Commun. Optim. Theory 2024 (2024) 4 https://doi.org/10.23952/cot.2024.4

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org

CONVERGENCE OF INEXACT ORBITS OF NONEXPANSIVE MAPPINGS IN COMPLETE METRIC SPACES

ALEXANDER J. ZASLAVSKI

Department of Mathematics, The Technion – Israel Institute of Technology, Haifa, Israel

Dedicated to the memory of Professor Naum Shor

Abstract. In this paper, we analyze the asymptotic behavior of inexact iterates of nonexpansive mappings which take a nonempty and closed subset of a complete metric space into the space, under the presence of errors, and generalize the results known in the literature.

Keywords. Complete metric space; Convergence analysis; Fixed point; Inexact iterate; Nonexpansive mapping.

2020 Mathematics Subject Classification. 47H04, 47H09, 47H10.

1. Introduction

During more than sixty years now, there has been a lot of activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [3, 5, 11, 13, 14, 17, 18, 19, 20, 23, 24, 25, 28, 29] and the references cited therein. This activity stems from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also covers the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility and common fixed point problems, which find important applications in engineering and medical sciences [8, 12, 15, 26, 27, 28, 29].

In [22], it was established the convergence of every inexact orbit of a strict contraction (that is, c-Lipschitz where $c \in (0,1)$) mapping with summable errors. This result was generalized in [5] where it was shown that if any exact orbit of a nonexpansive mapping converges to its fixed point, then this convergence property also holds for its inexact orbits with summable errors. This result was obtained for a self-mapping of a complete metric space X. In [30], the result of [5] was generalized for nonexpansive mappings which take a nonempty and closed subset K of the complete metric space X into X. The result of [30] was obtained for a sequence of inexact iterates such that an r-neighborhood of each iterate is contained in K where $r > 0$ is a constant which does not depend on an iterate but on the whole sequence of iterates. In the

E-mail address: ajzasl@technion.ac.il.
Received: March 8, 2023; Accepted: May 24, 2023.

present paper we prove the result of [30] under weaker assumptions. Here for a given sequence of inexact iterates $\{x_i\}_{i=0}^{\infty}$ we assume only that for a given natural number n_0 there exists $\delta > 0$ and an infinite set of natural numbers E such that for every $m \in E$ and every $i \in [m, m+n_0]$ the δ-neighborhood of x_i is contained in K.

In this section we discuss the results of [5, 30] and their applications. In out paper we use the following notation.

Let (X,ρ) be a complete metric space. For each $x \in X$ and each nonempty set $B \subset X$, put

$$\rho(x,B) = \inf\{\rho(x,y) :\ y \in B\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x,r) = \{y \in X :\ \rho(x,y) \le r\}.$$

For each mapping $A : X \to X$, let $A^0 x = x$ for all $x \in X$.

In [5], it was studied the influence of errors on the convergence of orbits of nonexpansive mappings in metric spaces and it was obtained the following result (see also Theorem 2.72 of [25]).

Theorem 1.1. *Let $A : X \to X$ satisfy*

$$\rho(Ax,Ay) \le \rho(x,y) \text{ for all } x,y \in X,$$

let $F(A)$ be the set of all fixed points of A and let for each $x \in X$, the sequence $\{A^n x\}_{n=1}^{\infty}$ converges in (X,ρ).

Assume that $\{x_n\}_{n=0}^{\infty} \subset X$, $\{r_n\}_{n=0}^{\infty} \subset (0,\infty)$ satisfies

$$\sum_{n=0}^{\infty} r_n < \infty$$

and that

$$\rho(x_{n+1},Ax_n) \le r_n,\ n = 0,1,\ldots.$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of A in (X,ρ).

In [30], we generalized this result for nonexpansive mappings which take a nonempty, closed subset of the complete metric space X into X. More precisely, the following result was obtained.

Theorem 1.2. *Assume that K is a nonempty closed subset of X, $A : K \to X$ satisfies*

$$\rho(Ax,Ay) \le \rho(x,y) \text{ for all } x,y \in K,$$

$F(A)$ is the set of all fixed points of A and that for each $x \in X$, if the sequence $\{A^n x\}_{n=1}^{\infty}$ is well-defined, then it converges in (X,ρ).

