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ON MATRICES WHOSE EXPONENTIAL IS A P-MATRIX

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Dedicated to Professor Ezra Zeheb on the occasion of his 85th birthday

Abstract. A matrix is called a P-matrix if all its principal minors are positive. P-matrices have found important applications in functional analysis, mathematical programming, and dynamical systems theory. We introduce a new class of real matrices denoted $\mathbb{E}^{\mathbb{P}}$. A matrix A is in $\mathbb{E}^{\mathbb{P}}$ if $\exp(At)$ is a P-matrix any $t \geq 0$. We analyze the properties of this new class of matrices and describe an application of the theoretical results to opinion dynamics.
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1. Introduction

A matrix $A \in \mathbb{R}^{n \times n}$ is called a P-matrix if every principal minor of A is positive. In particular, the diagonal entries of A and the determinant of A are positive. The class of P-matrices, denoted $\mathbb{P}$, includes important matrix classes such as positive-definite matrices, non-singular M-matrices [1, Chapter 2], B-matrices [2], totally positive matrices [3, 4], and diagonally dominant matrices with positive diagonal entries.

Fiedler and Ptak [5] presented the first systematic study of P-matrices. These matrices have found many applications in economics [6], dynamical systems [7], and mathematical programming [8]. For a survey on P-matrices, see [9, Ch. 4].

We briefly review some of these applications. P-matrices have been used to analyze the injectivity of nonlinear mappings. Consider a C^1 mapping $f : \Omega \to \mathbb{R}^n$ with $\Omega \subseteq \mathbb{R}^n$. Let $J(x) := \frac{\partial}{\partial x} f(x)$ denote the Jacobian of f. It is natural to speculate that if $\det(J(x)) \neq 0$ for all $x \in \mathbb{R}^n$ then f is injective. But this is not true in general. Gale and Nikaido [10] proved that if $J(x) \in \mathbb{P}$ for all $x \in \Omega$, and Ω is a rectangle then f is injective in Ω. For generalizations of the Gale and Nikaido theorem, see e.g., [11, 12].

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Another important application of P-matrices is in the field of mathematical programming, in particular, the *linear complementarity problem* (LCP), which is a generalization of both linear programming and quadratic programming. For a comprehensive treatment of the LCP and, in particular, its numerous applications, see [13]. Some recent results on the relations between LCP and totally positive matrices are given in [14]. Given $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, the LCP(q,M) is: find (or conclude there is no) $z \geq 0$ such that

$$\begin{aligned} w &:= Mz+q \geq 0, \\ w^T z &= 0. \end{aligned} \tag{1.1}$$

Condition (1.1) is called the complementarity condition, since it implies that if $z_i > 0$ $[w_i > 0]$ for some index i then $w_i = 0$ $[z_i = 0]$. The LCP admits a unique solution for every $q \in \mathbb{R}^n$ if and only if (iff) $M \in \mathbb{P}$ [15].

P-matrices have also found applications in dynamical systems theory. We describe their role in a specific model that highlights an intuitive interpretation of *P*-matrices. A fundamental model in mathematical ecology is the Lotka-Volterra equations [16, Part 5]:

$$\dot{x}_i = x_i(b_i + \sum_{j=1}^n a_{ij}x_j), \quad i = 1,\ldots,n. \tag{1.2}$$

Here $x_i(t)$ is the biomass of species i at time t, b_i is the growth rate of species i, and a_{ij} describes the interconnection between species j and species i. It is clear that the non-negative orthant $\mathbb{R}^n_+$ is an invariant set of the dynamics, and we assume that $x(0) \in \mathbb{R}^n_+$. Let $A := (a_{ij})_{i,j=1}^n$ and $b := \begin{bmatrix} b_1 & \ldots & b_n \end{bmatrix}^T$. Any equilibrium point e of (1.2) satisfies

$$\operatorname{diag}(e_1,\ldots,e_n)(b+Ae) = 0, \tag{1.3}$$

where $\operatorname{diag}(c_1,\ldots c_n)$ is the $n\times n$ diagonal matrix with diagonal entries $c_1,\ldots,c_n$. The Jacobian of the vector field in (1.2) is

$$J(x) = \operatorname{diag}(g_1(x),\ldots,g_n(x)) + \operatorname{diag}(x_1,\ldots,x_n)A, \tag{1.4}$$

where

$$g_i(x) := b_i + \sum_{j=1}^n a_{ij}x_j. \tag{1.5}$$

If e is an equilibrium of (1.2), and $e_i = 0$ for some index i then (1.3), (1.4), and (1.5) imply that

$$(\zeta^i)^T J(e) = g_i(e)(\zeta^i)^T,$$

where ζ^i is the ith canonical vector in $\mathbb{R}^n$. Thus, a necessary condition for the stability of e is that

$$g_i(e) = b_i + \sum_{j=1}^n a_{ij}e_j \leq 0, \tag{1.6}$$

for any i such that $e_i = 0$. By (1.3), Eq. (1.6) also holds (with an equality) for any i such that $e_i > 0$. Consider the *LCP*$(-b,-A)$, that is, find a vector $y \geq 0$ such that $-Ay-b \geq 0$ and $y^T(-Ay-b) = 0$. Then any stable equilibrium point e of (1.2) is a solution of this *LCP*. If $(-A) \in \mathbb{P}$ then the LCP admits a unique solution for any b, so (1.2) has no more than a single stable equilibrium, for any b.

Here, we introduce and analyze a new class of matrices: we call a matrix $A \in \mathbb{R}^{n\times n}$ an *exponential P-matrix* if $\exp(At)$ is a P-matrix for *all* $t \geq 0$. We denote this class of matrices by $\mathbb{E}^{\mathbb{P}}$. To the best of our knowledge, this type of matrices has not been studied before. Our work is motivated in part by the work of Binyamin Schwarz [17] on totally positive differential systems (TPDSs). Recall that a matrix is called totally positive (TP) if *all* its minors are positive [3, 18, 19]. Schwarz considered the linear time-varying system

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0, \tag{1.7}$$

where $t \to A(t)$ is continuous. The solution of this system is $x(t) = \Phi(t,t_0)x_0$ where $\Phi(t,t_0)$, the transition matrix from t_0 to t, is the solution at time t of the matrix differential equation

$$\frac{d}{ds}\Phi(s,t_0) = A(s)\Phi(s,t_0), \quad \Phi(t_0,t_0) = I.$$

System (1.7) is called a TPDS if $\Phi(t,t_0)$ is TP for any pair t_0,t with $t > t_0 \geq 0$. Schwarz showed that if (1.7) is a TPDS then the variation diminishing property of TP matrices [19] can be used to analyze the asymptotic behaviour of $x(t)$. If $A(t) \equiv A$ then (1.7) is a TPDS iff A is Jacobi, that is, A is tri-diagonal with positive entries on the super- and sub-diagonals. In other words,

$$\exp(At) \text{ is TP for any } t > 0 \text{ iff } A \text{ is Jacobi.}$$

It was recently shown [20] that TPDSs have important applications in the asymptotic analysis of time-varying nonlinear dynamical systems in the form $\dot{x} = f(t,x)$ whose Jacobian $J(t,x) := \frac{\partial}{\partial x} f(t,x)$ is a Jacobi matrix for all t,x. See also [21] for the analysis of discrete-time totally positive dynamical systems. The analysis of such systems builds on the use of the multiplicative- and additive-compounds of J [22].

Note that if $\Phi(t,t_0)$ is TP then in particular it is a P-matrix. Thus, if A is Jacobi then $A \in \mathbb{E}^{\mathbb{P}}$. However, the analysis of matrices whose exponential is a P-matrix seems to be more complicated than that of matrices whose exponential is TP due to the fact that TP matrices are closed under multiplication, whereas P-matrices are not. Another important difference is that the class of P-matrices is closed under matrix inversion, whereas the class of TP matrices is not.

The contributions of this paper include the following. We introduce the new class of matrices $\mathbb{E}^{\mathbb{P}}$, provide various conditions for a matrix to be in $\mathbb{E}^{\mathbb{P}}$, describe transformations that preserve $\mathbb{E}^{\mathbb{P}}$, describe an application of our results to consensus systems, and analyze existing consensus algorithms in this new framework.

The remainder of this paper is organized as follows. The next section reviews some results on P-matrices that are used later on. Section 3 describes our main results. Section 4 describes an application of our theoretical results in the context of consensus algorithms $\dot{x} = -Lx$, where L is a Laplacian matrix. We add a natural requirement on the dynamics of this system, called *non-sign reversal*, and show that it holds iff $L \in \mathbb{E}^{\mathbb{P}}$. The final section concludes and describes several directions for further research.

