $\widehat{x} = \frac{1}{2}$ and $\widehat{y} = -\frac{1}{2}$. Thus

$$\frac{1}{2}f(\widehat{x}) + \frac{1}{2}f(\widehat{y}) - f\left(\frac{1}{2}\widehat{x} + \frac{1}{2}\widehat{y}\right) = \left(-\frac{1}{4}, -\frac{1}{4}\right) - (0,0) = \left(-\frac{1}{4}, -\frac{1}{4}\right) \notin C,$$

which show that f is not C-convex. Now, let $\mu : V_2 \to V_2$ be defined by

$$\mu(x,y) = (\max\{|x|,|y|\},0) \quad \text{for } (x,y) \in V_2.$$

Then f is (C,μ)-adjconvex.

In Definition 1.1, if we take $V = V_1$, $V_2 = \mathbb{R}$, $K = [0,+\infty) \subset \mathbb{R}$, then we obtain the following concepts.

Definition 1.2 (see [5, Definition 2])**.** Let X be a nonempty convex subset of a vector space V and $\mu : \mathbb{R} \to \mathbb{R}$ be a mapping. A real-valued function $f : X \to \mathbb{R}$ is called

(i) *adjustability convex with respect to* μ (abbreviated as (μ)-*adjconvex*) if

$$f(tx + (1-t)y) \leq \mu(tf(x) + (1-t)f(y))$$

for all $x,y \in X$ and $t \in [0,1]$. In particular, if μ is an identity mapping on $\mathbb{R}$, then f is called *convex*.

(ii) *strictly adjustability convex with respect to* μ (abbreviated as *strictly* (μ)-*adjconvex*) if

$$f(tx + (1-t)y) < \mu(tf(x) + (1-t)f(y))$$

for all $x,y \in X$ with $x \neq y$ and $t \in (0,1)$. In particular, if μ is an identity mapping on $\mathbb{R}$, then f is called *strictly convex*.

It is well-known that semicontinuity plays an important role in nonlinear analysis, vector optimization and variational problems. Many authors have devoted their attention to investigate new generalizations of the concept and properties of semicontinuity; see, e.g., [1, 2, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] and references therein. In this paper, we introduce and study the new concepts of essential vectorial lower semicontinuity, essential vectorial upper semicontinuity and essential vectorial continuity (see Definition 2.1 below). Some properties and auxiliary results for vectorial lower semicontinuity, essential vectorial upper semicontinuity and essential vectorial continuity are establised in Section 2. In Section 3, we present new existence theorems for C-adjustability convex vector-valued functions in the weakly compact setting. As applications, in Section 4, we obtain a new fixed point theorem and existence theorems for root-finding problem, eigenvector problem and minimization problem, which are quite different from the known results in the corresponding literature.

2. Essential vectorial lower semicontinuity and upper semicontinuity

In this section, we introduce the concepts of essential vectorial lower semicontinuity, essential vectorial upper semicontinuity and essential vectorial continuity.

Definition 2.1. Let X be a topological space and Y be a t.v.s. with its zero vector θ, C a proper, closed and convex pointed cone in Y with $intC \neq \emptyset$ and $e \in intC$. A vector-valued function $f : X \to Y$ is called

(1) *essential vectorial lower semicontinuous* (*with respect to C and e*) (abbreviated as (EV)-*lsc*) *at* $z \in X$ if for any $\varepsilon > 0$, there exists an open neighborhood $\mathscr{N}(z)$ of z, such that
$$f(x) - f(z) + \varepsilon e \in intC \qquad \text{for all } x \in \mathscr{N}(z).$$

(2) *essential vectorial upper semicontinuous* (*with respect to C and e*) (abbreviated as (EV)-*usc*) *at* $z \in X$ if for any $\varepsilon > 0$, there exists an open neighborhood $\mathscr{N}(z)$ of z, such that
$$f(x) - f(z) - \varepsilon e \in -intC \qquad \text{for all } x \in \mathscr{N}(z).$$

(3) *essential vectorial lower semicontinuous* (*with respect to C and e*) *on X* if it is (EV)-lsc at every point of X.

(4) *essential vectorial upper semicontinuous* (*with respect to C and e*) *on X* if it is (EV)-usc at every point of X.

(5) *essential vectorial continuous* (*with respect to C and e*) (abbreviated as (EV)-*continuous*) *on X* if it is both (EV)-usc and (EV)-lsc on X.

Remark 2.1. If we take $Y = \mathbb{R}$, $C = 0, +\infty) \subset \mathbb{R}$ and $e = 1$, then the concepts in Definition 2.1 coincide with the usual concepts of lower semicontinuity, upper semicontinuity and continuity, respectively.

The following result is simple, but it is very useful in this paper.

Lemma 2.1. *Let X be a topological space and $z \in X$. Let Y be a t.v.s. with its zero vector θ, C a proper, closed and convex pointed cone in Y with $intC \neq \emptyset$ and $e \in intC$. Then $f : X \to Y$ is (EV)-usc at z if and only if $-f$ is (EV)-lsc at z.*

Proof. By the definitions of (EV)-usc and (EV)-lsc, we have

f is (EV)-usc at z

$\Longleftrightarrow$ for any $\varepsilon > 0$, there exists an open neighborhood $\mathscr{N}(z)$ of z, such that
$$f(x) - f(z) - \varepsilon e \in -intC \qquad \text{for all } x \in \mathscr{N}(z).$$

$\Longleftrightarrow$ for any $\varepsilon > 0$, there exists an open neighborhood $\mathscr{N}(z)$ of z, such that
$$(-f)(x) - (-f)(z) + \varepsilon e \in intC \qquad \text{for all } x \in \mathscr{N}(z).$$

$\Longleftrightarrow -f$ is (EV)-lsc at z.

The proof is completed. □

The following useful lemma was indeed proved in [12, Lemma 1], but we give its proof here for the sake of completeness and the readers convenience.