Let

$$\{x_n\}_{n=0}^{\infty} \subset K,$$

$\tilde{r} > 0$, $\{r_n\}_{n=0}^{\infty} \subset (0,\infty)$ satisfy

$$\sum_{n=0}^{\infty} r_n < \infty,$$

$$\rho(x_{n+1},Ax_n) \le r_n,\ n = 0,1,\ldots$$

and let

$$B(x_n,\tilde{r}) \subset K \text{ for all sufficiently large natural numbers } n.$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of A in (X,ρ).

Theorem 1.1 found interesting applications and is an important ingredient in superiorization and perturbation resilience of algorithms. See [2, 4, 6, 7, 9, 10, 16, 21] and the references mentioned therein. The superiorization methodology works by taking an iterative algorithm, investigating its perturbation resilience, and then using proactively such perturbations in order to "force" the perturbed algorithm to do in addition to its original task something useful. This methodology can be explained by the following result on convergence of inexact iterates.

Assume that $(X, \|\cdot\|)$ is a Banach space, $\rho(x,y) = \|x-y\|$ for all $x,y \in X$, for each $x \in X$, the sequence $\{A^n x\}_{n=1}^{\infty}$ converges in the norm topology. $x_0 \in X$, $\{\beta_k\}_{k=0}^{\infty}$ is a sequence of positive numbers satisfying

$$\sum_{k=0}^{\infty} \beta_k < \infty, \tag{1.1}$$

$\{v_k\}_{k=0}^{\infty} \subset X$ is a norm bounded sequence and that for any integer $k \geq 0$,

$$x_{k+1} = A(x_k + \beta_k v_k). \tag{1.2}$$

Then it follows from Theorem 1.1 that the sequence $\{x_k\}_{k=0}^{\infty}$ converges in the norm topology of X and its limit is a fixed point of A. In this case the mapping A is called bounded perturbations resilient (see [6] and Definition 10 of [9]). In other words, if exact iterates of a nonexpansive mapping converge, then its inexact iterates with bounded summable perturbations converge too.

Now assume that $x_0 \in X$ and the sequence $\{\beta_k\}_{k=0}^{\infty}$ satisfying (1.1) are given and we need to find an approximate fixed point of A. In order to meet this goal we construct a sequence $\{x_k\}_{k=1}^{\infty}$ defined by (1.2). Under an appropriate choice of the bounded sequence $\{v_k\}_{k=0}^{\infty}$, the sequence $\{x_k\}_{k=1}^{\infty}$ possesses some useful property. For example, the sequence $\{f(x_k)\}_{k=1}^{\infty}$ can be decreasing, where f is a given function.

2. The main results

Assume that (X, ρ) is a complete metric space, $K \subset X$ is a nonempty, closed set and that a mapping $T : K \to X$ satisfies

$$\rho(T(x), T(y)) \leq \rho(x,y) \text{ for each } x,y \in K. \tag{2.1}$$

Set

$$F(T) = \{x \in K : \; T(x) = x\} \tag{2.2}$$

and fix

$$\theta \in X.$$

We assume that

$$F(T) \neq \emptyset$$

and that the following assumption holds:

(A1) For each $M, \varepsilon > 0$, there exists a natural number $n(\varepsilon, M)$ such that for each finite sequence $\{x_i\}_{i=0}^{n} \subset K$ satisfying

$$\rho(x_0, \theta) \leq M$$

and

$$x_{i+1} = T(x_i), \; i = 0, \dots, n(M, \varepsilon) - 1$$

the inequality

$$\rho(x_{n(M,\varepsilon)}, F) \leq \varepsilon$$

is true.

It is well-known that (A1) holds if T is a contractive type mapping [25]. In this paper, we prove the following results.

Theorem 2.1. *Assume that $\{x_i\}_{i=0}^{K}$ satisfies*

$$\sum_{i=0}^{\infty} \rho(T(x_i), x_{i+1}) < \infty \tag{2.3}$$

and that the following property holds:

(P1) for each natural number n_0, there exists $\delta > 0$ such that, for each integer $m \geq 0$, there exists and integer $n \geq m$ such that

$$B(x_i, \delta) \subset K, \ i = n, \dots, n + n_0.$$

Then there exists

$$\lim_{i \to \infty} x_i \in F(T).$$

Theorem 2.1 is proved in Section 3. The next theorem is proved in Section 4.