Notation. For two integers i,j, with $i \leq j$, let $[i,j] := \{i, i+1, \dots, j\}$. Small [capital] letters denote column vectors [matrices]. For $A, B \in \mathbb{R}^{n \times m}$, the notation $A \geq B$ [$A \gg B$] implies that $a_{ij} \geq b_{ij}$ [$a_{ij} > b_{ij}$] for all i,j. We call $A \in \mathbb{R}^{n \times m}$ a non-negative [positive] matrix if $A \geq 0$ [$A \gg 0$]. The non-negative orthant in $\mathbb{R}^n$ is $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \,|\, x \geq 0\}$.

Consider a matrix $A \in \mathbb{R}^{n \times m}$ and fix an integer $k \in [1, \min\{n,m\}]$. Let $Q^{k,\ell}$ denote the set of increasing sequences of k numbers from $\{1, \dots, \ell\}$ ordered lexicographically. For example,

$$Q^{2,3} = \{\{1,2\},\{1,3\},\{2,3\}\}.$$

A k-minor of A is the determinant of some $k\times k$ submatrix of A. Each such submatrix is defined by a set of row indices $\alpha\in Q^{k,n}$ and column indices $\beta\in Q^{k,m}$. This submatrix is denoted by $A[\alpha|\beta]$, and the corresponding minor is

$$A(\alpha|\beta):=\det(A[\alpha|\beta]).$$

In particular, $A(\alpha|\beta)$ is called a principal minor if $\alpha=\beta$, and a leading principal minor if $\alpha=\beta=\{1,2,\dots,j\}$, for some $j\geq 1$. Similarly, $A[\alpha|\alpha]$ is called a principal submatrix of A. For $\alpha\in Q^{k,n}$, let $\bar\alpha:=\{1,\dots,n\}\setminus\alpha$ (we use set notation, but we always assume that the elements in $\bar\alpha$ are ordered in increasing order).

The *k-multiplicative compound matrix* of $A\in\mathbb{R}^{n\times m}$, denoted $A^{(k)}$, is the $\binom{n}{k}\times\binom{m}{k}$ matrix that includes all the k-minors ordered lexicographically. For example, if $n=m=3$ and $k=2$ then

$$A^{(2)}=\begin{bmatrix}A(\{12\}|\{12\}) & A(\{12\}|\{13\}) & A(\{12\}|\{23\})\\ A(\{13\}|\{12\}) & A(\{13\}|\{13\}) & A(\{13\}|\{23\})\\ A(\{23\}|\{12\}) & A(\{23\}|\{13\}) & A(\{23\}|\{23\})\end{bmatrix}.$$

Note that, by definition, $A^{(1)}=A$, and if $n=m$ then $A^{(n)}=\det(A)$. The *k-additive compound matrix* of $A\in\mathbb{R}^{n\times n}$ is

$$A^{[k]}:=\frac{d}{d\varepsilon}(I+\varepsilon A)^{(k)}|_{\varepsilon=0}. \tag{1.8}$$

Note that this implies that $A^{[k]}=\frac{d}{d\varepsilon}\left(\exp(A\varepsilon)\right)^{(k)}|_{\varepsilon=0}$. For an explicit formula for the entries of $A^{[k]}$ in terms of the entries of A, see, e.g., [17]. We refer to [23, 24] for more information on compound matrices and their applications to dynamical systems described by ordinary differential equations. See also [22] for a recent tutorial on applications of compound matrices in systems and control theory.

A matrix $A\in\mathbb{R}^{n\times n}$ is called totally positive (TP) [totally non-negative (TN)] if $A^{(k)}\gg 0$ [$A^{(k)}\geq 0$] for all $k\in[1,n]$. A matrix $A\in\mathbb{R}^{n\times n}$ is called *sign-symmetric* [25] if for any $k\in[1,n]$ and any $\alpha,\beta\in Q^{k,n}$ we have $A(\alpha|\beta)A(\beta|\alpha)\geq 0$. We say that $A\in\mathbb{R}^{n\times n}$ is *sign-pattern symmetric* if $a_{ij}a_{ji}\geq 0$ for all $i,j\in[1,n]$. Then A is sign-symmetric iff $A^{(k)}$ is sign-pattern symmetric for all $k\in[1,n-1]$.

2. PRELIMINARIES: P-MATRICES

A matrix $A\in\mathbb{R}^{n\times n}$ is called a P-matrix if all its principal minors are positive, that is, $A^{(k)}$ has positive diagonal entries for all $k\in[1,n]$. We list some transformations that preserve the P-matrix property. Recall that $S\in\mathbb{R}^{n\times n}$ is called a signature matrix if it is a diagonal matrix, and every diagonal entry is either one or minus one. Then $S^{-1}=S$. A matrix $V\in\mathbb{R}^{n\times n}$ is called a permutation matrix if it has exactly one entry of 1 in each row and each column, and zeros elsewhere. Then $V^{-1}=V^T$.

Theorem 2.1. [26] *Suppose that $A\in\mathbb{R}^{n\times n}$ is a P-matrix. Then*

(1) $A^T\in\mathbb{P}$.
(2) If V is a permutation matrix then $VAV^T\in\mathbb{P}$.
(3) If D_1,D_2 are diagonal matrices and D_1D_2 has positive diagonal entries then $D_1AD_2\in\mathbb{P}$.
(4) If S is a signature matrix then $SAS\in\mathbb{P}$.
(5) If D is a non-negative diagonal matrix then $A+D\in\mathbb{P}$.

(6) If D is a diagonal matrix with $0 \le D \le I$ then $D+(I-D)A \in \mathbb{P}$.
(7) $A^{-1} \in \mathbb{P}$.

Let $A \in \mathbb{R}^{n\times n}$. Since $\exp(At) = I + \sum_{k=1}^{\infty} \frac{A^k t^k}{k!}$, we have that $\exp(At) \in \mathbb{P}$ for all $t > 0$ sufficiently small. The next result provides more information on the value t for which $\exp(At) \in \mathbb{P}$ when all the entries of A are non-negative.

Theorem 2.2. [26] *If $A \ge 0$ then $\exp(At) \in \mathbb{P}$ for all $t \in [0, \frac{1}{\rho(A)})$, where $\rho(A)$ is the spectral radius of A.*

The next result summarizes several necessary and sufficient conditions for a matrix to be a P-matrix.

Theorem 2.3. [5, 27] *Let $A \in \mathbb{R}^{n\times n}$. The following six conditions are equivalent.*
(1) Any principal minor of A is positive, that is, $A \in \mathbb{P}$;
(2) For any $x \in \mathbb{R}^n \setminus \{0\}$, there exists an index $k \in \{1,\dots,n\}$ such that $x_k(Ax)_k > 0$;
(3) For any $x \in \mathbb{R}^n \setminus \{0\}$, there exists a positive diagonal matrix $D(x)$ such that $x^T D(x)Ax > 0$;
(4) For any $x \in \mathbb{R}^n \setminus \{0\}$, there exists a non-negative diagonal matrix $H(x)$ such that $x^T H(x)Ax > 0$;
(5) Every real eigenvalue of A and of any principal submatrix of A is positive;
(6) For any signature matrix S, there exists a vector $x \gg 0$ such that $SASx \gg 0$.

Remark 2.4. *For $A \in \mathbb{R}^{n\times n}$, let*

$$rev(A) := \{x \in \mathbb{R}^n \,|\, x_i(Ax)_i \le 0 \text{ for all } i \in [1,n]\}.$$

Roughly speaking, this is the set of vectors that A "sign reverses". Condition 2) in Theorem 2.3 implies that A is a P-matrix iff $rev(A) = \{0\}$. This is known as the sign non-reversal property (see, e.g., [28]*).*

Remark 2.5. *Recall that $A \in \mathbb{R}^{n\times n}$ is called stable (or Hurwitz) if $\operatorname{Re}(\lambda) < 0$ for any eigenvalue λ of A. It is called diagonally stable if there exists a positive diagonal matrix D such that $DA + A^T D$ is negative-definite* [29]*. Condition 3) in Theorem 2.3 implies that if A is diagonally stable then $(-A) \in \mathbb{P}$. Recall* [30] *that if A is Hurwitz and Metzler then it is diagonally stable, so $(-A) \in \mathbb{P}$. If A is Metzler then $(-A) \in \mathbb{P}$ is in fact a necessary and sufficient condition for Hurwitz of A, refereed to as the Hicksian stability condition, see e.g.* [31] *(see also* [32] *for an application of this property to determining box invariance of dynamical systems). The class of matrices*

$$\{A \in \mathbb{R}^{n\times n} \,|\, (-A) \in \mathbb{P}\}$$

is sometimes denoted by $\mathbb{P}^{(-)}$ (see [33]*).*

Recall that $A \in \mathbb{R}^{n\times n}$ is called a Q-matrix, denoted $A \in \mathbb{Q}$, if for any $k \in \{1,\dots,n\}$ the sum of all the k principal minors of A is positive. In particular, any P-matrix is a Q-matrix. The next result provides a necessary spectral condition for $A \in \mathbb{R}^{n\times n}$ to be a Q-matrix. For $z \in \mathbb{C}$, let $\arg(z) \in (-\pi, \pi]$ denote the argument of z.