Lemma 2.2. *Let V be a t.v.s. with its zero vector θ_V and C be a nonempty subset of V with $intC \neq \emptyset$. Then the following statements hold.*

(a) *If C is a cone, then $\lambda intC = intC$ for all $\lambda > 0$.*
(b) *If C is a convex cone, then $intC + C = intC + intC = intC$.*

Proof. First, we verify conclusion (a). Assume that C is a cone. Then

$$\lambda intC = int\,(\lambda intC) \subseteq int\,(\lambda C) \subseteq intC \quad \text{for any } \lambda > 0. \tag{2.1}$$

Let $\lambda > 0$ be given. So $\lambda^{-1} > 0$. By using (2.1), we obtain

$$intC = \lambda\left(\lambda^{-1} intC\right) \subseteq \lambda intC. \tag{2.2}$$

Combining (2.1) with (2.2), we show that $\lambda intC = intC$. To see (b), we first claim

$$intC + C = intC. \tag{2.3}$$

Since C is a convex cone, we have

$$intC + C = int\,(intC + C) \subseteq int\,(C + C) \subseteq intC.$$

Due to $\theta_V \in C$, we obtain

$$intC = \{\theta_V\} + intC \subseteq C + int\,(C).$$

Hence our claim (2.3) is proved. Next, we prove that $intC + intC = intC$. By (2.3), we obtain

$$intC + intC \subseteq intC + C \subseteq intC. \tag{2.4}$$

On the other hand, by using (a), we have

$$intC \subseteq \frac{1}{2} intC + \frac{1}{2} intC = intC + intC. \tag{2.5}$$

Taking into account (2.4) with (2.5), we show that $intC + intC = intC$. The proof is completed. □

The following Lemmas are crucial to our main results.

Lemma 2.3. *Let X be a topological space and Y be a t.v.s. with its zero vector θ, C a proper, closed and convex pointed cone in Y with $intC \neq \emptyset$ and $e \in intC$. Suppose that $f, g : X \to Y$ are two vector-valued functions. Then the following statements hold:*
(a) *If f and g are (EV)-lsc (resp. (EV)-usc) at $z \in X$, then $f + g$ is (EV)-lsc (resp. (EV)-usc) at z.*
(b) *Let $\lambda \geq 0$. If f is (EV)-lsc (resp. (EV)-usc) at $z \in X$, then λf is (EV)-lsc (resp. (EV)-usc) at z.*

Proof. (a) (i) Suppose that f and g are (EV)-lsc at $z \in X$. Let $\varepsilon > 0$ be given. Then there exist open neighborhoods $\mathscr{N}_f(z)$ and $\mathscr{N}_g(z)$ of z such that

$$f(x) - f(z) + \frac{\varepsilon}{2} e \in intC \quad \text{for all } x \in \mathscr{N}_f(z) \tag{2.6}$$

and

$$g(x) - g(z) + \frac{\varepsilon}{2} e \in intC \quad \text{for all } x \in \mathscr{N}_g(z). \tag{2.7}$$

Let $\mathscr{N}(z) := \mathscr{N}_f(z) \cap \mathscr{N}_g(z)$. Thus $\mathscr{N}(z)$ is an open neighborhood of z. For any $x \in \mathscr{N}(z)$, taking into account (2.6) with (2.7) and applying Lemma 2.2, we get

$$(f+g)(x) - (f+g)(z) + \varepsilon e \in intC + intC = intC \quad \text{for all } x \in \mathscr{N}(z).$$

This shows that $f + g$ is (EV)-lsc at z.

(ii) Assume that f and g are (EV)-usc at $z \in X$. By using Lemma 2.1, $-f$ and $-g$ are (EV)-lsc at z. From (i), we obtain $-(f+g) = (-f)+(-g)$ is (EV)-lsc at z. Applying Lemma 2.1 again, we prove that $f+g$ is (EV)-usc at z.

Hence, by cases (i) and (ii), we prove (a).

(b) Clearly, the conclusion is true for $\lambda = 0$. So it suffices to verify the conclusion is true for $\lambda > 0$.

(i)′ Suppose that f is (EV)-usc at $z \in X$. Let $\varepsilon > 0$ be given. Then there exists an open neighborhood $\mathscr{N}(z)$ of z such that

$$f(x) - f(z) - \frac{\varepsilon}{\lambda} e \in -intC \quad \text{for all } x \in \mathscr{N}(z). \tag{2.8}$$

By (2.8) and using Lemma 2.2, we obtain

$$(\lambda f)(x) - (\lambda f)(z) - \varepsilon e \in -\lambda intC = -intC \quad \text{for all } x \in \mathscr{N}(z).$$

So λf is (EV)-lsc at z.

(ii)′ Assume that f is (EV)-lsc at $z \in X$. Applying Lemma 2.1, $-f$ is (EV)-usc at z. From (i)′, we obtain $-(\lambda f) = \lambda(-f)$ is (EV)-usc at z. Applying Lemma 2.1 again, we show that λf is (EV)-lsc at z.

Therefore, by cases (i)′ and (ii)′, the conclusion of (b) is proved.

The proof of Lemma 2.3 is completed. □

Lemma 2.4. *Let $\lambda \in \mathbb{R}$ and V be a t.v.s. with its zero vector θ, C a proper, closed and convex pointed cone in V with $intC \neq \emptyset$ and $e \in intC$. Let $f : V \to V$ be defined by $f(x) = \lambda x$ for all $x \in V$. Then f is essential vectorial continuous (with respect to C and e) on V.*

Proof. Let $I_V : V \to V$ be defined by $I_V(x) = x$ for all $x \in V$. Then $f = \lambda I_V$. It is easy to see that I_V is both (EV)-usc and (EV)-lsc on X, so it is (EV)-continuous on X. By Lemma 2.1, $-I_V$ is also both (EV)-usc and (EV)-lsc on X. Hence $-I_V$ is (EV)-continuous on X. We now consider the following two possible cases:

(i) If $\lambda \geq 0$, then, by Lemma 2.3, $f = \lambda I_V$ is both (EV)-usc and (EV)-lsc on X, so it is (EV)-continuous on X.

