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COOPERATIVE-TRAJECTORY CHARACTERISTIC FUNCTION IN NETWORK DIFFERENTIAL GAMES WITH TERMINAL PAYOFFS

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In Memory of Professor Josef Shinar

Abstract. This paper extends the cooperative-trajectory characteristic function approach to differential network games with terminal payoffs. We derive the corresponding dynamic Shapley value solution and imputation distribution procedure. The salient features of this type of characteristic function include measuring the marginal contributions of individual players are evaluated based on their cooperative actions/strategy, and the simplicity in evaluating the cooperative-trajectory characteristic function. This expands the application of the approach to a much wider class of games, especially real-life games that involve a sizable number of players and terminal payoffs.

Keywords. Characteristic function; Imputation distribution; Network differential games; Shapley value.

2020 Mathematics Subject Classification. 91A25, 91B06, 90B10, 91-08.

1. Introduction

This paper is dedicated to the memory of our highly esteemed scholar, colleague, and friend – Professor Josef Shinar. His seminal works on control and game theory have and will continue to shed light on the development of control and differential game theories. A fast-growing branch of game theory is network games or games on networks which have become a field that attracts theoretical and technical developments. Mazalov and Chirkova [1] provided a comprehensive disquisition on the theory and applications of networking games. Network differential games have become a field that attracts theoretical and technical developments. The classic book of N. N. Krasovskii's [2] on the fundamentals of differential games provided profound impact to the field. Wie [3, 4] developed differential game models for studying traffic network. Zhang et

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al. [5] presented a differential game of network defense. Pai [6] provided a differential game formulation of a controlled network. Meza and Lopez-Barrientos [7] examined a differential game of a duopoly with network externalities. Cao et al. [8] studied the minimax equilibrium of network differential games. Petrosyan [9] developed cooperative differential games on networks. Petrosyan et al. [10], Petrosyan and Yeung [11], Petrosyan et al. [12], and Petrosyan et al. [13] developed theories in network differential games. Yeung et al. [14], Yeung et al. [15] presented network dynamic game applications.

Designing solutions for cooperative network differential games is a classic problem in the field. The Shapley [16] value appears to be one of the best solutions in attributing a fair gain to each player in a complex situation like a network. However, the determination of the worth of the subsets of players (characteristic function) in the Shapley value is not indisputably unique. In addition, in a differential game, the worth of the coalitions of players change as the game evolves. The concept of cooperative-trajectory characteristic function was proposed as a novel form for measuring the worth of coalitions for the Shapley value imputation (see Petrosyan and Yeung [17], Petrosyan et al. [10], Petrosyan and Yeung [11], and Bulgakova and Petrosyan [18]). Cooperative-trajectory characteristic functions are used to generate a time-consistent Shapley value solution in a class of network differential games. This new class of characteristic functions is evaluated along the cooperative trajectory. It measures the worth of coalitions under the process of cooperation instead of under minimax confrontation or the Nash non-cooperative stance. The rationale for such formulation is to attribute the contributions of the players in the process of cooperation. The existing approaches to construct the characteristic function require the minimax or Nash equilibrium computations for each possible coalition, which may be unrealistic. Many real-life games may involve a sizable number of players. The number of possible coalitions in a n-player game is 2^n. For instance, a cooperative project involving 27 EU nations would have 134217728 (more than a hundred and thirty million) coalitions. This may require the solving of a large number of game equilibria. The cooperative trajectory characteristic function approach did not require such type of calculations. In fact, it just requires one global optimization. The resultant dynamic Shapley value imputation yields a new cooperative solution in differential network games. The salient features of the cooperative-trajectory characteristic function include

(i) measuring the worth of coalitions in the process of cooperation along the cooperative trajectory,

(ii) generating time-consistent Shapley value imputation in a dynamic framework,

(iii) the marginal contributions of individual players are evaluated based on their cooperative actions / strategy, and

(iv) the simplicity in evaluating the cooperative-trajectory characteristic function.