Theorem 2.6. [34] *Let $A \in \mathbb{R}^{n\times n}$, with $n \ge 2$, be a Q-matrix. Then any eigenvalue λ of A satisfies*

$$|\arg(\lambda)| < \pi - \frac{\pi}{n}. \tag{2.1}$$

In other words, there is a wedge around the negative x-axis which is free from eigenvalues of A. In particular, a P-matrix cannot have a real and negative eigenvalue (the latter is also implied by Condition 2) in Theorem 2.3).

The next result provides another necessary and sufficient condition for a matrix to be a P-matrix.

Theorem 2.7. [26] *Suppose that $A = BC^{-1}$, with $B, C \in \mathbb{R}^{n\times n}$. Then $A \in \mathbb{P}$ iff the matrix $SC + (I-S)B$ is non-singular for any matrix $S = \operatorname{diag}(s_1,\dots,s_n)$ with $s_i \in [0,1]$.*

The problem of testing whether a given $n\times n$ matrix is a P-matrix, formulated as a decision problem, is co-NP-complete in n [35].

The next section describes our main results.

3. Main Results

We begin by introducing a new class of matrices that we call *exponential P-matrices*.

Definition 3.1. *The set $\mathbb{E}^{\mathbb{P}} \subset \mathbb{R}^{n\times n}$ includes all the matrices $A \in \mathbb{R}^{n\times n}$ satisfying*

$$\exp(At) \in \mathbb{P} \text{ for all } t \geq 0.$$

One motivation for introducing this notion is a dynamical systems generalization of the sign-reversal property described in Remark 2.4. Indeed, if $\dot{x} = Ax$ then the following two properties are equivalent:

(1) $A \in \mathbb{E}^{\mathbb{P}}$;
(2) for any $x(0) \in \mathbb{R}^n \setminus \{0\}$ and any time $t \geq 0$ there exists at least one index $i = i(t, x(0))$ such that

$$x_i(0)x_i(t) > 0. \tag{3.1}$$

In other words, at each time t there is at least one state-variable that has the same sign at time t as in time 0. We describe an application of this property to opinion dynamics in Section 4 below.

To gain more perspective, consider also the discrete-time linear consensus system

$$x(j+1) = Ax(j),$$

and assume that we require that for any $x(0) \in \mathbb{R}^n \setminus \{0\}$ the following property holds: for any time $k = 0, 1, \dots$ there exists an index i (that may depend on $x(0)$ and k) such that

$$x_i(0)x_i(k) > 0. \tag{3.2}$$

This is the discrete-time analogue of condition (3.1). Since $x(k) = A^k x(0)$, (3.2) is equivalent to the requirement that $A^k \in \mathbb{P}$. Recall that a matrix $A \in \mathbb{R}^{n\times n}$ is called a PM-matrix if $A^k \in \mathbb{P}$ for all $k = 0, 1, \dots$ (see, e.g., [36]). In this sense, $\mathbb{E}^{\mathbb{P}}$-matrices can be viewed as the continuous-time analogue of PM-matrices.

Definition 3.1 only considers non-negative times. However, using Condition 7) in Theorem 2.1 yields the following result.

Corollary 3.2. $A \in \mathbb{E}^{\mathbb{P}}$ *iff*

$$\exp(At) \in \mathbb{P} \text{ for all } t \in \mathbb{R}. \tag{3.3}$$

Proof. If (3.3) holds then clearly $A \in \mathbb{E}^{\mathbb{P}}$. Now suppose that $A \in \mathbb{E}^{\mathbb{P}}$, that is, $\exp(At) \in \mathbb{P}$ for any $t \geq 0$. Fix $s < 0$. Then $\exp(sA) = (\exp(-sA))^{-1}$, and since $(-s) > 0$ and the inverse of a P-matrix is a P-matrix, we conclude that $\exp(sA) \in \mathbb{P}$. □

Example 3.3. *Consider* $A = \begin{bmatrix} 0 & -1 & 0 \\ -2 & 0 & -2 \\ 0 & -1 & 0 \end{bmatrix}$. *Note that* $(-A)$ *is Jacobi. A calculation gives*

$$\exp(At) = \begin{bmatrix} \cosh^2(t) & -\sinh(t)\cosh(t) & \sinh^2(t) \\ -\sinh(2t) & \cosh(2t) & -\sinh(2t) \\ \sinh^2(t) & -\sinh(t)\cosh(t) & \cosh^2(t) \end{bmatrix},$$

and

$$(\exp(At))^{(2)} = \begin{bmatrix} \cosh^2(t) & -\sinh(2t) & \sinh^2(t) \\ -\sinh(t)\cosh(t) & \cosh(2t) & -\sinh(t)\cosh(t) \\ \sinh^2(t) & -\sinh(2t) & \cosh^2(t) \end{bmatrix}.$$

The diagonal entries of $(\exp(At))^{(k)}$ *are the principal minors of order* k *of* $\exp(At)$. *We see that the principal minors of order one and two of* $\exp(At)$ *are positive for all* $t \geq 0$, *so* $A \in \mathbb{E}^{\mathbb{P}}$. *Note that the principal minors are positive for any* $t \leq 0$ *as well. However, some non-principal minors of* $\exp(At)$ *may take negative values, as* A *is not Jacobi.*

More generally, since $\exp(-At)$ is the inverse of $\exp(At)$, it follows from Jacobi's identity [37, Chapter 0] that for any $k \in [1, n-1]$ and any $\alpha \in Q^{k,n}$, we have

$$(\exp(-At))(\alpha|\alpha) = \frac{(\exp(At))(\bar{\alpha}|\bar{\alpha})}{\det(\exp(At))},$$

where $\bar{\alpha} := \{1, \dots, n\} \setminus \alpha$. This shows, in particular, that $A \in \mathbb{E}^{\mathbb{P}}$ iff $-A \in \mathbb{E}^{\mathbb{P}}$.

If $A \in \mathbb{E}^{\mathbb{P}}$ then $\exp(At) \in \mathbb{P}$ for any $t \in \mathbb{R}$ and writing $\exp(At) = \exp(At_1)(\exp(A(t_1 - t)))^{-1}$, with an arbitrary t_1, and using Theorem 2.7 yields the following result.

Theorem 3.4. *A matrix* $A \in \mathbb{E}^{\mathbb{P}}$ *iff the matrix*

$$S\exp(At_2) + (I - S)\exp(At_1)$$

is non-singular for any $t_1, t_2 \in \mathbb{R}$ *and any matrix* $S = \operatorname{diag}(s_1, \dots, s_n)$ *with* $s_i \in [0, 1]$.

The next result lists some transformations that preserve the $\mathbb{E}^{\mathbb{P}}$-property.

Theorem 3.5. *Suppose that* $A \in \mathbb{E}^{\mathbb{P}}$. *Then*

(1) $cA \in \mathbb{E}^{\mathbb{P}}$ *for any* $c \in \mathbb{R}$.
(2) $A^T \in \mathbb{E}^{\mathbb{P}}$.
(3) *If* Q *is a permutation matrix then* $QAQ^T \in \mathbb{E}^{\mathbb{P}}$.
(4) *If* D *is a positive diagonal matrix then* $DAD^{-1} \in \mathbb{E}^{\mathbb{P}}$.
(5) *If* S *is a signature matrix then* $SAS \in \mathbb{E}^{\mathbb{P}}$.
(6) *If* D *is a diagonal matrix and* $DA = AD$ *then* $A + D \in \mathbb{E}^{\mathbb{P}}$.

Proof. To prove 1), fix $c \in \mathbb{R}$. Then $\exp((cA)t) = \exp(A(ct))$, and since $A \in \mathbb{E}^{\mathbb{P}}$, $\exp(A(ct)) \in \mathbb{P}$. The proof of 2) follows from the fact that $\exp(A^T t) = (\exp(At))^T$. To prove 3), note that if Q is a permutation matrix then

$$\exp(QAQ^T t) = Q\exp(At)Q^T,$$

and since $\exp(At) \in \mathbb{P}$, this implies that $QAQ^T \in \mathbb{E}^{\mathbb{P}}$. The proofs of 4) and 5) are similar to the proof of 3). To prove 6), let $B := A + D$. Then $\exp(Bt) = \exp(Dt)\exp(At)$. Since $\exp(Dt)$ is a positive diagonal matrix, and $\exp(At) \in \mathbb{P}$, we conclude from Condition 3) in Theorem 2.1 that $\exp(Bt) \in \mathbb{P}$. □

The following example shows that $\mathbb{E}^{\mathbb{P}}$ is not invariant under similarity transformations.