(ii) If $\lambda < 0$, then $-\lambda > 0$. By Lemma 2.3 again, $f = \lambda I_V = (-\lambda)(-I_V)$ is both (EV)-usc and (EV)-lsc on X, so it is (EV)-continuous on X.

Therefore, by cases (i) and (ii), we prove that f is (EV)-continuous on V. The proof is completed. □

In the following, unless otherwise specified, we always suppose that X is a topological space and Y is a real locally convex Hausdorff t.v.s. with its zero vector θ, C a proper, closed and convex pointed cone in Y with $intC \neq \emptyset$, $e \in intC$ and $\precsim_C$ a partial ordering with respect to C. The Gerstewitz's nonlinear scalarization function [1, 2, 3, 4, 5, 10, 13, 14, 15, 16, 17, 18, 19, 20] $\xi_e : Y \to \mathbb{R}$ is defined by

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - C\}, \text{ for all } y \in Y.$$

Lemma 2.5 (see [1, 2, 3, 4, 5, 10, 13, 14, 15, 16, 17, 18, 19, 20])**.** *For each $r \in \mathbb{R}$ and $y \in Y$, the following statements are satisfied:*

(i) $\xi_e(y) \leq r \iff y \in re - C;$

(ii) $\xi_e(y) > r \iff y \notin re - C;$

(iii) $\xi_e(y) \geq r \iff y \notin re - intC$;
(iv) $\xi_e(y) < r \iff y \in re - intC$;
(v) $\xi_e(y) = r \iff y \in re - \partial(C)$;
(vi) $\xi_e(re) = r$;
(vii) $\xi_e(y + re) = \xi_e(y) + r$;
(viii) $\xi_e(\cdot)$ *is positively homogeneous and continuous on* Y;
(ix) *If* $y_1 \in y_2 + C$ *(i.e.* $y_2 \precsim_C y_1$*), then* $\xi_e(y_2) \leq \xi_e(y_1)$;
(x) *If* $y_1 \in y_2 + intC$ *(i.e.* $y_2 \ll_C y_1$*), then* $\xi_e(y_2) < \xi_e(y_1)$
(xi) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$ *for all* $y_1, y_2 \in Y$.

Theorem 2.1. *Suppose that* $f : X \to Y$ *is* (EV)*-usc on* X*. Then the following statements hold:*
(a) *For any* $\lambda \in \mathbb{R}$, $\{x \in X : f(x) \in \lambda e - intC\}$ *is open in* X.
(b) *For any* $\lambda \in \mathbb{R}$, $\{x \in X : f(x) \notin \lambda e - intC\}$ *is closed in* X.

Proof. The conclusions (a) and (b) are obviously equivalent, so it is sufficient to verify (a). Let $\lambda \in \mathbb{R}$ be given. Take $z \in \{x \in X : f(x) \in \lambda e - intC\}$. Then we have

$$f(z) \in \lambda e - intC \iff \xi_e \circ f(z) < \lambda. \quad \text{(by Lemma 2.5 (iv))}$$

Let $\varepsilon_0 := \lambda - \xi_e \circ f(z) > 0$. Since f is (EV)-usc at z, there exists an open neighborhood $\mathscr{N}(z)$ of z such that

$$\begin{aligned} & f(x) - f(z) - \varepsilon_0 e \in -intC \\ \iff & f(x) \in f(z) - (\xi_e \circ f(z))e + \lambda e - intC \end{aligned} \tag{2.9}$$

for all $x \in \mathscr{N}(z)$. We now claim that $f(z) - (\xi_e \circ f(z))e \in -C$. On the contrary, assume that $f(z) - (\xi_e \circ f(z))e \notin -C$. Applying Lemma 2.5 (ii), we have

$$\begin{aligned} & f(z) \notin (\xi_e \circ f(z))e - C \\ \iff & \xi_e \circ f(z) > \xi_e \circ f(z), \end{aligned}$$

which leads a contradiction. Hence it must be $f(z) - (\xi_e \circ f(z))e \in -C$. So for any $x \in \mathscr{N}(z)$, by (2.9) and using Lemma 2.2, we obtain

$$f(x) \in f(z) - (\xi_e \circ f(z))e + \lambda e - intC \subseteq \lambda e - intC - C = \lambda e - intC,$$

which implies $\mathscr{N}(z) \subseteq \{x \in X : f(x) \in \lambda e - intC\}$. This shows $\{x \in X : f(x) \in \lambda e - intC\}$ is open in X. Therefore the conclusion (a) holds. It is obvious that (b) is immediate from (a). The proof is completed. □

Recall that a vector-valued function $f : X \to Y$ is said to be (e,C)-*lower semicontinuous* on X [1, 2, 16] if for each $r \in \mathbb{R}$, the set $\{x \in X : f(x) \in re - C\}$ is closed.

Theorem 2.2. *Suppose that* $f : X \to Y$ *is* (EV)*-lsc on* X*. Then the following statements holds:*
(a) *For any* $\lambda \in \mathbb{R}$, $\{x \in X : f(x) \notin \lambda e - C\}$ *is open in* X.
(b) *For any* $\lambda \in \mathbb{R}$, $\{x \in X : f(x) \in \lambda e - C\}$ *is closed in* X.
(c) f *is* (e,C)*-lower semicontinuous on* X.