This paper deals with the cooperative trajectory characteristic function approach to differential network games with terminal payoffs. Formal proof and an application example are presented. This expands the application of the approach to a much wider class of games. The organization of the paper is as follows. Section 2 presents the game formulation and section 3 develops the cooperative-strategy characteristic function for games with terminal payoffs. Section 4 derives time-consistent dynamic Shapley value. Section 5 provides an illustrative example. Section 6 concludes the paper.

2. GAME FORMULATION

Consider a n-player differential network game with a game horizon $[t_0, T]$. Let $N = \{1, 2, \cdots, n\}$ be the set of players in a network game. The arcs in network L are $arc(i, j) \in L$ for players $i, j \in N$. The set of players connected to player i is $\widetilde{K}(i) = \{j : arc(i, j) \in L\}$, and the set $K(i) = \widetilde{K}(i) \cup \{i\}$, for $i \in N$. Every player $i \in N$ can cut the connection with any other players at any instant of time.

Let $x^i \in R^m$ denote the state variables of player $i \in N$, and $u^i \in U^i \subset R^k$ the control variable of player $i \in N$. The state dynamics of the game is

$$\dot{x}^i = f^i(x^i, u^i), x^i(t_0) = x_0^i, \tag{2.1}$$

for $t \in [t_0, T]$ and $i \in N$.

The payoff function of player i depends upon his state variable, the state variables of players from the set $K(i)$ and his own control variable:

$$H_i(x_0^i, x_0^{(K)i}, u^i) = \sum_{j \in K(i)} \int_{t_0}^{T} h_i^j(x^i(\tau), x^j(\tau), u^i(\tau)) d\tau + q^i(x^i(T)), \tag{2.2}$$

where $h_i^i(x^i(\tau), x^j(\tau), u^i(\tau)) = h_i^i(x^i(\tau), u^i(\tau))$ and $h_i^j \geq 0$.

The players cooperate to maximize their joint payoff by solving the problem

$$\max_{u^1, u^2, \ldots, u^n} \sum_{i \in N} (\sum_{j \in K(i)} \int_{t_0}^{T} h_i^j(x^i(\tau), x^j(\tau), u^i(\tau)) d\tau + q^i(x^i(T)), \tag{2.3}$$

subject to

$$\dot{x}^i = f^i(x^i, u^i), x^i(t_0) = x_0^i, \tag{2.4}$$

for $t \in [t_0, T]$ and $i \in N$.

We use $\bar{u}^i(t)$, for $t \in [t_0, T]$ and $i \in N$, to denote the optimal cooperative strategies of player from the solution to (2.3)-(2.4). Substituting $\bar{u}^i(t)$ into (2.4), we obtain the cooperative state dynamics

$$\dot{x}^i = f^i(x^i, \bar{u}^i), \tag{2.5}$$

for $i \in N$.

Let the cooperative trajectory be denoted by

$$\bar{x}(t) = (\bar{x}^1(t), \bar{x}^2(t), \cdots, \bar{x}^n(t)).$$

The network differential game (2.1)-(2.2) reflects cases like large-scale inter-connected electricity networks. Each electricity producer would operate on its own to produce electricity if left alone. Maintaining a connection with another producer j allows producer i to use both its transmission system $\bar{x}^i(t)$ and $\bar{x}^j(t)$. It can also reflect cases like vertical integration of firms. The payoff of a firm depends on its controls and gains with connection with upstream or downstream firms through the state of the connected firm like goodwill, knowledge and technology.

3. Characteristic Function

Now, we consider the case where the players cooperate to maximize their joint payoff. The joint cooperative payoff can be expressed as

$$\begin{aligned}&\sum_{i\in N}\Big(\sum_{j\in K(i)}\int_{t_0}^{T} h_i^j(\bar{x}^i(\tau),\bar{x}^j(\tau),\bar{u}^i(\tau))d\tau+q_i(x^i(T))\\&=\max_{u^1,u^2,\dots,u^n}\sum_{i\in N}\Big(\sum_{j\in K(i)}\int_{t_0}^{T} h_i^j(x^i(\tau),x^j(\tau),u^i(\tau))d\tau+q_i(x^i(T)),\end{aligned}\tag{3.1}$$

subject to

$$\dot{x}^i=f^i(x^i,u^i),x^i(t_0)=x_0^i,\tag{3.2}$$

for $t\in[t_0,T]$ and $i\in N$.