Example 3.6. *The matrix* $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *is in* $\mathbb{E}^{\mathbb{P}}$*, as* $\exp(At) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$*, but for* $T = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ *we have*

$$B := TAT^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

and $B \notin \mathbb{E}^{\mathbb{P}}$*, as*

$$\exp(Bt) = T\exp(At)T^{-1} = \begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}$$

and for any $t \geq 1$ *this matrix has a non-positive diagonal entry, so it is not a P matrix.*

3.1. **Some classes of $\mathbb{E}^{\mathbb{P}}$ matrices.** In general, it seems difficult to determine if a given matrix is in $\mathbb{E}^{\mathbb{P}}$. However, we show that some specific classes of matrices are in $\mathbb{E}^{\mathbb{P}}$.

3.1.1. *The* 2×2 *case.* We first study the case of 2×2 matrices and provide a necessary and sufficient sign-pattern condition for a matrix to be in $\mathbb{E}^{\mathbb{P}}$.

Proposition 3.7. *Suppose that* $A \in \mathbb{R}^{2\times 2}$*. Then* $A \in \mathbb{E}^{\mathbb{P}}$ *iff* $a_{12}a_{21} \geq 0$.

In other words, $A \in \mathbb{E}^{\mathbb{P}}$ iff A has the sign pattern

$$\begin{bmatrix} * & \geq 0 \\ \geq 0 & * \end{bmatrix} \text{ or } \begin{bmatrix} * & \leq 0 \\ \leq 0 & * \end{bmatrix}, \tag{3.4}$$

where $*$ denotes "don't care", that is, iff $A \in \mathbb{R}^{2\times 2}$ is sign-pattern symmetric. Note that the first [second] sign pattern here corresponds to the linear dynamical system $\dot{x} = Ax$ being a cooperative [competitive] system [38].

Proof. If $a_{12}a_{21} = 0$ then A is triangular and so is $\exp(At)$. In this case, the diagonal entries of $\exp(At)$ are $\exp(a_{ii}t)$, $i = 1, 2$. Hence, $A \in \mathbb{E}^{\mathbb{P}}$.

If $a_{12} > 0$ and $a_{21} > 0$ then A is Jacobi, so $A \in \mathbb{E}^{\mathbb{P}}$. If $a_{12} < 0$ and $a_{21} < 0$, then $-A \in \mathbb{E}^{\mathbb{P}}$ and thus $A \in \mathbb{E}^{\mathbb{P}}$.

To complete the proof, we need to show that if

$$a_{12}a_{21} < 0 \tag{3.5}$$

then $A \notin \mathbb{E}^{\mathbb{P}}$.

Let $B(t) := \exp(At)$. Since $A \in \mathbb{R}^{2\times 2}$, $B(t)$ is a P-matrix iff its diagonal entries $b_{11}(t)$ and $b_{22}(t)$ are positive. Let $s := \text{trace}(A)/2$, and write $A = sI + (A - sI)$. Note that $\text{trace}(A - sI) = 0$. Then

$$B(t) = \exp(st)\exp((A - sI)t).$$

The term $\exp(st)$ has no effect on the signs of $b_{11}(t)$, $b_{22}(t)$, so we may assume that

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{bmatrix}.$$

The eigenvalues of A are $\pm\sqrt{a_{11}^2 + a_{12}a_{21}}$. Assume that (3.5) holds. We consider three cases.

Case 1. Suppose that $a_{12}a_{21} < -a_{11}^2$. In this case, the eigenvalues of A are purely imaginary and Theorem 3.11 below implies that $A \notin \mathbb{E}^{\mathbb{P}}$.

Case 2. Suppose that $a_{12}a_{21} = -a_{11}^2$. Then $\det(A) = \text{trace}(A) = 0$, so $A^2 = 0$ and thus

$$B(t) = I + At = \begin{bmatrix} 1 + a_{11}t & a_{12}t \\ a_{21}t & 1 - a_{11}t \end{bmatrix}.$$

Since $a_{12}a_{21} < 0$, $a_{11} \neq 0$, and we conclude that $B(t) \notin \mathbb{P}$ for any $t \geq 0$ sufficiently large, so $A \notin \mathbb{E}^{\mathbb{P}}$.

Case 3. Suppose that $a_{12}a_{21} > -a_{11}^2$. Then $\det(A) = -a_{11}^2 - a_{12}a_{21} < 0$, and the diagonal entries of $B(t)$ are

$$b_{11}(t) = \cosh(st)\left(1 + \frac{a_{11}}{s}\tanh(st)\right),$$
$$b_{22}(t) = \cosh(st)\left(1 - \frac{a_{11}}{s}\tanh(st)\right),$$

where $s := \sqrt{-\det(A)}$. Since $a_{12}a_{21} < 0$, $\frac{|a_{11}|}{s} > 1$, so $B(t) \notin \mathbb{P}$ for all $t > 0$ sufficiently large, so $A \notin \mathbb{E}^{\mathbb{P}}$. □

Example 3.8. *Consider the matrix* $A(w) := \begin{bmatrix} -1 & w \\ -w & -1 \end{bmatrix}$, *with* $w > 0$. *Then*

$$\exp(At) = \exp(-t)\begin{bmatrix} \cos(wt) & \sin(wt) \\ -\sin(wt) & \cos(wt) \end{bmatrix},$$

so $\exp(At) \in \mathbb{P}$ *for* $t \in [0, \frac{\pi}{2w})$, *but*

$$\exp(A\frac{\pi}{2w}) = \exp(-\frac{\pi}{2w})\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \notin \mathbb{P},$$

so $A \notin \mathbb{E}^{\mathbb{P}}$. *Note that when* $w \to 0$, *the* L_2 *induced norm of* A *converges to one, yet the first time where we "loose" the* $\mathbb{E}^{\mathbb{P}}$ *property, namely,* $t = \frac{\pi}{2w}$ *converges to infinity.*

3.1.2. *The* 3×3 *case.* Analyzing $\mathbb{E}^{\mathbb{P}}$ for 3×3 matrices seems to be more difficult than the 2×2 case. We give here some partial results.

Let $A \in \mathbb{R}^{3\times 3}$. Since $A \in \mathbb{E}^{\mathbb{P}}$ iff $(A - \frac{1}{3}\text{trace}(A)I_3) \in \mathbb{E}^{\mathbb{P}}$, we may assume that $\text{trace}(A) = 0$. Thus, consider the matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ h & i & -a-e \end{bmatrix}. \tag{3.6}$$

The characteristic polynomial of A is

$$\begin{aligned} P(s) &:= \det(sI_3 - A) \\ &= s^3 + d_1 s + d_0, \end{aligned} \tag{3.7}$$

with

$$d_1 := -a^2 - e^2 - ae - bd - ch - fi,$$
$$d_0 := -\det(A) = ae^2 + a^2e - abd - bde + afi - bfh - cdi + ceh. \tag{3.8}$$

Recall that the (polynomial) discriminant of a cubic polynomial $F(x) := d_3x^3 + d_2x^2 + d_1x_1 + d_0$, with $d_3 \neq 0$, is

$$\Delta(F) = -27d_3^2d_0^2 + 18d_3d_2d_1d_0 - 4d_3d_1^3 - 4d_2^3d_0 + d_2^2d_1^2,$$

and that all the roots of $F(x) = 0$ are real iff $\Delta(F) \geq 0$. For the polynomial $P(s)$, we have

$$\Delta(P) = -27d_0^2 - 4d_1^3,$$

Thus Theorem 3.11 below implies that a necessary condition for $A \in \mathbb{E}^{\mathbb{P}}$ is

$$d_1^3 \leq -\frac{27}{4} d_0^2. \tag{3.9}$$

In particular, we must have $d_1 \leq 0$. The next result analyzes the case where $d_1 = 0$.

Proposition 3.9. *Consider the matrix A in* (3.6)*, and suppose that $d_1 = 0$. Then $A \in \mathbb{E}^{\mathbb{P}}$ iff there exists a permutation matrix P such that PAP^T is strictly triangular (i.e., triangular with zeros on the main diagonal).*

In other words, when $d_1 = 0$ the only 3×3 $\mathbb{E}^{\mathbb{P}}$ matrices are the "trivial" ones.