Proof. It is sufficient to verify (a). Let $\lambda \in \mathbb{R}$ be given. Put

$$\mathscr{U}_\lambda := \{x \in X : f(x) \notin \lambda e - C\}.$$

Take $z \in \mathscr{U}_\lambda$. Thus we have

$$f(z) \notin \lambda e - C \iff \xi_e \circ f(z) > \lambda. \qquad \text{(by Lemma 2.5 (ii))}$$

Let $\widehat{\varepsilon} := \xi_e \circ f(z) - \lambda > 0$. Since f is (EV)-lsc at z, there exists an open neighborhood $\mathscr{N}(z)$ of z such that

$$f(x) - f(z) + \widehat{\varepsilon} e \in intC$$
$$\iff f(x) \in f(z) - (\xi_e \circ f(z))e + \lambda e + intC$$

for all $x \in \mathscr{N}(z)$. For any $x \in \mathscr{N}(z)$, by (vii) and (x) in Lemma 2.5, we get

$$\begin{aligned} \xi_e(f(x)) &> \xi_e(f(z) + (\lambda - (\xi_e \circ f(z))e) \\ &= \xi_e \circ f(z) + \lambda - \xi_e \circ f(z) \\ &= \lambda. \end{aligned} \tag{2.10}$$

Hence, by (2.10) and applying Lemma 2.5 (ii), we obtain

$$f(x) \notin re - C \quad \text{for all } x \in \mathscr{N}(z),$$

which means that $\mathscr{N}(z) \subseteq \mathscr{U}_\lambda$. Therefore $\mathscr{U}_\lambda$ is open in X and the conclusion (a) is proved. It is obvious that (b) is immediate from (a) and (c) is immediate from (b). The proof is completed. □

3. EXISTENCE THEOREMS FOR ESSENTIAL VECTORIAL SEMICONTINUOUS AND (C,κ)-ADJCONVEX VECTOR-VALUED FUNCTIONS

We start with the following crucial existence theorem which will be used for proving our main results.

Theorem 3.1. *Let $\beta \in \mathbb{R}$ and $\kappa : Y \to Y$ be a vector-valued function satisfying the following condition:*

(S) For any $\varepsilon > 0$, there exists $\gamma > 0$ such that

$$x \notin \beta e - C \text{ and } x \in (\beta + \gamma)e - intC \implies \kappa(x) \in (\beta + \varepsilon)e - intC.$$

Then there exists a strictly decreasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive real numbers such that

$$\kappa((\beta + \lambda_{n+1})e) \in (\beta + \lambda_n)e - intC$$

for all $n \in \mathbb{N}$ and $\lambda_n \downarrow 0$ as $n \to \infty$.

Proof. Given $\lambda_1 > 0$. Thus, according to (S), there exists $\delta_1 > 0$ such that

$$x \notin \beta e - C \text{ and } x \in (\beta + \delta_1)e - intC \implies \kappa(x) \in (\beta + \lambda_1)e - intC. \tag{3.1}$$

Let $\lambda_2 = \min\left\{\frac{\delta_1}{2}, \frac{\lambda_1}{2}\right\}$ and take $v_1 = (\beta + \lambda_2)e \in Y$. Then, by Lemma 2.1, we have the following:

- $v_1 \notin \beta e - C$;
- $v_1 \in (\beta + \delta_1)e - intC$;
- $\lambda_2 < \lambda_1$.

So, by using (3.1), we have

$$\kappa(v_1) \in (\beta+\lambda_1)e - intC.$$

For λ_2, it must exist $\delta_2 > 0$ such that

$$x \notin \beta e - C \text{ and } x \in (\beta+\delta_2)e - intC \quad \Longrightarrow \quad \kappa(x) \in (\beta+\lambda_2)e - intC. \tag{3.2}$$

Put $\lambda_3 = \min\left\{\frac{\delta_2}{2}, \frac{\lambda_2}{2}\right\}$ and $v_2 = (\beta+\lambda_3)e \in Y$. Thus we have the following:

- $v_2 \notin \beta e - C$;
- $v_2 \in (\beta+\delta_2)e - intC$;
- $\lambda_3 < \lambda_2$.

By (3.2), we obtain $\kappa(v_2) \in (\beta+\lambda_2)e - intC$. Continuing this process, for λ_j, $j \in \mathbb{N}$ with $j \geq 2$, it must exist $\delta_j > 0$ such that

$$x \notin \beta e - C \text{ and } x \in (\beta+\delta_j)e - intC \quad \Longrightarrow \quad \kappa(x) \in (\beta+\lambda_j)e - intC. \tag{3.3}$$

Take

$$\lambda_{j+1} = \min\left\{\frac{\delta_j}{2}, \frac{\lambda_j}{2}\right\} \tag{3.4}$$

and

$$v_j = \left(\beta+\lambda_{j+1}\right)e \in Y. \tag{3.5}$$

Taking into account (3.3), (3.4) and (3.5), we deduce that $\lambda_{j+1} < \lambda_j$ and $\kappa(v_j) \in (\beta+\lambda_j)e - intC$. So, we can construct a strictly decreasing sequences $\{\lambda_n\}$ of positive real numbers such that

$$\kappa((\beta+\lambda_{n+1})e) \in (\beta+\lambda_n)e - intC$$

Thanks to (3.4), we obtain $0 < \lambda_{n+1} \leq \frac{\lambda_1}{2^n}$ for $n \in \mathbb{N}$, which yields $\lambda_n \downarrow 0$ as $n \to \infty$. The proof is completed. □

Remark 3.1. [5, Lemma 3] is a special case of Theorem 3.1.

The following result is immediate from Theorem 3.1, if we take $Y = \mathbb{R}$, $K = [0,+\infty) \subset \mathbb{R}$ and $e = 1$.

Corollary 3.1. *Let $\beta \in \mathbb{R}$ and $\tau : \mathbb{R} \to \mathbb{R}$ be a function satisfying the following condition:*

($S_{\mathbb{R}}$) For any $\varepsilon > 0$, there exists $c > 0$ such that

$$\beta < x < \beta + c \quad \textit{implies} \quad \tau(x) < \beta + \varepsilon.$$

Then there exists a strictly decreasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $\tau(\beta+\lambda_{n+1}) < \beta+\lambda_n$ for all $n \in \mathbb{N}$ and $\lambda_n \downarrow 0$ as $n \to \infty$.