Then, we consider allotting the cooperative gains to individual players according to the Shapley value. One of the issues in using the Shapley value is the determination of the worth of coalitions (which is often called characteristic function). In cooperative differential games, the characterization of the worth of a coalition of players S is not indisputably unique except for the case of zero-sum games in which coalition S seeks to maximize its payoff while coalition $N\backslash S$ seeks to minimize the payoff. In this section, we present a new formulation of the worth of coalition $S\subset N$ as

$$V(S;x_0,T-t_0)=\sum_{l\in S}\Big(\sum_{j\in K(l)\cap S}\int_{t_0}^{T} h_i^j(\bar{x}^l(\tau),\bar{x}^j(\tau),\bar{u}^l(\tau))d\tau+q_l(\bar{x}^l(T)).\tag{3.3}$$

Note that the worth of coalition S is measured by the sum of the payoffs of the players in the coalition in the cooperation process with the exclusion of the gains from players outside coalition S. Thus, the characteristic function reflecting the worth of coalition S in (3.3) is formulated along the cooperative trajectory $\bar{x}(t)$. This is a novel feature and we name it as cooperative-trajectory characteristic function.

For simplicity in notation, we denote

$$\alpha_{lj}(x_0,T-t_0)=\int_{t_0}^{T} h_i^j(\bar{x}^l(\tau),\bar{x}^j(\tau),\bar{u}^l(\tau))d\tau+q_l(\bar{x}^l(T)),$$

and

$$\alpha_{lj}(\bar{x}(t),T-t)=\int_{t}^{T} h_l^j(\bar{x}^l(\tau),\bar{x}^j(\tau),\bar{u}^l(\tau))d\tau+q_l(\bar{x}^l(T)),\tag{3.4}$$

for $t\in[t_0,T]$.

Using the notations in (3.4), we can express (3.3) as

$$V(S;x_0,T-t_0)=\sum_{l\in S}\sum_{j\in K(l)\cap S}\alpha_{lj}(x_0,T-t_0),\tag{3.5}$$

and similarly, along the cooperative trajectory $\bar{x}(t)$,

$$V(S;\bar{x}(t),T-t)=\sum_{l\in S}\sum_{j\in K(l)\cap S}\alpha_{lj}(\bar{x}(t),T-t), \tag{3.6}$$

for $t\in[t_0,T]$.

Proposition 3.1. *The following inequalities hold for cooperative-trajectory characteristic function along the cooperative trajectory:*

$$V(S_1\cup S_2;\bar{x}(t),T-t)\geq V(S_1;\bar{x}(t),T-t)+V(S_2;\bar{x}(t),T-t)-V(S_1\cap S_2;\bar{x}(t),T-t), \tag{3.7}$$

for $t\in[t_0,T]$.

Proof. See Appendix A. □

The inequalities in Proposition 3.1 imply that the game is convex and so are the subgames along the cooperative trajectory. This also means that the core of the game is not void and the Shapley value belongs to the core. In the case where S_1 and S_2 are distinct subsets, we obtain the super-additivity of the cooperative-trajectory characteristic function as

$$V(S_1\cup S_2;\bar{x}(t),T-t)\geq V(S_1;\bar{x}(t),T-t)+V(S_2;\bar{x}(t),T-t).$$

The cooperative-trajectory characteristic function in (2.5) can be expressed as

$$\begin{aligned}V(S;x_0,T-t_0)&=\sum_{l\in S}(\sum_{j\in K(l)\cap S}\int_{t_0}^{T}h_l^j(\bar{x}^l(\tau),\bar{x}^j(\tau),\bar{u}^l(\tau))d\tau+q_l(x^l(T))\\&=\sum_{l\in S}\sum_{j\in K(l)\cap S}\int_{t_0}^{T}h_l^j(\bar{x}^l(\tau),\bar{x}^j(\tau),\bar{u}^l(\tau))d\tau+V(S;\bar{x}(t),T-t),\end{aligned} \tag{3.8}$$

for $S\subset N$.