Proof. If there exists a permutation matrix P such that PAP^T is strictly triangular then Theorems 3.5 and 3.10 imply that $A \in \mathbb{E}^{\mathbb{P}}$. Thus, we only need to prove the converse implication. To do this, assume that

$$A \in \mathbb{E}^{\mathbb{P}}. \tag{3.10}$$

Since we assume that $d_1 = 0$, Eq. (3.9) implies that $d_0 = 0$. Thus, $P(s) = s^3$, so

$$\exp(At) = I + At + A^2t^2/2. \tag{3.11}$$

Since all the eigenvalues of A are zero, $\operatorname{trace}(A^2) = 0$. If one of the diagonal entries of A^2 is negative then (3.11) implies that for sufficiently large t, a diagonal entry of $\exp(At)$ is negative, and this contradicts (3.10). Thus, all the diagonal entries of A^2 are non-negative, while their sum is zero. Therefore, each diagonal entry of A^2 must be zero. Then the diagonal entries of $\exp(At)$ are the diagonal entries of $I + At$, that is, $1 + at$, $1 + et$, and $1 - (a+e)t$. Combining this with the fact that $A \in \mathbb{E}^{\mathbb{P}}$ gives $a = e = 0$. Then the diagonal entries of A^2 are $bd + ch$, $bd + fi$, $ch + fi$ and these are all zero, so $bd = ch = fi = 0$. Also, the fact that $d_0 = 0$ gives $bfh + cdi = 0$. A straightforward derivation shows that these conditions imply that PAP^T is strictly triangular for some permutation matrix P. □

The next result describes several classes of matrices that are $\mathbb{E}^{\mathbb{P}}$.

Theorem 3.10. *Let $A \in \mathbb{R}^{n \times n}$. Any one of the following three conditions implies that $A \in \mathbb{E}^{\mathbb{P}}$.*

1) A is lower triangular or upper triangular.
2) A is symmetric.
3) A is Jacobi.

Proof. Pick an arbitrary $t \geq 0$ and let $B := \exp(At)$. We will show that if any one of the conditions in the theorem holds then $B \in \mathbb{P}$.

To prove 1), note that the (lower or upper) triangular structure is preserved under summation and multiplication. Hence, $B = \sum_{\ell=0}^{\infty} \frac{A^\ell t^\ell}{\ell!}$ is also a triangular matrix. Since the eigenvalues of a triangular matrix are its diagonal entries, the diagonal entries of B are $\exp(a_{ii}t)$, $i = 1, \ldots, n$. Pick $k \in [1, n]$ and $\alpha \in Q^{k,n}$. Then $B(\alpha|\alpha) = \prod_{i \in \alpha} \exp(a_{ii}t) > 0$, so $B \in \mathbb{P}$.

To prove 2), suppose that A is symmetric. Then all the eigenvalues λ_i of A are real, B is symmetric and the eigenvalues of B are $\exp(\lambda_i t) > 0$, $i = 1, \ldots, n$. Thus, B is positive-definite, and by Theorem 2.3, $B \in \mathbb{P}$.

To prove 3), suppose that A is Jacobi. Then $\exp(At)$ is TP for all $t > 0$, and in particular A is $\mathbb{E}^{\mathbb{P}}$. Here, we also give a more direct proof. Since A is Jacobi, it has the form

$$A = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_1 & a_2 & b_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_2 & a_3 & b_3 & \cdots & 0 & 0 & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & c_{n-1} & a_n \end{bmatrix},$$

with $b_i, c_i > 0$, $i = 1, \ldots, n-1$.

Consider the positive diagonal matrix $D := \operatorname{diag}(1, \sqrt{\frac{b_1}{c_1}}, \ldots, \sqrt{\frac{b_1 b_2 \cdots b_{n-1}}{c_1 c_2 \cdots c_{n-1}}})$. Then DAD^{-1} is symmetric. Hence, $\exp(DAD^{-1}t) = D\exp(At)D^{-1} \in \mathbb{P}$. Theorem 2.1 implies that $\exp(At) \in \mathbb{P}$. □

The three classes of matrices in Theorem 3.10 all have real eigenvalues. One may conjecture that if all the eigenvalues of $A \in \mathbb{R}^{n\times n}$ are real then $A \in \mathbb{E}^{\mathbb{P}}$. However, the matrix B in Example 3.6 above shows that this conjecture is false.

The next result shows that having all real eigenvalues is a necessary condition for EP-matrices.

Theorem 3.11. *If $A \in \mathbb{R}^{n\times n}$ has a complex (non-real) eigenvalue then $A \notin \mathbb{E}^{\mathbb{P}}$.*

Proof. Let $\lambda = a + jb$, where $b \neq 0$ and $j := \sqrt{-1}$, be a complex eigenvalue of A. Since the complex eigenvalues of a real matrix always occur in conjugate pairs, we can assume that $b > 0$. Now, $\exp(At)$ has the eigenvalue

$$\exp(\lambda t) = \exp(at)\exp(jbt).$$

For $t = \frac{(n-1)\pi}{nb} > 0$, we have $\arg(\exp(\lambda t)) = \pi - \frac{\pi}{n}$, and Theorem 2.6 implies that $\exp(At) \notin \mathbb{Q}$, and thus $\exp(At) \notin \mathbb{P}$. □

The next result shows that A having all real eigenvalues does imply a property of the matrix exponential $\exp(At)$, albeit one that is weaker than $\mathbb{E}^{\mathbb{P}}$. To state it we require the following definition.

Definition 3.12. *A matrix $A \in \mathbb{R}^{n\times n}$ is called an exponential Q-matrix, denoted $A \in \mathbb{E}^{\mathbb{Q}}$, if $\exp(At)$ is a Q-matrix for any $t \geq 0$.*

Proposition 3.13. *Let $A \in \mathbb{R}^{n\times n}$. Then $A \in \mathbb{E}^{\mathbb{Q}}$ iff all the eigenvalues of A are real.*

Proof. We already know that if A has a complex (non-real) eigenvalue then there exists a time $t > 0$ such that $\exp(At) \notin Q$.

To prove the converse implication, let $B \in \mathbb{R}^{n\times n}$ be a matrix with eigenvalues λ_i, $i = 1, \ldots, n$, with every λ_i real and positive. Then the characteristic polynomial of B is

$$\prod_{i=1}^{n}(s - \lambda_i) = s^n - (\lambda_1 + \cdots + \lambda_n)s^{n-1} + (\lambda_1\lambda_2 + \cdots + \lambda_{n-1}\lambda_n)s^{n-2} + \cdots + (-1)^n\lambda_1 \ldots \lambda_n.$$

It follows that the sum of the principal minors of order one of B is $\lambda_1 + \cdots + \lambda_n > 0$, the sum of the principal minors of order two of B is $\lambda_1\lambda_2 + \cdots + \lambda_{n-1}\lambda_n > 0$, and so on. Thus, B is a Q-matrix. We can now complete the proof of the proposition. Let $s_i \in \mathbb{R}$, $i = 1, \ldots, n$, denote the eigenvalues of A. Then the eigenvalues of $\exp(At)$ are $\exp(s_i t)$, $i = 1, \dots, n$, and these are real and positive for all $t \in \mathbb{R}$, so $\exp(At)$ is a Q-matrix. $\square$

Remark 3.14. *An immediate implication of Theorem 3.11 is that if $A \in \mathbb{R}^{n\times n}$ is skew-symmetric then $A \notin \mathbb{E}^{\mathbb{P}}$. If $A \in \mathbb{E}^{\mathbb{P}}$, then Theorem 3.11 also implies that the linear dynamical system $\dot{x} = Ax$ does not have limit cycles.*

The following result provides a necessary and sufficient condition for $A \in \mathbb{E}^{\mathbb{P}}$ in terms of the diagonal entries of additive compound matrices of A. Recall that if $A \in \mathbb{R}^{n\times n}$ then $A^{[k]} \in \mathbb{R}^{r\times r}$, with $r := \binom{n}{k}$.

Proposition 3.15. *Let $A \in \mathbb{R}^{n\times n}$. Then $A \in \mathbb{E}^{\mathbb{P}}$ iff for any $t \geq 0$ and any $k \in [1, n-1]$, we have*

$$(\exp(A^{[k]}t))_{ii} > 0, \text{ for all } i \in [1, \binom{n}{k}].$$

Proof. Recall (see e.g. [23]) that

$$(\exp(A))^{(k)} = \exp(A^{[k]}).$$

Thus, if α is the ith element in $Q^{k,n}$ then the corresponding principal minor of $\exp(At)$ satisfies

$$(\exp(At))(\alpha|\alpha) = (\exp(A^{[k]}t))_{ii},$$

and this completes the proof. $\square$

The next result provides a sufficient condition for $\mathbb{E}^{\mathbb{P}}$ based on sign-symmetry of $\exp(At)$.

Proposition 3.16. *Suppose that $A \in \mathbb{R}^{n\times n}$ has only real eigenvalues, and that*

$$\exp(At) \text{ is sign-symmetric for all } t \geq 0. \tag{3.12}$$

Then $A \in \mathbb{E}^{\mathbb{P}}$.

Note that by Theorem 3.11, the assumption that A has only real eigenvalues is a necessary condition for $\mathbb{E}^{\mathbb{P}}$.