Theorem 2.2. *Assume that $M > 0$, a sequence $\{x_i\}_{i=0}^{\infty} \subset K$ satisfies*

$$\lim_{i \to \infty} \rho(T(x_i), x_{i+1}) = 0 \tag{2.4}$$

and that the following assumption holds:

(A2) for each natural number n_0 there exist $\delta > 0$ and a strictly increasing sequence of natural numbers $\{k_i\}_{i=1}^{\infty}$ such that for each integer $i \geq 1$,

$$\rho(x_{k_i}, \theta) \leq M$$

and

$$B(x_j, \delta) \subset K, \ j = k_i, \dots, k_i + n_0.$$

Then there exists

$$\liminf_{i \to \infty} \rho(x_i, F(T)) = 0.$$

Theorem 2.3. *Assume that $M > 0$, a sequence $\{x_i\}_{i=0}^{\infty} \subset K$ satisfies*

$$\lim_{i \to \infty} \rho(T(x_i), x_{i+1}) = 0 \tag{2.5}$$

and that the following assumption holds:

(A3) for each natural number n_0 there exist $\delta > 0$ and a strictly increasing sequence of natural numbers $\{k_i\}_{i=1}^{\infty}$ such that

$$\sup\{k_{i+1} - k_i : \ i = 1, 2, \dots\} < \infty$$

and for each integer $i \geq 1$,

$$\rho(x_{k_i}, \theta) \leq M$$

and

$$B(x_j, \delta) \subset K, \ j = k_i, \dots, k_i + n_0.$$

Then there exists

$$\lim_{i \to \infty} \rho(x_i, F(T)) = 0.$$

Theorem 2.3 is proved in Section 5. Note that property (P1) and assumptions (A2) and (A3) hold for a sequence of iterates such that there distances to the boundary of K are bounded by a positive constant.

3. Proof of Theorem 2.1

Fix
$$z \in F(T). \tag{3.1}$$
By (2.1) and (3.1), for each integer $i \geq 0$,
$$\rho(z, x_{i+1}) \leq \rho(z, T(x_i)) + \rho(T(x_i), x_{i+1}) \leq \rho(z, x_i) + \rho(T(x_i), x_{i+1})). \tag{3.2}$$
In view of (3.2), for each integer $j \geq 1$,
$$\rho(z, x_j) \leq \rho(z, x_0) + \sum_{i=0}^{\infty} \rho(x_{i+1}, T(x_i)). \tag{3.3}$$
Set
$$M = \rho(z, x_0) + \sum_{i=0}^{\infty} \rho(x_{i+1}, T(x_i)). \tag{3.4}$$
Let $\varepsilon \in (0,1)$. By assumption (A1), there exists a natural number n_0 such that the following property holds:

(P2) for each finite sequence $\{y_i\}_{i=0}^{n_0} \subset K$ satisfying
$$\rho(y_0, z) \leq M$$
and
$$y_{i+1} = T(y_i),\ i = 0, \dots, n_0 - 1$$
the inequality
$$\rho(y_{n_0}, F) \leq \varepsilon/4$$
is true.

Property (P1) implies that there exists
$$\delta \in (0, \varepsilon/8)$$
such that the following property holds:

(P3) for each integer $m \geq 0$ there exists an integer $n \geq m$ such that
$$B(x_i, \delta) \subset K,\ i = n, \dots, n + n_0.$$

By (2.3), there exists an integer $n_1 > 0$ such that
$$\sum_{i=n_1}^{\infty} \rho(T(x_i), x_{i+1}) < \delta. \tag{3.5}$$
Property (P3) implies that there exists an integer
$$n_2 > n_1$$
such that
$$B(x_i, \delta) \subset K,\ i = n_2, \dots, n_2 + n_0. \tag{3.6}$$
Set
$$y_{n_2} = x_{n_2}. \tag{3.7}$$
It follows from (3.3), (3.4) and (3.7) that
$$\rho(y_{n_2}, z) = \rho(x_{n_2}, z) \leq M. \tag{3.8}$$

Set

$$y_{n_2+1} = T(y_{n_2}) \tag{3.9}$$

which is well defined in view of (3.6). Equations (3.5)-(3.7) and (3.9) imply that

$$\rho(x_{n_2+1}, y_{n_2+1}) = \rho(x_{n_2+1}, T(x_{n_2})) < \delta \tag{3.10}$$

and

$$y_{n_2+1} \in K. \tag{3.11}$$

By induction, we set for each integer $p > n_2$

$$y_{p+1} = T(y_p) \tag{3.12}$$

is y_p exists and belongs to K.