Corollary 3.2 (see [6, Lemma 3.1])**.** *Let $\beta \in \mathbb{R}$ and $\tau : \mathbb{R} \to \mathbb{R}$ be a function satisfying $\lim\limits_{x\to\beta^+} \tau(x) = \beta$. Then there exists a strictly decreasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $\tau(\beta+\lambda_{n+1}) < \beta+\lambda_n$ for all $n \in \mathbb{N}$ and $\lambda_n \downarrow 0$ as $n \to \infty$.*

Proof. For any $\varepsilon > 0$, since $\lim\limits_{x\to\beta^+} \tau(x) = \beta$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\beta < x < \beta+\delta \quad \text{implies} \quad \tau(x) < \beta+\varepsilon.$$

Therefore the conclusion is immediate from Corollary 3.1. □

Now, we establish the following existence theorem which is one of the main results of this paper and will be applied to eigenvector problem, fixed point problem, root-finding problem and minimization problem in Section 4.

Theorem 3.2. *Let $(E,\|\cdot\|)$ be a normed linear space, Y be a locally convex Hausdorff t.v.s. with its zero vector θ, C be a proper, closed and convex pointed cone in Y with $intC \neq \emptyset$, and let $e \in intC$ be fixed. Let $\beta \in \mathbb{R}$, W be a nonempty weakly compact and convex subset of E, $\kappa : Y \to Y$ be a $\precsim_C$-nondecreasing vector-valued function satisfying the condition (S) as in Theorem 3.1 and $\varphi : W \to Y$ be a vector-valued function. Assume that*

- $(P1)$ *for any positive real number γ, $\{x \in W : \varphi(x) \in (\beta+\gamma)e - C\}$ is a nonempty closed subset of W,*
- $(P2)$ *φ is (C,κ)-adjconvex.*

Then there exists $v \in W$ such that $\varphi(v) \in \beta e - C$.

Proof. By applying Theorem 3.1, there exists a strictly decreasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive real numbers such that

$$\kappa((\beta+\lambda_{n+1})e) \in (\beta+\lambda_n)e - intC$$

$$\iff \quad \kappa((\beta+\lambda_{n+1})e) \ll_C (\beta+\lambda_n)e \tag{3.6}$$

for all $n \in \mathbb{N}$, and $\lambda_n \downarrow 0$ as $n \to \infty$. For any $n \in \mathbb{N}$, let

$$\begin{aligned} H_n &= \{x \in W : \varphi(x) \in (\beta+\lambda_n)e - C\} \\ &= \{x \in W : \varphi(x) \precsim_C (\beta+\lambda_n)e\}. \end{aligned}$$

By $(P1)$, H_n is a nonempty closed subset of W. Define $F : W \to \mathbb{R}$ by $\Gamma(x) = \xi_e \circ \varphi(x)$ for $x \in W$. Applying Lemma 2.5 (i), we have

$$H_n = \{x \in W : \Gamma(x) \leq \beta + \lambda_n\}.$$

Clearly, $H_{n+1} \subseteq H_n$ for all $n \in \mathbb{N}$. We choose a an arbitrary point z_n from H_n for all $n \in \mathbb{N}$. For any $m,n \in \mathbb{N}$ with $m \geq n$, let

$$D_{m,n} = \{z_i : n+1 \leq i \leq m+1\}.$$

We first claim that

$$co(D_{m,n}) \subseteq H_n \quad \text{for all } m,n \in \mathbb{N} \text{ with } m \geq n. \tag{3.7}$$

Indeed, let $m,n \in \mathbb{N}$ with $m \geq n$. If $m = n$, then

$$co(D_{n,n}) = \{z_{n+1}\} \subseteq H_{n+1} \subseteq H_n$$

and (3.7) is true. For $m \geq 2$ and $n = m-1$, $co(D_{m,m-1}) = co\{z_m, z_{m+1}\}$. If $x \in co(D_{m,m-1})$, then there exists $t \in [0,1]$ such that

$$x = tz_m + (1-t)z_{m+1}. \tag{3.8}$$

Since $z_m, z_{m+1} \in H_m$, $\varphi(z_m), \varphi(z_{m+1}) \in (\beta+\lambda_m)e - C$. Since C is a convex cone, we obtain

$$t\varphi(z_m) + (1-t)\varphi(z_{m+1}) \in (\beta+\lambda_m)e - C.$$

which deduces

$$t\varphi(z_m) + (1-t)\varphi(z_{m+1}) = (\beta+\lambda_m)e - \zeta$$

for some $\zeta \in C$. Since $(\beta+\lambda_m)e-\zeta \precsim_C (\beta+\lambda_m)e$ and κ is $\precsim_C$-nondecreasing, we get

$$\kappa(t\varphi(z_m)+(1-t)\varphi(z_{m+1})) = \kappa((\beta+\lambda_m)e-\zeta) \precsim_C \kappa((\beta+\lambda_m)e) \tag{3.9}$$

Taking into account $(P2)$, (3.6), (3.8) and (3.9), we get

$$\varphi(x) \precsim_C \kappa(t\varphi(z_m)+(1-t)\varphi(z_{m+1})) \precsim_C \kappa((\beta+\lambda_m)e) \ll_C (\beta+\lambda_{m-1})e$$

$$\iff \quad \Gamma(x) = \xi_e \circ \varphi(x) < \beta+\lambda_{m-1}. \quad \text{(by Lemma 2.5 (iv))}$$

which implies $x \in H_{m-1}$. Hence we prove $co(D_{m,m-1}) \subseteq H_{m-1}$ and (3.7) is true for $m \geq 2$ and $n = m-1 < m$. Assume that (3.7) is valid for $n = k < m$. Note first that

$$\begin{aligned} co(D_{m,k-1}) &= co(\{z_i : k \leq i \leq m+1\}) \\ &= co(\{z_k\} \cup \{z_{k+1}, \cdots, z_{m+1}\}) \\ &= co(\{z_k\} \cup D_{m,k}). \end{aligned}$$