Proof. Fix $t \geq 0$. Since A has real eigenvalues, $\exp(At)$ has real and positive eigenvalues. In particular $\exp(At)$ is positively stable (i.e., $(-\exp(At))$ is a Hurwitz matrix). Recall that for the class of sign-symmetric matrices, positivity of principal minors and positive stability are equivalent (see, e.g., [25]), so we conclude that $\exp(At) \in \mathbb{P}$. $\square$

Remark 3.17. *Fix $k \in [1, n]$. It follows from (1.8) that*

$$(\exp(At))^{(k)} = I_r + tA^{[k]} + o(t),$$

with $r := \binom{n}{k}$, so if condition (3.12) holds then $I_r + tA^{[k]}$ is sign-pattern symmetric for any $t > 0$ sufficiently small, that is, $A^{[k]}$ is sign-pattern symmetric.

It is natural to ask whether a sufficient condition for $A \in \mathbb{E}^{\mathbb{P}}$ is that A has real eigenvalues and $A^{[k]}$ is sign-pattern symmetric for all $k \in [1, n-1]$. We already know that this is true for $n = 2$, but the next example demonstrates that this is not a sufficient condition for $A \in \mathbb{E}^{\mathbb{P}}$ already for $n = 3$.

Example 3.18. *Let* $A = \begin{bmatrix} 0.7 & 8 & 6 \\ 0.5 & 9 & 5 \\ 5 & 1 & 0.1 \end{bmatrix}$. *It is straightforward to verify that all the eigenvalues of* A *are real, and that* $A^{[1]} = A$ *and* $A^{[2]}$ *are sign-pattern symmetric, yet* $\exp(3A)$ *is not a P-matrix, so* $A \notin \mathbb{E}^{\mathbb{P}}$.

Proposition 3.16 suggests the following question: suppose that $A \in \mathbb{R}^{n\times n}$ has only real eigenvalues and is sign-symmetric. Is $\exp(At)$ sign-symmetric for all $t \geq 0$? The following example shows that in general the answer is no.

Example 3.19. *Consider the matrix* $A = \begin{bmatrix} -1/2 & 1 & 1 \\ 1 & 10 & 1 \\ 1 & 0 & 20 \end{bmatrix}$. *It is straightforward to verify that:* A *has three real eigenvalues,* A *is sign-symmetric, but* $\exp(0.1A)$ *is not sign-symmetric.*

There are however special cases where sign-symmetry of A does allow us to apply Proposition 3.16. Recall that a matrix is called P_0 if all its principal minors are non-negative (see, e.g., [25]).

Corollary 3.20. *Suppose that* $A \in \mathbb{R}^{n\times n}$ *is sign-symmetric, and is a projection matrix (i.e.,* $A^2 = A$*). Then* $A \in \mathbb{E}^{\mathbb{P}}$.

Note that we do not require here explicitly that A has only real eigenvalues. Indeed, since A is a projection matrix, all its eigenvalue are either one or zero.

Proof. Fix $t \geq 0$, and let $B := \exp(At) - I$. Since $A^2 = A$,

$$\begin{aligned} B &= \sum_{k=1}^{\infty} \frac{At^k}{k!} \\ &= (\exp(t) - 1)A. \end{aligned}$$

Since A is sign-symmetric, A^2 is a P_0-matrix [25, Lemma 2.1], and since $A^2 = A$, A is a P_0-matrix. Hence, B is also a P_0-matrix. Now [39, Theorem 1.5] implies that $\exp(At) = I + B$ is a P-matrix. □

Example 3.21. *Consider the matrix* $A = (1/4)\begin{bmatrix} 4 & -3 & -3 \\ 0 & 1 & -3 \\ 0 & -1 & 3 \end{bmatrix}$. *This is a sign-symmetric projection matrix. A calculation gives*

$$\exp(At) = (1/4)\begin{bmatrix} 4\exp(t) & 3-3\exp(t) & 3-3\exp(t) \\ 0 & 3+\exp(t) & 3-3\exp(t) \\ 0 & 1-\exp(t) & 1+3\exp(t) \end{bmatrix},$$

and

$$(\exp(At))^{(2)} = (\exp(t)/4)\begin{bmatrix} 3+\exp(t) & * & * \\ * & 1+3\exp(t) & * \\ * & * & 4 \end{bmatrix},$$

where $'*'$ *denotes values that are not relevant. Since all the diagonal entries of* $\exp(At)$ *and* $(\exp(At))^{(2)}$ *are positive for all* t*, we see that indeed* $A \in \mathbb{E}^{\mathbb{P}}$.

3.2. **Generalizations of Theorem 3.10.** This section provides several generalizations of Theorem 3.10. Specifically, it is straightforward to verify that the three classes of matrices in Theorem 3.10 are special cases of Theorems 3.22, 3.25, and 3.27 below, respectively. Furthermore, all the matrix classes in Theorem 3.10 are special cases of the result described in Theorem 3.28 below.

Theorem 3.22. *Suppose that*

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}, \tag{3.13}$$

where $B \in \mathbb{R}^{n\times n}$*,* $C \in \mathbb{R}^{n\times m}$*,* $D \in \mathbb{R}^{m\times m}$*, and* 0 *denotes an* $m\times n$ *matrix of zeros. Then* $A \in \mathbb{E}^{\mathbb{P}}$ *iff* $B, D \in \mathbb{E}^{\mathbb{P}}$.

Proof. First, we show $A \in \mathbb{E}^{\mathbb{P}}$ implies $B, D \in \mathbb{E}^{\mathbb{P}}$. Note that

$$\exp(At) = \begin{bmatrix} \exp(Bt) & * \\ 0 & \exp(Dt) \end{bmatrix},$$

where $*$ denotes "don't care". Since $\exp(At)$ is a P-matrix, every principal submatrix of $\exp(At)$ is also a P-matrix. Hence, $B, D \in \mathbb{E}^{\mathbb{P}}$.

To prove the converse implication, assume that $B, D \in \mathbb{E}^{\mathbb{P}}$. Fix $t \geq 0$ and $v \in \mathbb{R}^{n+m} \setminus \{0\}$. Let $w := \exp(At)v$, i.e. the solution at time t of $\dot{x} = Ax$, $x(0) = v$. By Theorem 2.3, it is enough to show that there exists $k \in [1, n+m]$ such that $w_k v_k > 0$. We consider two cases.
Case 1. Suppose that at least one of $v_{n+1}, \dots, v_{n+m}$ is non zero. Then

$$\begin{bmatrix} w_{n+1} \\ \vdots \\ w_{n+m} \end{bmatrix} = \exp(Dt) \begin{bmatrix} v_{n+1} \\ \vdots \\ v_{n+m} \end{bmatrix}, \tag{3.14}$$

and the fact that $D \in \mathbb{E}^{\mathbb{P}}$ implies that there exists $i \in [n+1, n+m]$ such that $w_i v_i > 0$.
Case 2. Suppose that $v_{n+1} = \dots = v_{n+m} = 0$. Then

$$\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \exp(Bt) \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \tag{3.15}$$

and since $v \neq 0$ and $B \in \mathbb{E}^{\mathbb{P}}$, there exists $i \in [1, n]$ such that $w_i v_i > 0$. Hence, $A \in \mathbb{E}^{\mathbb{P}}$. □

Remark 3.23. *Any reducible matrix is similar to a matrix of the form* (3.13) *up to a permutation similarity. Since* $\mathbb{E}^{\mathbb{P}}$ *is preserved under permutation similarity (see Theorem 3.5), Proposition 3.22 shows that when analyzing* $\mathbb{E}^{\mathbb{P}}$ *it is enough to study irreducible matrices.*

Example 3.24. *Let*

$$A = \begin{bmatrix} -4 & 1 & -6 \\ 2 & -5 & 6 \\ 0 & 0 & -9 \end{bmatrix}. \tag{3.16}$$

Note that A does not belong to any of the three matrix classes in Theorem 3.10, but it has the form (3.13) *with* $B = \begin{bmatrix} -4 & 1 \\ 2 & -5 \end{bmatrix}$ *and* $D = -9$*. Since* $B, D \in \mathbb{E}^{\mathbb{P}}$*, so is A. Indeed, a calculation* *gives that*

$$3\exp(At) = \begin{bmatrix} 2\exp(-3t)+\exp(-6t) & * & * \\ * & \exp(-3t)+2\exp(-6t) & * \\ * & * & 3\exp(-9t) \end{bmatrix},$$

and

$$3(\exp(At))^{(2)} = \begin{bmatrix} 3\exp(-9t) & * & * \\ * & 2\exp(-12t)+\exp(-15t) & * \\ * & * & \exp(-12t)+2\exp(-15t) \end{bmatrix},$$

so all the principal minors of $\exp(At)$ *are positive for all* t.