Assume that $p \in \{n_2+1, \dots, n_0+n_2\}$,

$$y_p \in K \tag{3.13}$$

exists and

$$\rho(y_p, x_p) \le \sum_{i=n_2}^{p-1} \rho(x_{i+1}, T(x_i)). \tag{3.14}$$

(In view of (3.10) and (3.11), our assumption holds for $p = n_2+1$.) By (3.12) and (3.13), y_{p+1} is defined. Equations (2.1), (3.12) and (3.14) imply that

$$\begin{aligned} \rho(y_{p+1}, x_{p+1}) &= \rho(T(y_p), T(x_p)) + \rho(T(x_p), x_{p+1}) \\ &\le \rho(y_p, x_p) + \rho(T(x_p), x_{p+1}) \\ &\le \sum_{i=n_2}^{p-1} \rho(T(x_i), x_{i+1}) + \rho(T(x_p), x_{p+1}) = \sum_{i=n_2}^{p} \rho(T(x_i), x_{i+1}). \end{aligned} \tag{3.15}$$

In view of (3.5) and (3.15), we have

$$\rho(y_{p+1}, x_{p+1}) \le \sum_{i=n_1}^{\infty} \rho(T(x_i), x_{i+1}) < \delta.$$

If $p < n_0+n_2$, we obtain from (3.6) that $y_{p+1} \in K$. Thus, by induction, we have that

$$y_p \in K,\ p = n_2, \dots, n_0+n_2$$

are well defined and (3.14) holds for all $p = n_2+1, \dots, n_2+n_0$. By (3.5), (3.14) and inequality $n_2 > n_1$, we have

$$\rho(x_{n_2+n_0}, y_{n_2+n_0}) \le \sum_{i=n_2}^{\infty} \rho(T(x_i), x_{i+1}) \le \sum_{i=n_1}^{\infty} \rho(T(x_i), x_{i+1}) < \delta < \varepsilon/8. \tag{3.16}$$

Property (P2) and (3.8) and (3.12) imply that $\rho(y_{n_0+n_2}, F(T)) \le \varepsilon/4$. This together with (3.16) implies that

$$\rho(x_{n_2+n_0}, F) \le \varepsilon/4 + \varepsilon/8. \tag{3.17}$$

By (3.17), there exists

$$z \in F(T) \tag{3.18}$$

such that

$$\rho(x_{n_0+n_2}, z) < \varepsilon/2. \tag{3.19}$$

It follows from (2.1) and (3.18) that for each integer $p \geq n_0 + n_2$

$$\rho(x_{p+1}, z) \leq \rho(T(x_p), x_{p+1}) + \rho(T(x_p), z) \leq \rho(T(x_p), x_{p+1}) + \rho(x_p, z). \tag{3.20}$$

By (3.5), (3.19) and (3.20), for each integer $p \geq n_0 + n_2$,

$$\rho(x_p, z) \leq \rho(x_{n_0+n_2}, z) + \sum_{i=n_1+n_0}^{\infty} \rho(T(x_i), x_{i+1}) < \varepsilon/2 + \varepsilon/8.$$

Since ε is an arbitrary element of the interval $(0,1)$, we conclude that $\{x_i\}_{i=0}^{\infty}$ is a Cauchy sequence and there exits $x_* = \lim_{i\to\infty} x_i$. Evidently, $T(x_*) = x_*$. Theorem 2.1 is proved.

4. Proof of Theorem 2.2

Let $\varepsilon > 0$. By assumption (A1), there exists a natural number n_0 such that the following property holds:

(a) for each finite sequence $\{y_i\}_{i=0}^{n_0} \subset K$ satisfying $\rho(y_0, \theta) \leq M$ and $y_{i+1} = T(y_i)$, $i = 0, \dots, n_0 - 1$, the inequality $\rho(y_{n_0}, F(T)) \leq \varepsilon/4$ is true.