Condition (3.8) exhibits the time consistent property of the cooperative-trajectory characteristic function $V(S;\bar{x}(t),T-t)$. This property has not been shared by any existing characteristic functions in differential games. Finally, any individual player attempting to act independently will have the links to other players in the network being cut off.

4. Characteristic Function

Now, we consider allocating the grand coalition cooperative network gain $V(N;x_0,T-t_0)$ according to the Shapley value using the cooperative-trajectory characteristic function. Player i's payoff under cooperation would be the Shapley value

$$Sh_i(x_0,T-t_0)=\sum_{\substack{S\subset N\\ i\in S}}\frac{(|S|-1)!(n-|S|)!}{n!}[V(S;x_0,T-t_0)-V(S\backslash\{i\};x_0,T-t_0)], \tag{4.1}$$

for $i\in N$.

Invoking (3.6), the difference $V(S;x_0,T-t_0)-V(S\backslash\{i\};x_0,T-t_0)$ can be expressed as

$$\begin{aligned}&\sum_{l\in S}\sum_{j\in K(l)\cap S}\alpha_{lj}(x_0,T-t_0)-\sum_{l\in S\backslash\{i\}}\sum_{j\in K(l)\cap S\backslash\{i\}}\alpha_{lj}(x_0,T-t_0)\\&=\sum_{j\in K(l)\cap S}\alpha_{ij}(x_0,T-t_0).\end{aligned} \tag{4.2}$$

Therefore, we can obtain the cooperative payoff of player i as the Shapley value

$$Sh_i(x_0, T-t_0) = \sum_{\substack{S\subset N\\ i\in S}} \frac{(|S|-1)!(n-|S|)!}{n!}[\sum_{j\in K(l)\cap S\backslash\{i\}} \alpha_{ij}(x_0, T-t_0)]. \tag{4.3}$$

For subgames along the cooperative trajectory, the cooperative payoff of player i can be obtained as:

$$Sh_i(\bar{x}(t), T-t) = \sum_{\substack{S\subset N\\ i\in S}} \frac{(|S|-1)!(n-|S|)!}{n!}[\sum_{j\in K(l)\cap S} \alpha_{ij}(\bar{x}(t), T-t)], \tag{4.4}$$

for $t \in [t_0, T]$, where

$$\alpha_{ij}(\bar{x}(t), T-t) = \int_t^T h_l^j(\bar{x}^i(\tau), \bar{x}^j(\tau), \bar{u}^i(\tau))d\tau + q_i(\bar{x}^i(T)).$$

Proposition 4.1. *The Shapley value imputation in (4.3)-(4.4) is time consistent.*

Proof. Substituting the values of $\alpha_{ij}(x_0, T-t_0)$ into (4.3), we obtain

$$\begin{aligned}
&Sh_i(x_0, T-t_0)\\
&= \sum_{\substack{S\subset N\\ i\in S}} \frac{(|S|-1)!(n-|S|)!}{n!} \times \sum_{j\in K(i)\cap S} (\int_{t_0}^T h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau), \bar{u}^i(\tau))d\tau + q_i(\bar{x}^i(T)))\\
&= \sum_{\substack{S\subset N\\ i\in S}} \frac{(|S|-1)!(n-|S|)!}{n!} \sum_{j\in K(i)\cap S} \int_{t_0}^T h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau), \bar{u}^i(\tau))d\tau\\
&+ \sum_{\substack{S\subset N\\ i\in S}} \frac{(|S|-1)!(n-|S|)!}{n!} \sum_{j\in K(i)\cap S} (\int_t^T h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau), \bar{u}^i(\tau))d\tau + q_i(\bar{x}^i(T))).
\end{aligned} \tag{4.5}$$