The next result provides a sufficient condition for $A = TDT^{-1}$, with D a diagonal matrix, to be $\mathbb{E}^{\mathbb{P}}$. This condition is a kind of symmetry condition for the minors of T. For $\alpha, \beta \in Q^{k,n}$, let

$$s(\alpha,\beta) := (-1)^{\sum_{j\in\alpha} j + \sum_{i\in\beta} i},$$

that is, the signature of $\alpha + \beta$.

Theorem 3.25. *Suppose that* $A \in \mathbb{R}^{n\times n}$ *has real eigenvalues* $\lambda_1, \dots, \lambda_n$ *and is diagonalizable, that is, there exists a non-singular matrix* $T \in \mathbb{R}^{n\times n}$ *such that* $A = TDT^{-1}$ *with* $D := \operatorname{diag}(\lambda_1, \dots, \lambda_n)$*. If for any* $k \in [1, \lceil \frac{n-1}{2} \rceil]$ *and any* $\alpha \in Q^{k,n}$ *we have*

$$\det(T) s(\alpha,\beta) T(\alpha|\beta) T(\bar\alpha|\bar\beta) \geq 0 \text{ for all } \beta \in Q^{k,n}, \tag{3.17}$$

and this holds with an inequality for at least one $\beta \in Q^{k,n}$*, then* $A \in \mathbb{E}^{\mathbb{P}}$.

Example 3.26. *Let* $T = \begin{bmatrix} 3 & 3 & -3 \\ 3 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$. *Then* $\det(T) = -24 < 0$. *Calculating*

$$r(\alpha,\beta) := s(\alpha,\beta) T(\alpha|\beta) T(\bar\alpha|\bar\beta)$$

for all possible $\alpha, \beta \in Q^{1,3}$ *yields:*

$$\begin{aligned}
r(\{1\},\{1\}) &= (-1)^2 T(\{1\}|\{1\}) T(\{2,3\}|\{2,3\}) = 0, \\
r(\{1\},\{2\}) &= (-1)^3 T(\{1\}|\{2\}) T(\{2,3\}|\{1,3\}) = -12, \\
r(\{1\},\{3\}) &= (-1)^4 T(\{1\}|\{3\}) T(\{2,3\}|\{1,2\}) = -12, \\
r(\{2\},\{1\}) &= (-1)^3 T(\{2\}|\{1\}) T(\{1,3\}|\{2,3\}) = -18, \\
r(\{2\},\{2\}) &= (-1)^4 T(\{2\}|\{2\}) T(\{1,3\}|\{1,3\}) = -6, \\
r(\{2\},\{3\}) &= (-1)^5 T(\{2\}|\{3\}) T(\{1,3\}|\{1,2\}) = 0, \\
r(\{3\},\{1\}) &= (-1)^4 T(\{3\}|\{1\}) T(\{1,2\}|\{2,3\}) = -6, \\
r(\{3\},\{2\}) &= (-1)^5 T(\{3\}|\{2\}) T(\{1,2\}|\{1,3\}) = -6, \\
r(\{3\},\{3\}) &= (-1)^6 T(\{3\}|\{3\}) T(\{1,2\}|\{1,2\}) = -12.
\end{aligned}$$

Thus, condition (3.17) *holds, and Theorem 3.25 implies that* $TDT^{-1} \in \mathbb{E}^{\mathbb{P}}$ *for any diagonal matrix* $D \in \mathbb{R}^{3\times 3}$.

Proof of Theorem 3.25. Since $A = TDT^{-1}$, $\exp(At) = T\exp(Dt)T^{-1}$. Pick $k \in [1, \lceil \frac{n-1}{2} \rceil]$ and $\alpha \in Q^{k,n}$. Then the corresponding principal minor of $\exp(At)$ satisfies

$$\begin{aligned}(\exp(At))(\alpha|\alpha) &= \sum_{\beta,\gamma \in Q^{k,n}} T(\alpha|\beta)(\exp(Dt))(\beta|\gamma)T^{-1}(\gamma|\alpha)\\ &= \sum_{\beta \in Q^{k,n}} T(\alpha|\beta)(\exp(Dt))(\beta|\beta)T^{-1}(\beta|\alpha)\\ &= \sum_{\beta \in Q^{k,n}} \left(T(\alpha|\beta)T^{-1}(\beta|\alpha)\prod_{i\in\beta}\exp(\lambda_i t)\right),\end{aligned}$$

where we used the fact that D is diagonal. By Jacobi's identity [37, Chapter 0],

$$T^{-1}(\beta|\alpha) = s(\alpha,\beta)T(\bar{\alpha}|\bar{\beta})/\det(T).$$

Note that $s(\alpha,\beta) = s(\bar{\alpha},\bar{\beta})$. Thus,

$$(\exp(At))(\alpha|\alpha) = \frac{1}{\det(T)} \sum_{\beta \in Q^{k,n}} \Big(s(\alpha,\beta)T(\alpha|\beta)T(\bar{\alpha}|\bar{\beta})\prod_{i\in\beta}\exp(\lambda_i t)\Big),$$

and (3.17) implies that $(\exp(At))(\alpha|\alpha) > 0$. □

Theorem 3.27. *Suppose that $A \in \mathbb{R}^{n\times n}$ satisfies the following property: for any $t \geq 0$ there exists an integer $k > 0$ such that*

$$I + \frac{At}{\ell} \text{ is TN for any } \ell \geq k. \tag{3.18}$$

Then $A \in \mathbb{E}^{\mathbb{P}}$.

For example, we say that $A \in \mathbb{R}^{n\times n}$ is a *weak Jacobi matrix* if it is tri-diagonal and all the entries on the super- and sub-diagonals are non-negative. If A is a weak Jacobi matrix then it satisfies the property in Theorem 3.27 (see [18, p. 6]), so $A \in \mathbb{E}^{\mathbb{P}}$. In particular, weak Jacobi matrices are the generators (in the Lie-algebraic sense) of the group of non-singular TN matrices [40].

Proof. Fix $t \geq 0$. Recall that

$$\exp(At) = \lim_{\ell\to\infty}(I + \frac{At}{\ell})^{\ell} \tag{3.19}$$

(see e.g. [41]). For any ℓ sufficiently large, $I + \frac{At}{\ell}$ is TN, and using the fact that the product of two TN matrices is a TN matrix implies that $(I+\frac{At}{\ell})^j$ is TN for any $j \geq 1$. Let $B := (I+\frac{At}{\ell})^{\ell}$. Using the Cauchy-Binet formula gives that any principal minor of B satisfies

$$B(\alpha|\alpha) \geq \left((I + \frac{At}{\ell})(\alpha|\alpha)\right)^{\ell},$$

so

$$B(\alpha|\alpha) \geq \left(1 + \frac{t}{\ell}\sum_{i\in\alpha} a_{ii} + o(\frac{1}{\ell})\right)^{\ell},$$

and taking $\ell \to \infty$ gives

$$(\exp(At))(\alpha|\alpha) \geq \exp(t\sum_{i\in\alpha} a_{ii}) > 0,$$

so $A \in \mathbb{E}^{\mathbb{P}}$. □

Theorem 3.28. *Let $A \in \mathbb{R}^{n\times n}$. If there exists an $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in [0,\bar{\varepsilon}]$ and any integer $\ell > 0$,*

$$(I+\varepsilon A)^{\ell} \text{ is sign-symmetric.} \tag{3.20}$$

Then $A \in \mathbb{E}^{\mathbb{P}}$.

Proof. Suppose that $B \in \mathbb{R}^{n\times n}$ is a P-matrix and that B^{ℓ} is sign symmetric for any integer $\ell > 0$. Pick $k \in [1,n]$ and $\alpha \in Q^{k,n}$. Then the Cauchy-Binet formula gives:

$$\begin{aligned}(B^2)(\alpha|\alpha) &= \sum_{\gamma\in Q^{k,n}} B(\alpha|\gamma)B(\gamma|\alpha)\\ &\geq (B(\alpha|\alpha))^2\\ &> 0,\end{aligned}$$

so B^2 is also a P-matrix and, by assumption, also sign-symmetric. Continuing in this fashion implies that

$$(B^{2^{\ell}})(\alpha|\alpha) \geq (B(\alpha|\alpha))^{2^{\ell}} \text{ for any integer } \ell > 0. \tag{3.21}$$

Now suppose that (3.20) holds. Fix $t \geq 0$. Then for any $\ell > 0$ sufficiently large the matrix $B := I + \frac{At}{2^{\ell}}$ is a P-matrix and (3.20) implies that B^{ℓ} is sign-symmetric, so (3.21) gives

$$\begin{aligned}((I+\frac{At}{2^{\ell}})^{2^{\ell}})(\alpha|\alpha) &\geq ((I+\frac{At}{2^{\ell}})(\alpha|\alpha))^{2^{\ell}}\\ &\geq (1+\frac{t}{2^{\ell}}\sum_{i\in\alpha} a_{ii} + o(\frac{1}{2^{\ell}}))^{2^{\ell}}.\end{aligned}$$

Taking $\ell \to \infty$ gives

$$(\exp(At))(\alpha|\alpha) \geq \exp(t\sum_{i\in\alpha} a_{ii}) > 0,$$

and this completes the proof. □

Using the relation between sign-symmetric matrices and sign-pattern symmetric matrices, we can restate Theorem 3.28 as follows.