By (A2), there exist $\delta \in (0, \varepsilon/2)$ and a strictly increasing sequence of natural numbers $\{k_i\}_{i=1}^{\infty}$ such that for each integer $i \geq 1$,

$$\rho(x_{k_i}, \theta) \leq M \tag{4.1}$$

and

$$B(x_j, \delta) \subset K,\ j = k_i, \dots, k_i + n_0. \tag{4.2}$$

By (2.4), there exists an integer $n_1 > n_0$ such that

$$\rho(T(x_j), x_{j+1}) \leq \delta(2n_0)^{-1} \text{ for each integer } j \geq n_1. \tag{4.3}$$

There exists an integer $q_0 \geq 1$ such that

$$k_{q_0} \geq n_1. \tag{4.4}$$

Let $q \geq q_0$ be an integer. Set

$$y_{k_q} = x_{k_q}, \tag{4.5}$$

$$y_{k_q+1} = T(y_{k_q}). \tag{4.6}$$

In view of (4.2) and (4.5), y_{k_q+1} is well defined. It follows from (4.3)-(4.6) that

$$\rho(x_{k_q+1}, y_{k_q+1}) = \rho(x_{k_q+1}, T(x_{k_q})) \leq \delta(2n_0)^{-1} \tag{4.7}$$

and in view of (4.2), $y_{k_q+1} \in K$. By induction for each integer $p \geq k_{q+1}$, set

$$y_{p+1} = T(y_p) \tag{4.8}$$

is $y_p \in K$. Assume that an integer p satisfies $k_q + 1 \leq p < k_q + n_0$, exists $y_p \in K$ and

$$\rho(x_p, y_p) \leq \delta(p - k_q)(2n_0)^{-1}. \tag{4.9}$$

(In view of (4.7), inequality (4.9) holds for $p = k_q + 1$.) By (4.2) and (4.9), $y_p \in K$ and $y_{p+1} = T(y_p)$ is well defined. It follows from (2.1), (4.3) and (4.4) that

$$\rho(x_{p+1}, y_{p+1}) \leq \rho(x_{p+1}, T(x_p)) + \rho(T(x_p), T(y_p))$$
$$\leq \delta(2n_0)^{-1} + \rho(x_p, y_p) \leq \delta(p - k_q + 1)(2n_0)^{-1},$$
$$\rho(x_{p+1}, y_{p+1}) \leq \delta$$

and in view of (4.2), $y_{p+1} \in K$. Thus we proved by induction that for all $p = k_q, \dots, k_q + n_0$ $y_p \in K$ is well defined and

$$\rho(x_p, y_p) \le \delta(p - k_q)(2n_0)^{-1} \le \delta. \tag{4.10}$$

Property (a), (4.1) and (4.5) imply that

$$\rho(y_{k_q+n_0}, F(T)) < \varepsilon/4. \tag{4.11}$$

By (4.10) and (4.11), $\rho(x_{k_q+n_0}, F(T)) < \varepsilon$. Since ε is an arbitrary positive number we conclude that $\liminf_{i\to\infty} \rho(x_i, F(T)) = 0$. Theorem 2.2 is proved.

5. Proof of Theorem 2.3

Let $\varepsilon > 0$. By assumption (A1), there exists a natural number n_0 such that the following property holds:

(a) for each finite sequence $\{y_i\}_{i=0}^{n_0} \subset K$ satisfying $\rho(y_0, \theta) \le M$ and $y_{i+1} = T(y_i)$, $i = 0, \dots, n_0 - 1$ the inequality $\rho(y_{n_0}, F) < \varepsilon/8$ is true.