Let $p \in co(D_{m,k-1})$ be given. If $p = z_i$ for some $i_0 \in \{k, k+1, \cdots, m+1\}$, then $p \in H_{i_0} \subseteq H_{k-1}$. Suppose $p \neq z_i$ for all $i \in \{k, k+1, \cdots, m+1\}$. Thus there exist $\gamma_k, \gamma_{k+1}, \cdots, \gamma_{m+1} \in [0,1)$ with $\sum_{i=k}^{m+1} \gamma_i = 1$, such that $p = \sum_{i=k}^{m+1} \gamma_i z_i$. Let

$$w = \sum_{i=k+1}^{m+1} \frac{\gamma_i}{1-\gamma_k} z_i.$$

Due to $\sum_{i=k+1}^{m+1} \frac{\gamma_i}{1-\gamma_k} = 1$ and applying the induction hypothesis, we know $w \in co(D_{m,k}) \subseteq H_k$ and

$$p = \sum_{i=k}^{m+1} \gamma_i z_i = \gamma_k z_k + (1-\gamma_k)w.$$

Since $z_k, w \in H_k$, we have $\varphi(z_k), \varphi(w) \in (\beta+\lambda_k)e - C$ and deduce that

$$\gamma_k \varphi(z_k) + (1-\gamma_k)\varphi(w) \in (\beta+\lambda_k)e - C.$$

So $\gamma_k \varphi(z_k) + (1-\gamma_k)\varphi(w) = (\beta+\lambda_k)e - d$ for some $d \in C$. Since φ is (C,κ)-adjconvex and κ is $\precsim_C$-nondecreasing, we obtain

$$\varphi(p) \precsim_C \kappa(\gamma_k\varphi(z_k)+(1-\gamma_k)\varphi(w)) = \kappa((\beta+\lambda_k)e-d) \precsim_C \kappa((\beta+\lambda_k)e) \ll_C (\beta+\lambda_{k-1})e.$$

This shows $p \in H_{k-1}$. Hence we conclude $co(D_{m,k-1}) \subseteq H_{k-1}$. Therefore (3.7) is true by mathematic induction. For any $n \in \mathbb{N}$, let

$$U_n = \{x_i : i \geq n+1\}.$$

Then $co(U_n) \subseteq H_n$ for all $n \in \mathbb{N}$. Indeed, assume that $co(U_{j^*}) \nsubseteq H_{j^*}$ for some $j^* \in \mathbb{N}$. So there exists $z_{k_1}, z_{k_2}, \cdots, z_{k_s} \in U_{j^*}$ and $\alpha_1, \alpha_2, \cdots, \alpha_s \geq 0$ with $\sum_{i=1}^{s} \alpha_i = 1$, such that $\sum_{i=1}^{s} \alpha_i z_{k_i} \in co\left(D_{k_t-1,k_1-1}\right)$ and $\sum_{i=1}^{s} \alpha_i z_{k_i} \notin H_{j^*}$.

On the other hand, since $k_i \geq j^*+1$ for all $1 \leq i \leq s$, we have $D_{k_s-1,k_1-1} \subseteq D_{k_s-1,j^*}$ and hence deduces from (3.7) that

$$co\left(D_{k_s-1,k_1-1}\right) \subseteq co\left(D_{k_s-1,j^*}\right) \subseteq H_{j^*},$$

which leads to a contradiction. Hence $co(U_n) \subseteq H_n$ for all $n \in \mathbb{N}$. By the closedness of H_n, we have $\overline{co}(U_n) \subseteq H_n$ for all $n \in \mathbb{N}$. Since $\overline{co}(U_{n+1}) \subseteq \overline{co}(U_n)$ and $\overline{co}(U_n)$ is weakly compact for all $n \in \mathbb{N}$, $\{\overline{co}(U_n) : n \in \mathbb{N}\}$ is a family of closed subsets of the weakly compact set $\overline{co}(U_1)$ which has the finite intersection property. Hence, we get

$$\emptyset \neq \bigcap_{n\in\mathbb{N}} \overline{co}(U_n) \subseteq \bigcap_{n\in\mathbb{N}} H_n.$$

So we can take $v \in \bigcap_{n\in\mathbb{N}} H_n(x) \subseteq W$. Accordingly, $\Gamma(v) \leq \beta + \lambda_n$ for all $n \in \mathbb{N}$. Since $\lambda_n \downarrow 0$ as $n \to \infty$, we obtain

$$\Gamma(v) \leq \beta \quad \Longleftrightarrow \quad \varphi(v) \in \beta - C.$$

The proof is completed. □

Corollary 3.3. *Let W be a nonempty weakly compact and convex subset of a normed linear space $(E, \|\cdot\|)$ with origin θ, $\tau : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function satisfying $\lim_{x\to\beta^+} \tau(x) = \beta$ and $h : W \to \mathbb{R}$ be a function. Let $\beta \in \mathbb{R}$. Suppose that*

- (a) *for any positive real number γ, $\{x \in W : h(x) \leq \beta + \gamma\}$ is a nonempty closed subset of W,*
- (b) *h is (τ)-adjconvex.*

Then there exists $v \in W$ such that $h(v) \leq \beta$.

Proof. Take $Y = \mathbb{R}$, $C = [0, +\infty) \subset \mathbb{R}$ and $e = 1$. Then Y is a locally convex Hausdorff t.v.s. with its zero vector $\theta = 0$, C is a proper, closed and convex pointed cone in Y with $intC = (0, +\infty) \neq \emptyset$, and $1 \in intC$. Define a partial ordering $\lesssim_K$ with respect to C by

$$x \precsim_C y \Longleftrightarrow y - x \in C.$$

Then h is a mapping from W into Y and $\tau : Y \to Y$ is a $\precsim_C$-nondecreasing function satisfying the condition (S) as in Theorem 3.1. Clearly, conditions (a) and (b) respectively imply conditions $(P1)$ and $(P2)$ as in Theorem 3.2. Hence all the assumptions of Theorem 3.2 are satisfied and the required conclusion follows directly from Theorem 3.2. The proof is completed. □

Theorem 3.3. *In Theorem 3.2, if the condition $(P1)$ is replaced with condtions $(q1)$ and $(q2)$, where*

- $(q1)$ *φ is (e, C)-lower semicontinuous on W;*
- $(q2)$ *for any positive real number γ, there exists $x \in W$ such that $\varphi(x) \in (\beta + \gamma)e - C$.*

Then there exists $v \in W$ such that $\varphi(v) \in \beta - C$.