Invoking (4.4), we have

$$\begin{aligned}
Sh_i(x_0, T-t_0) = &\sum_{\substack{S\subset N\\ i\in S}} \frac{(|S|-1)!(n-|S|)!}{n!} \sum_{j\in K(i)\cap S} \int_{t_0}^t h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau), \bar{u}^i(\tau))d\tau\\
&+ Sh_i(\bar{x}(t), T-t),
\end{aligned} \tag{4.6}$$

which exhibits the time consistency property of the Shapley value imputation $Sh_i(\bar{x}(t), T-t)$, for $t \in [t_0, T]$. □

Using the Shapley value, the imputation to player i along the cooperative trajectory $\{\bar{x}(t)\}_{t=0}^T$ is:

$$Sh_i(\bar{x}(t), T-t) = \sum_{S\subseteq N} \frac{(s-1)!(n-s)!}{n!}[V(S;t,\bar{x}) - V(S\backslash\{i\};t,\bar{x})].$$

Recall that $V(S;x_0,T-t_0)-V(S\backslash\{i\};x_0,T-t_0)$ can be expressed as

$$\begin{aligned}&\sum_{l\in S}\sum_{j\in K(l)\cap S}\alpha_{lj}(x_0,T-t_0)-\sum_{l\in S\backslash\{i\}}\sum_{j\in K(l)\cap S\backslash\{i\}}\alpha_{lj}(x_0,T-t_0)\\&=\sum_{j\in K(i)\cap S}\alpha_{ij}(x_0,T-t_0)\\&=\sum_{j\in K(i)\cap S}(\int_t^T h_i^j(\bar{x}^i(\tau),\bar{x}^j(\tau),\bar{u}^i(\tau))d\tau+q_i(\bar{x}^i(T)).\end{aligned}$$

Hence

$$Sh_i(\bar{x}(t),T-t)=\sum_{S\subseteq N}\frac{(s-1)!(n-s)!}{n!}\sum_{j\in K(i)\cap S}(\int_t^T h_i^j(\bar{x}^i(\tau),\bar{x}^j(\tau),\bar{u}^i(\tau))d\tau+q_i(\bar{x}^i(T))),\quad(4.7)$$

for $i\in N$ and $t\in[t_0,T]$. Moreover,

$$\sum_{S\subseteq N}\frac{(s-1)!(n-s)!}{n!}q_i(\bar{x}^i(T))=q_i(\bar{x}^i(T)).\quad(4.8)$$

Therefore, we can express

$$Sh_i(\bar{x}(t),T-t)=\sum_{S\subseteq N}\frac{(s-1)!(n-s)!}{n!}\sum_{j\in K(i)\cap S}(\int_t^T h_i^j(\bar{x}^i(\tau),\bar{x}^j(\tau),\bar{u}^i(\tau))d\tau+q_i(\bar{x}^i(T)).\quad(4.9)$$

Crucial to the analysis is the design of an IDP such that the Shapley imputation (4.3)-(4.4) can be realized. We define an IDP $\beta_i(\tau)$ such that

$$\int_{t_0}^T\beta_i(\tau)d\tau+q_i(\bar{x}^i(T)=Sh_i(\bar{x}(t),T-t_0),\quad(4.10)$$

for $i\in N$.

Theorem 4.2. *An imputation distribution procedure (IDP) prescribing player $i\in N$ at time $t\in[t_0,T]$ an allotment*

$$\beta_i(t)=\sum_{\substack{S\subseteq N\\ i\in S}}\frac{(|S|-1)!(n-|S|)!}{n!}\sum_{j\in K(i)\cap S}h_i^j(\bar{x}^i(t),\bar{x}^j(t),\bar{u}^i(t)),\quad(4.11)$$

would lead to the realization of the Shapley value imputation $Sh_i(\bar{x}(t),T-t)$ in (4.3)-(4.4).