By (A3), there exist $\delta_0 \in (0, \varepsilon/8)$ and a strictly increasing sequence of natural numbers $\{k_i\}_{i=1}^{\infty}$ such that

$$\sup\{k_{i+1} - k_i :\ i = 1, 2, \dots\} < \infty \tag{5.1}$$

and for each integer $i \ge 1$,

$$\rho(x_{k_i}, \theta) \le M \tag{5.2}$$

and

$$B(x_j, \delta_0) \subset K,\ j = k_i, \dots, k_i + n_0. \tag{5.3}$$

By (5.1), there exists a natural number $n_1 > n_0$ such that

$$k_{i+1} - k_i < n_1,\ i = 1, 2, \dots. \tag{5.4}$$

Choose

$$\delta \in (0, \delta_0(8n_1)^{-1}). \tag{5.5}$$

In view of (2.5), there exists an integer $n_2 > n_1$ such that

$$\rho(T(x_j), x_{j+1}) \le \delta(2n_1)^{-1} \text{ for each integer } j \ge n_2. \tag{5.6}$$

There exists an integer $q_0 \ge 1$ such that

$$k_{q_0} \ge n_2. \tag{5.7}$$

Let $q \ge q_0$ be an integer. Set

$$y_{k_q} = x_{k_q},\ y_{k_q+1} = T(y_{k_q}). \tag{5.8}$$

In view of (5.3) and (5.8), y_{k_q+1} is well defined. It follows from (5.6)-(5.8) that

$$\rho(x_{k_q+1}, y_{k_q+1}) = \rho(x_{k_q+1}, T(x_{k_q})) \le \delta(2n_1)^{-1}. \tag{5.9}$$

Equations (5.3), (5.5) and (5.9) imply that $y_{k_q+1} \in K$. By induction for each integer $p \ge k_{q+1}$, set

$$y_{p+1} = T(y_p) \tag{5.10}$$

if $y_p \in K$. Assume that an integer p satisfies

$$k_q + 1 \le p < k_q + n_0, \tag{5.11}$$

exists $y_p \in K$ and

$$\rho(x_p, y_p) \leq \delta(p-k_q)(2n_1)^{-1}. \tag{5.12}$$

(In view of (5.9), our assumption holds for $p = k_q + 1$.) It follows from (2.1), (5.3), (5.6), (5.7), (5.10) and (5.12) that

$$\begin{aligned} \rho(x_{p+1}, y_{p+1}) &\leq \rho(x_{p+1}, T(x_p)) + \rho(T(x_p), T(y_p)) \\ &\leq \delta(2n_1)^{-1} + \delta(p-k_q)(2n_1)^{-1} \leq \delta(p-k_q+1)(2n_1)^{-1} \leq \delta. \end{aligned}$$

Thus we showed by induction that for all $p = k_q, \ldots, k_q + n_0$, $y_p \in K$ is well defined and (5.12) holds. Property (a), (5.2) and (5.8) imply that

$$\rho(y_{k_q+n_0}, F(T)) < \varepsilon/8. \tag{5.13}$$

By (5.12) and (5.13), we have

$$\begin{aligned} \rho(x_{k_q+n_0}, F(T)) &\leq \rho(x_{k_q+n_0}, y_{k_q+n_0}) + \rho(y_{k_q+n_0}, F(T)) \\ &\leq n_0\delta(2n_1)^{-1} + \varepsilon/8 \leq \varepsilon/8 + \varepsilon/64. \end{aligned}$$

This implies that there exists $z \in F(T)$ such that

$$\rho(x_{k_q+n_0}, z) \leq \varepsilon/2. \tag{5.14}$$

By (2.1), (5.6), (5.7) and (5.14), for each integer i satisfying $k_q + n_0 \leq i < k_q + n_1 + n_0$, we have

$$\rho(x_{i+1}, z) \leq \rho(x_{i+1}, T(x_i)) + \rho(T(x_i), z) \leq \delta(2n_1)^{-1} + \rho(x_i, z),$$

$$\begin{aligned} \rho(x_i, z) &\leq \rho(x_{k_q+n_0}, z) + (i - k_q - n_0)\delta(2n_1)^{-1} \\ &\leq \varepsilon/8 + \varepsilon/32 \leq \varepsilon/4. \end{aligned}$$

Thus $\rho(x_i, F(T)) \leq \varepsilon/2$ for each integer $i \geq k_{q_0}$. Since ε is an arbitrary positive number Theorem 2.3 is proved.

REFERENCES

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922), 133–181.

[2] H. H. Bauschke, V. R. Koch, Projection methods: Swiss army knives for solving feasibility and best approximation problems with half-spaces, Contemporary Mathematics 636 (2015), 1–40.