Proof. For any positive real number γ, by $(q1)$, $(q2)$, the set $\{x \in W : \varphi(x) \in (\beta + \gamma)e - C\}$ is a nonempty closed subset of W. So the condition $(P1)$ as in Theorem 3.2 holds. Applying Theorem 3.2, we can derive the conclusion immediately. □ □

Remark 3.2. In Theorem 3.3, the condition $(q1)$ can be replaced by $(q1)'$, where

$(q1)'$ φ is (EV)-lsc on W.

Corollary 3.4. *In Corollary 3.3, if the condition (a) is replaced with condtions $(a1)$ and $(a2)$, where*

- $(a1)$ *h is lower semicontinuous;*

$(a2)$ *for any positive real number* γ*, there exists* $x \in W$ *such that* $h(x) \leq \gamma$.

Then there exists $v \in W$ *such that* $h(v) \leq \beta$.

4. Some applications

Let M be a nonempty subset of a real linear space V with origin θ_V and $T : M \to V$ a mapping. A point $z \in M$ is called a *fixed point* of T if $T(z) = z$. A real number λ and a point $x \in M$ with $x \notin \theta_V$ are called an *eigenvalue* and the corresponding *eigenvector* of T, respectively, if $T(x) = \lambda x$.

In this section, we will study the following eigenvector problem for a given real number λ (abbreviated as $[EIVP(\lambda)]$):

$[EIVP(\lambda)]$ Given a real number λ, find $v \in M$ with $v \neq \theta_V$ such that $T(v) = \lambda v$.

Theorem 4.1. *Let* $(\mathbb{B}, \|\cdot\|)$ *be a normed linear space with its zero vector* $\mathbf{0}_{\mathbb{B}}$*,* C *be a proper, closed and convex pointed cone in* $\mathbb{B}$ *with* $intC \neq \emptyset$*, and let* $e \in intC$ *be fixed. Let* $\beta = 0$ *and* W *be a nonempty weakly compact and convex subset of* $\mathbb{B}$*,* $\kappa : \mathbb{B} \to \mathbb{B}$ *be a* $\precsim_C$*-nondecreasing linear vector-valued function satisfying the condition (S) as in Theorem 3.1 and* $\kappa(y) \in y - C$ *for all* $y \in \mathbb{B}$*. Assume that* $\lambda \geq 0$ *and* $T : W \to \mathbb{B}$ *is a vector-valued function satisfying the following conditions:*

$(V1)$ T *is* (EV)*-lsc on* W*,*

$(V2)$ T *is* (C, κ)*-adjconvex,*

$(V3)$ *for any positive real number* γ*, there exists* $x \in W$ *such that* $(T - \lambda I)(x) \in \gamma e - C$*,*

$(V4)$ $(T - \lambda I)(y) \in C$ *for all* $y \in W$*, where* $I(a) = a$ *for all* $a \in W$.

Then there exists $v \in W$ *such that* $T(v) = \lambda v$.

Moreover, if we further assume $\mathbf{0}_{\mathbb{B}} \notin W$*, then the problem* $[EIVP(\lambda)]$ *has a solution.*

Proof. Applying Lemma 2.4, the mapping $-\lambda I$ is (EV)-lsc on W. Using $(V1)$ and Lemma 2.3, $T - \lambda I = T + (-\lambda I)$ is also (EV)-lsc on W. By Theorem 2.2, $T - \lambda I$ is (e, C)-lower semicontinuous on W. We now claim that $T - \lambda I$ is (C, κ)-adjconvex. Let $x, y \in W$ and $t \in [0, 1]$ be given. By $(V2)$, we have

$$\kappa(tT(x) + (1-t)T(y)) - T(tx + (1-t)y) \in C. \tag{4.1}$$

By our hypothesis,

$$\kappa(tx + (1-t)y) \in tx + (1-t)y - C. \tag{4.2}$$

Since κ is linear and $\lambda \geq 0$, by taking into account (4.1) and (4.2), we get

$$\begin{aligned}
&\kappa(t(T - \lambda I)(x) + (1-t)(T - \lambda I)(y)) - (T - \lambda I)(tx + (1-t)y) \\
&= \kappa(tT(x) + (1-t)T(y) - \lambda(tx + (1-t)y)) - T(tx + (1-t)y) + \lambda(tx + (1-t)y) \\
&= \kappa(tT(x) + (1-t)T(y)) - \lambda\kappa(tx + (1-t)y) - T(tx + (1-t)y) + \lambda(tx + (1-t)y) \\
&= [\kappa(tT(x) + (1-t)T(y)) - T(tx + (1-t)y)] - \lambda[\kappa(tx + (1-t)y) - (tx + (1-t)y)] \\
&\in C + \lambda C \subseteq C,
\end{aligned}$$

which shows that $T - \lambda I$ is (C, κ)-adjconvex. Hence all the assumptions of Theorem 3.2 are satisfied. Applying Theorem 3.2, there exists $v \in W$ such that $(T - \lambda I)(v) \in -C$. Therefore, by our hypothesis $(V4)$, we get

$$(T - \lambda I)(v) \in C \cap (-C) = \{\mathbf{0}_{\mathbb{B}}\},$$

which deduces $(T-\lambda I)(v)=\mathbf{0}_{\mathbb{B}}$ or $T(v)=\lambda v$. Moreover, if $\mathbf{0}_{\mathbb{B}}\notin W$, then $v\neq \mathbf{0}_{\mathbb{B}}$ and hence v is a solution of $[EIVP(\lambda)]$. The proof is completed. □

As a direct consequence of Theorem 4.1 with $\lambda=1$, we obtain the following new fixed point theorem.