Proof. Comparing (4.9) with (4.10) shows that

$$\beta_i(t)=\sum_{\substack{S\subseteq N\\ i\in S}}\frac{(|S|-1)!(n-|S|)!}{n!}\sum_{j\in K(i)\cap S}h_i^j(\bar{x}^i(t),\bar{x}^j(t),\bar{u}^i(t)),$$

for $i\in N$ and $t\in[t_0,T]$. □

5. An Application Example

Consider a large interconnected network in which connectivity would yield positive externalities, like in the case of a large-scale interconnected electricity network.

The term $h_i^i(x^i(\tau),u^i(\tau)) = p^i[(u^i(t))^{1/2}(x^i(t))^{1/2}] - c^i u^i(t) - m^i v^i(t)$ is the gain to player i, where $c^i u^i(t)$ is the cost of control $u^i(t)$ and $m^i v^i(t)$ is cost of control $v^i(t)$.

The term

$$\begin{aligned}
&h_i^j(x^i(\tau),x^j(\tau),u^i(\tau))\\
&= p^{i(j)}[(u^{i(j)j}(t))^{1/2}(\varpi^{i(j)j}x^j(t))^{1/2} + (u^{i(j)i}(t))^{1/2}(\varpi^{i(j)i}x^i(t))^{1/2}]\\
&\quad - c^{i(j)j}u^{i(j)j}(t) - c^{i(j)i}u^{i(j)i}(t)
\end{aligned}$$

is the gain that player i can obtain by connecting with player j, where $c^{i(j)j}u^{i(j)j}(t)$ is the cost of control $u^{i(j)j}(t)$ and $c^{i(j)i}u^{i(j)i}(t)$ is the cost of control $u^{i(j)i}(t)$, for $j \in \tilde{K}(i)$.

The term $[Q^i x^i(T) + M^i]$ is the terminal payoff of player i. The payoff of player i is

$$\begin{aligned}
&H_i(x_0^i, x_0^{K(i)}, u^i)\\
&= \sum_{j \in K(i)} \int_{t_0}^{T} h_i^j(x^i(\tau),x^j(\tau),u^i(\tau))d\tau + q_i(x^i(T))\\
&= \int_{t_0}^{T} \{[p^i[(u^i(t))^{1/2}(x^i(t))^{1/2}] - c^i u^i(t) - m^i v^i(t)\\
&+ \sum_{j \in \widetilde{K}(i)} (p^{i(j)}[(u^{i(j)j}(t))^{1/2}(\varpi^{i(j)j}x^j(t))^{1/2}\\
&+ (u^{i(j)i}(t))^{1/2}(\varpi^{i(j)i}x^i(t))^{1/2}] - c^{i(j)j}u^{i(j)j}(t) - c^{i(j)i}u^{i(j)i}(t))]dt + [Q^i x^i(T) + M^i]\}.
\end{aligned} \tag{5.1}$$

The dynamics of the state $x^i(t)$ is:

$$\dot{x}^i(t) = a^i((v^i(t)x^i(t))^{1/2} - \sigma^i x^i(t), \tag{5.2}$$

for $i \in N$. The maximized payoff of the grand coalition can be obtained by maximizing

$$\sum_{i \in N} H_i(x_0^i, x_0^{K(i)}, u^i)$$

with the controls u^i, v^i, $u^{i(j)j}$, $u^{i(j)i}$. The payoff of the grand coalition can be obtained as:

$$V(N;\bar{x},T-t) = (\sum_{l \in N} A^{(N)l}(t)x^i + C^N(t))e^{-rt}, \tag{5.3}$$