Corollary 3.29. *Suppose that $A \in \mathbb{R}^{n\times n}$ satisfies the following property: there exists an $\bar{\varepsilon} > 0$ such that $((I+\varepsilon A)^{(k)})^{\ell}$ is sign-pattern symmetric for any $\varepsilon \in [0,\bar{\varepsilon}]$, $k \in [1,n-1]$, and any integer $\ell > 0$. Then $A \in \mathbb{E}^{\mathbb{P}}$.*

Remark 3.30. *It follows from* (1.8) *that*

$$((I+\varepsilon A)^{(k)})^{\ell} = I + \varepsilon\ell A^{[k]} + o(\varepsilon),$$

so the condition in the corollary implies that

$$A^{[k]} \text{ is sign-pattern symmetric for any } k \in [1,n-1]. \tag{3.22}$$

The next example demonstrates that Theorem 3.28 is indeed more general than Theorem 3.10.

Example 3.31. *Consider again the matrix A in* (3.16)*. Note that A does not belong to any of the three matrix classes in Theorem 3.10. A computation yields*

$$(I+\varepsilon A)^{\ell} = \frac{1}{3}\begin{bmatrix} * & a_1^{\ell}-a_2^{\ell} & * \\ 2a_1^{\ell}-2a_2^{\ell} & * & * \\ 0 & 0 & * \end{bmatrix},$$

$$\left((I+\varepsilon A)^{(2)}\right)^{\ell} = \frac{1}{3}\begin{bmatrix} * & * & * \\ 0 & * & a_3^{\ell}-a_4^{\ell} \\ 0 & 2a_3^{\ell}-2a_4^{\ell} & * \end{bmatrix},$$

where $a_1 := 1-3\varepsilon$*,* $a_2 := 1-6\varepsilon$*,* $a_3 := 1-12\varepsilon+27\varepsilon^2$*, and* $a_4 := 1-15\varepsilon+54\varepsilon^2$*. Note that both matrices are sign-pattern symmetric. By Corollary 3.29,* $A \in \mathbb{E}^{\mathbb{P}}$*.*

The next example shows that (3.20) is not a necessary condition for $\mathbb{E}^{\mathbb{P}}$.

Example 3.32. *Consider the matrix*

$$A = \begin{bmatrix} 15 & -9 & -18 \\ 1 & 3 & -12 \\ -1 & 3 & 12 \end{bmatrix}. \tag{3.23}$$

Let $D = \operatorname{diag}(0,12,18)$ *and consider the matrix* $T \in \mathbb{R}^{3\times 3}$ *given in Example 3.26. Then* $A = TDT^{-1}$*. Hence, Theorem 3.25 implies that* $A \in \mathbb{E}^{\mathbb{P}}$*. However, A is not sign-pattern symmetric, that is, condition* (3.20) *is not satisfied.*

Remark 3.33. *Recall that any principal submatrix of a P-matrix is also a P-matrix. A similar conclusion generally does not hold for the* $\mathbb{E}^{\mathbb{P}}$ *case. For example, the matrix A in* (3.23) *is* $\mathbb{E}^{\mathbb{P}}$ *and has the principal submatrix* $A[\{1,2\}|\{1,2\}] = \begin{bmatrix} 15 & -9 \\ 1 & 3 \end{bmatrix}$*, which is not* $\mathbb{E}^{\mathbb{P}}$ *according to Proposition 3.7.*

4. AN APPLICATION: OPINION DYNAMICS WITH AT LEAST ONE STUBBORN AGENT

Opinion dynamics studies how local interactions between social actors lead to the global formation of opinions. Applications include group decision making, the propagation of rumors, successful marketing, the emergence of extremism, and reaching consensus. For a recent survey, see e.g., [42].

Opinion dynamics models can be classified into several groups. There are models where the opinions are considered discrete (often accepting only two different results, say, voting either for the Republicans or the Democrats). Examples include using the Ising model to describe the behavior of laborers in a strike, where the two options are either to work or to strike [43], and other models for binary-state dynamics on a network like the Sznajd model [44].

A second group of models considers opinions as continuous variables that can take values in an interval. The value of the variables may represent the worthiness of a choice. Examples include the well-known Deffuant model [45] and the Hegselmann-Krause [46] models.

A third group of models considers opinions that are observed as discrete actions, but are represented internally by each agent as a continuous variable. A typical example is to model the opinion of agent i as a variable x_i taking values in $[-1,1]$, such that $x_i > 0$ is interpreted as voting for one candidate and $x_i < 0$ represents voting for the other candidate [47].

Many algorithms have been presented to solve distributed problems with many cooperating agents, such as average consensus, rendezvous, and sensor coverage [48, 49, 50, 51]. Here, we consider the well-known linear consensus algorithm:

$$\dot{x} = -Lx, \tag{4.1}$$

where $L \in \mathbb{R}^{n\times n}$ is a Laplacian matrix. We introduce the following requirement.

Definition 4.1. *We say that* (4.1) *reaches consensus* without complete sign-reversal *if for any* $x(0) \in \mathbb{R}^n \setminus \{0\}$ *the following properties hold:*

(1) The solution $x(t)$ converges, as $t \to \infty$, to $c1_n$ for some $c \in \mathbb{R}$;
(2) for any $t \geq 0$ there exists an index $i \in \{1,\dots,n\}$ (that may depend on $x(0)$ and t) such that $x_i(t)x_i(0) > 0$.

The second requirement aims to prevent a situation where there is a sign-reversal between the vectors $x(0)$ and $x(t)$. For example, if $n = 3$ and $x(0)$ has the sign pattern $\begin{bmatrix}+ & - & +\end{bmatrix}^T$ we do not allow $x(t) = \begin{bmatrix}- & + & -\end{bmatrix}^T$ at any time $t \geq 0$. This has a natural interpretation. We do not allow that *all* the agents "change their mind" along the course of reaching consensus. This new requirement is related to a well-known notion in opinion dynamics called "stubborn agents" [52], that is, an agent that does not change its opinion over time (the terminology in this field is not uniform and such agents are also called a leader [53], social media [54], closed-minded [55], and inflexible agent [56]). However, the requirement in Definition 4.1 is novel, as it does not pinpoint a specific agent, but rather focuses on the global behaviour.

The conditions needed for reaching consensus are well-known, so we focus on the second requirement in Definition 4.1. The next result follows immediately from the fact that the solution of (4.1) is $x(t) = \exp(-Lt)x(0)$.

Proposition 4.2. *The second requirement in Definition 4.1 holds iff $L \in \mathbb{E}^{\mathbb{P}}$.*

If L represents an undirected connectivity graph then it is symmetric, and Theorem 3.10 implies that $L \in \mathbb{E}^{\mathbb{P}}$. But a non-symmetric Laplacian matrix L may have complex eigenvalues and then Theorem 3.11 implies that $L \notin \mathbb{E}^{\mathbb{P}}$, so there exists $t > 0$ such that $\exp(-Lt) \notin \mathbb{P}$, and by Theorem 2.3 there exists $x(0) \in \mathbb{R}^n \setminus \{0\}$ such that $x_i(0)x_i(t) \leq 0$ for all i.

5. Conclusion

P-matrices play an important role in many fields of applied mathematics. Here, we defined the new notion of an $\mathbb{E}^{\mathbb{P}}$-matrix i.e., a matrix A such that $\exp(At)$ is a P-matrix for all $t \geq 0$ (and then also for all $t \leq 0$). We show that $\mathbb{E}^{\mathbb{P}}$-matrices must have all real eigenvalues and provide several conditions guaranteeing that a matrix is $\mathbb{E}^{\mathbb{P}}$. For $A \in \mathbb{R}^{2\times 2}$, a simple sign pattern condition on the matrix entries is proved to be necessary and sufficient for $A \in \mathbb{E}^{\mathbb{P}}$. Using the sign non-reversal property of P-matrices, we also described a natural application of $\mathbb{E}^{\mathbb{P}}$ matrices to linear consensus algorithms.

Interesting topics for further research include: completing the analysis of $\mathbb{E}^{\mathbb{P}}$ in the case of 3×3 matrices, and identifying sign patterns that guarantee that a matrix is $\mathbb{E}^{\mathbb{P}}$. Several authors studied dynamical systems that include complementarity constraints [57, 58], and the notion of $\mathbb{E}^{\mathbb{P}}$-matrices may also find applications in such systems.

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