Theorem 4.2. *In Theorem 4.1 , if $\lambda=1$ and the conditions $(V3)$ and $(V4)$ are replaced by $(V3)_p$ and $(V4)_p$ respectively, where*

$(V3)_p$ for any positive real number γ, there exists $x\in W$ such that $(T-I)(x)\in \gamma e-C$,
$(V4)_p$ $(T-I)(y)\in C$ for all $y\in W$, where $I(a)=a$ for all $a\in W$.

Then there exists $v\in W$ such that $T(v)=v$.

Applying Theorem 4.1 with $\lambda=0$, we establish the following existence and uniqueness theorem for zeros of vector-valued functions.

Theorem 4.3. *In Theorem 4.1, if $\lambda=0$ and the conditions $(V3)$ and $(V4)$ are replaced by $(V3)_e$ and $(V4)_e$ respectively, where*

$(V3)_e$ for any positive real number γ, there exists $x\in W$ such that $T(x)\in \gamma e-C$,
$(V4)_e$ $T(y)\in C$ for all $y\in W$.

Then there exists $v\in W$ such that $T(v)=\mathbf{0}_{\mathbb{B}}$.

Moreover, if we further assume $\kappa(\mathbf{0}_{\mathbb{B}})=\mathbf{0}_{\mathbb{B}}$ and the condition $(V2)$ is replaced with $(V2)_e$, where

$(V2)_e$ T is strictly (C,κ)-adjconvex,

then the equation $T(x)=\mathbf{0}_{\mathbb{B}}$ has a unique root in W.

Proof. Applying Theorem 4.1 with $\lambda=0$, there exists $v\in W$ such that $T(v)=\lambda v=\mathbf{0}_{\mathbb{B}}$. This means that the equation $T(x)=\mathbf{0}_{\mathbb{B}}$ has at least one root in W. Assume that $u,v\in W$ are two distinct roots of $T(x)=\mathbf{0}_{\mathbb{B}}$. Since W is convex and $\kappa(\mathbf{0}_{\mathbb{B}})=\mathbf{0}_{\mathbb{B}}$, we have $\frac{1}{2}u+\frac{1}{2}v\in W$ and $\kappa\left(\frac{1}{2}T(u)+\frac{1}{2}T(v)\right)=\mathbf{0}_{\mathbb{B}}$. By $(V2)_e$, we obtain

$$\kappa\left(\frac{1}{2}T(u)+\frac{1}{2}T(v)\right)-T\left(\frac{1}{2}u+\frac{1}{2}v\right)\in intK,$$

which implies

$$T\left(\frac{1}{2}u+\frac{1}{2}v\right)\in -intC\cap C=\emptyset,$$

a contradiction. So, the equation $T(x)=\mathbf{0}_{\mathbb{B}}$ has a unique root in W. The proof is completed. □

As an interesting application of Corollary 3.3 (or Theorem 3.2), we derive the following new minimization theorem.

Theorem 4.4. *Let W be a nonempty weakly compact and convex subset of a normed linear space $(E,\|\cdot\|)$ with origin θ and $h:W\to\mathbb{R}$ be a lower semicontinuous and bounded below function with $\beta:=\inf_{x\in W}h(x)$. If there exists a nondecreasing function $\tau:\mathbb{R}\to\mathbb{R}$ satisfying $\lim_{x\to\beta^+}\tau(x)=\beta$ such that h is (τ)-adjconvex, then h attains its minimum on W.*

Moreover, if we further assume $\tau(\beta)=\beta$ and h is strictly (τ)-adjconvex, then there exists a unique point v in W, such that $h(v)=\inf_{x\in W}h(x)$.

Proof. For any $\gamma > 0$, since h is lower semicontinuous and $\beta = \inf_{x\in W} h(x)$, the set $\{x \in M : f(x) \leq \beta + \gamma\}$ is a nonempty closed subset of M. Applying Corollary 3.3, there exists $v \in M$ such that $h(v) \leq \beta = \inf_{x\in W} h(x)$. This shows that $h(v) = \inf_{x\in W} h(x)$. Moreover, we assume that $\tau(\beta) = \beta$ and h is strictly (τ)-adjconvex. Suppose that u, v are two distinct points of W such that $h(u) = h(v) = \inf_{x\in W} h(x) = \beta$. Since W is convex, we have $\frac{1}{2}u + \frac{1}{2}v \in W$. By the strictly (τ)-adjconvexity of h, we get

$$\beta \leq h\left(\frac{1}{2}u + \frac{1}{2}v\right) < \tau\left(\frac{1}{2}h(x) + \frac{1}{2}h(y)\right) = \tau(\beta) = \beta,$$

a contradiction. Therefore, there exists a unique point v in W, such that $h(v) = \inf_{x\in W} h(x)$. The proof is completed. □

5. Conclusions

In this paper, inspired by the literature on generalizations of the concept and properties of semicontinuity, we introduce and investigate new concepts of essential vectorial lower semicontinuity, essential vectorial upper semicontinuity and essential vectorial continuity (see Definition 2.1) and obtain new properties and features for vectorial lower semicontinuity, essential vectorial upper semicontinuity and essential vectorial continuity. Utilizing these new concepts and results, we propose new existence theorems for C-adjustability convex vector-valued functions in the weakly compact setting. Additionally, more applications such as new fixed point theorem and existence theorems for root-finding problem, eigenvector problem and minimization problem are provided (for more details, see Theorems 4.1, 4.2, 4.3 and 4.4). In summary, our results in this paper are significantly original. We believe our results will be widely applied in nonlinear analysis, optimization and various fields of science. Therefore, further research on the new concepts and results presented in this paper would be of interest.

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