where $A^{(N)l}(t)$, for $l \in N$, and $C^N(t)$ satisfy

$$\begin{aligned}
\dot{A}^{(N)l}(t) &= rA^{(N)l}(t) + A^{(N)l}(t)\sigma^l - \frac{(p^l)^2}{4c^l} + \frac{(a^l A^{(N)l}(t))^2}{4m^l} \\
&- \sum_{j \in \widetilde{K}(l) \cap S} p^{l(j)} \frac{(p^{l(j)})^2}{4c^{l(j)l}} \varpi^{l(j)l} - \sum_{\varsigma \in \widetilde{K}(l) \cap S} p^{\varsigma(l)} \frac{(p^{\varsigma(l)})^2}{4c^{\varsigma(l)\varsigma}} \varpi^{\varsigma(l)\varsigma} \\
&- A^{(N)l}(t) a^l \frac{a^l A^{(N)l}(t)}{2m^l}; \text{and} \\
\dot{C}^{(N)}(t) &= rC^N(t) \text{ with } A^{(N)l}(T) = Q^l \text{ and } C^N(t) = \sum_{\varsigma \in N} M^\varsigma.
\end{aligned} \tag{5.4}$$

The optimal strategies under the grand coalition can be obtained as

$$\begin{aligned}
\bar{u}^i(t) &= (\frac{p^i}{2c^i})^2 \bar{x}^i(t), \\
\bar{u}^{i(j)j}(t) &= (\frac{p^{i(j)}}{2c^{i(j)j}})^2 \varpi^{i(j)j} \bar{x}^j(t), \\
\bar{u}^{i(j)i}(t) &= (\frac{p^{i(j)}}{2c^{i(j)i}})^2 \varpi^{i(j)i} \bar{x}^i(t), \\
\bar{v}^i(t) &= \frac{(a^i A^{(N)i}(t))^2}{4(m^i)^2} \bar{x}^i(t),
\end{aligned} \tag{5.5}$$

for $i \in N$. The optimal cooperative state dynamics under the grand coalition:

$$\dot{\bar{x}}^i(t) = a^i \frac{a^i A^{(N)i}(t)}{2m} \bar{x}^i(t) - \sigma^i \bar{x}^i(t), \tag{5.6}$$

for $i \in N$. The cooperative-trajectory characteristic function:

$$\begin{aligned}
& V(S; \bar{x}(t), T-t, \bar{x}) \\
&= \sum_{l \in S} \{ \int_t^T [p^l [(\bar{u}^l(\tau))^{1/2} (\bar{x}^l(\tau))^{1/2} - c^l \bar{u}^l(\tau) - m^l \bar{v}^l(\tau) \\
&+ \sum_{j \in \widetilde{K}(l) \cap s} (p^{l(j)} [(u^{l(j)j}(\tau))^{1/2} (\varpi^{l(j)j} \bar{x}^j(\tau))^{1/2} \\
&+ (\bar{u}^{l(j)l}(\tau))^{1/2} (\varpi^{l(j)l} \bar{x}^l(\tau))^{1/2}] \\
&- c^{l(j)l} \bar{v}^{l(j)l}(\tau) - c^{l(j)l} \bar{v}^{l(j)l}(\tau))] d\tau + [Q^l \bar{x}^l(T) + M^l] \}.
\end{aligned} \tag{5.7}$$

for $S \subset N$. Substituting (5.7) into the dynamic Shapley value, we obtain

$$\begin{aligned}
& Sh_i(\bar{x}(t), T-t) \\
& = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!} [V(S; \bar{x}(t), T-t) - V(S \backslash \{i\}; \bar{x}(t), T-t)] \\
& = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!} \{ \int_t^T [p^i[(\bar{u}^i(\tau))^{1/2} (\bar{x}^i(\tau))^{1/2}] \\
& - c^i \bar{u}^i(\tau) - m^i \bar{v}^i(\tau) + \sum_{j \in \widetilde{K}(i) \cap s} (p^{i(j)}[(\bar{u}^{i(j)j}(\tau))^{1/2} (\varpi^{i(j)j} \bar{x}^j(\tau))^{1/2} \\
& + (\bar{u}^{i(j)i}(\tau))^{1/2} (\varpi^{i(j)i} \bar{x}^i(\tau))^{1/2}] - c^{i(j)j} \bar{v}^{i(j)j}(\tau) - c^{i(j)i} \bar{v}^{i(j)i}(\tau))] d\tau \\
& + [Q^i \bar{x}^i(T) + M^i] \}.
\end{aligned} \tag{5.8}$$

An IDP

$$\begin{aligned}
\beta_i(\tau) = \sum_{\substack{S \subset N \\ i \in S}} & \frac{(|S|-1)!(n-|S|)!}{n!} \{ [p^i[(\bar{u}^i(\tau))^{1/2} (\bar{x}^i(\tau))^{1/2}] \\
& - c^i \bar{u}^i(\tau) - m^i \bar{v}^i(\tau) \\
& + \sum_{j \in \widetilde{K}(i) \cap s} (p^{i(j)}[(\bar{u}^{i(j)j}(\tau))^{1/2} (\varpi^{i(j)j} \bar{x}^j(\tau))^{1/2} \\
& + (\bar{u}^{i(j)i}(\tau))^{1/2} (\varpi^{i(j)i} \bar{x}^i(\tau))^{1/2}] \\
& - c^{i(j)j} \bar{v}^{i(j)j}(\tau) - c^{i(j)i} \bar{v}^{i(j)i}(\tau))] + [Q^i \bar{x}^i(T) + M^i] \}
\end{aligned} \tag{5.9}$$

will given to player i at time instant t. Note that $\int_t^T \beta_i(\tau) d\tau = Sh_i(\bar{x}(t), T-t)$, for any $t \in [0, T]$ and $i \in N$. Hence the cooperative solution $Sh_i(\bar{x}(t), T-t)$ is time consistent.

6. Conclusion

This paper analyzes the cooperative-trajectory characteristic function approach in differential network games with terminal payoffs. The corresponding time-consistent dynamic Shapley value solution and imputation distribution procedure are developed. The salient features of this type of characteristic function include measuring the marginal contributions of individual players are evaluated based on their cooperative actions/strategy, and the simplicity in evaluating the cooperative-trajectory characteristic function. This expands the applicability of cooperative-strategy characteristic function to a much wider class of games. Future applications of the approach, especially in real-life games that involve a sizable number of players and terminal payoffs are expected.

7. APPENDIX A: PROOF OF PROPOSITION 3.1

Using (3.6), we have

$$
\begin{aligned}
&V(S_1 \cup S_2; x_t, T-t) \\
&= \sum_{i \in S_1 \cup S_2} \sum_{j \in K(i) \cap (S_1 \cup S_2)} \alpha_{ij}(x_t, T-t) \\
&= \sum_{i \in S_1} \sum_{j \in K(i) \cap S_1} \alpha_{ij}(x_t, T-t) + \sum_{i \in S_2} \sum_{j \in K(i) \cap S_2} \alpha_{ij}(x_t, T-t) \\
&- \sum_{i \in S_1 \cap S_2} \sum_{j \in K(i) \cap (S_1 \cap S_2)} \alpha_{ij}(x_t, T-t) + \sum_{i \in S_1} \sum_{j \in K(i) \cap S_2} \alpha_{ij}(x_t, T-t) \\
&+ \sum_{i \in S_2} \sum_{j \in K(i) \cap S_1} \alpha_{ij}(x_t, T-t) \\
&\geq \sum_{i \in S_1} \sum_{j \in K(i) \cap S_1} \alpha_{ij}(x_t, T-t) + \sum_{i \in S_2} \sum_{j \in K(i) \cap S_2} \alpha_{ij}(x_t, T-t) \\
&- \sum_{i \in S_1 \cap S_2} \sum_{j \in K(i) \cap (S_1 \cap S_2)} \alpha_{lj}(x_t, T-t) \\
&= V(S_1; x_t, T-t) + V(S_2; x_t, T-t) - V(S_1 \cap S_2; x_t, T-t).
\end{aligned}
$$

Hence Proposition 3.1 follows.

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