[3] A. Betiuk-Pilarska, T. Domínguez Benavides, Fixed points for nonexpansive mappings and generalized nonexpansive mappings on Banach lattices, Pure Appl. Func. Anal. 1 (2016), 343–359.

[4] D. Butnariu, R. Davidi, G. T. Herman, I. G. Kazantsev, Stable convergence behavior under summable perturbations of a class of projection methods for convex feasibility and optimization problems, IEEE J. Selected Topics Signal Process. 1 (2007) 540–547.

[5] D. Butnariu, S. Reich, A. J. Zaslavski, Convergence to fixed points of inexact orbits of Bregman-monotone and of nonexpansive operators in Banach spaces, pp. 11–32, Proceedings of Fixed Point Theory and its Applications, Mexico, Yokahama Publishers, 2006.

[6] Y. Censor, R. Davidi, G. T. Herman, Perturbation resilience and superiorization of iterative algorithms, Inverse Probl. 26 (2010), 065008.

[7] Y. Censor, R. Davidi, G. T. Herman, R. W. Schulte, L. Tetruashvili, Projected subgradient minimization versus superiorization, J. Optim. Theory Appl. 160 (2014) 730-747.

[8] Y. Censor, M. Zaknoon, Algorithms and convergence results of projection methods for inconsistent feasibility problems: a review, Pure Appl. Func. Anal. 3 (2018), 565–586.

[9] Y. Censor, A. J. Zaslavski, Convergence and perturbation resilience of dynamic string-averaging projection methods, Comput. Optim. Appl. 54 (2013) 65–76.

[10] Y. Censor, A. J. Zaslavski, Strict Fejer monotonicity by superiorization of feasibility-seeking projection methods, J. Optim. Theory Appl. 165 (2015) 172–187.

[11] F. S. de Blasi, J. Myjak, S. Reich, A. J. Zaslavski, Generic existence and approximation of fixed points for nonexpansive set-valued maps, Set-Valued Var. Anal. 17 (2009) 97–112.

[12] A. Gibali, A new split inverse problem and an application to least intensity feasible solutions, Pure Appl. Funct. Anal. 2 (2017) 243–258.

[13] K. Goebel, W. A. Kirk, Topics in Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990.

[14] K. Goebel, S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, New York and Basel, 1984.

[15] L. G. Gurin, B. T. Poljak, E. V. Raik, ‘ The method of projections for finding the common point of convex sets, USSR Comput. Math. Math. Phys. 7 (1967) 1-24.

[16] G. T. Herman, R. Davidi, Image reconstruction from a small number of projections, Inverse Problems 2 (2008) 045011.

[17] J. Jachymski, Extensions of the Dugundji-Granas and Nadler's theorems on the continuity of fixed points, Pure Appl. Funct. Anal. 2 (2017) 657–666.

[18] M. A. Khamsi, W. M. Kozlowski, Fixed Point Theory in Modular Function Spaces, Birkhäuser/Springer, Cham, 2015.

[19] W. A. Kirk, Contraction mappings and extensions, Handbook of Metric Fixed Point Theory, Kluwer, Dordrecht, 2001, 1–34.

[20] R. Kubota, W. Takahashi, Y. Takeuchi, Extensions of Browder's demiclosedness principle and Reich's lemma and their applications, Pure Appl. Func. Anal. 1 (2016) 63–84.

[21] T. Nikazad, R. Davidi, G. T. Herman, Accelerated perturbation-resilient block-iterative projection methods with application to image reconstruction, Inverse Problems 28 (2012) 035005

[22] A. M. Ostrowski, The round-off stability of iterations, Z. Angew. Math. Mech. 47 (1967) 77–81.

[23] Z. D. Mitrović, S. Radenović, Reich, A. J. Zaslavski, Iterating nonlinear contractive mappings in Banach spaces, Carpatian J. Math. 36 (2020) 287–294.

[24] S. Reich, A. J. Zaslavski, Generic aspects of metric fixed point theory, Handbook of Metric Fixed Point Theory, Kluwer, Dordrecht, 2001 557–575.

[25] S. Reich, A. J. Zaslavski, Genericity in Nonlinear Analysis, Springer, New York, 2014.

[26] W. Takahashi, The split common fixed point problem and the shrinking projection method for new nonlinear mappings in two Banach spaces, Pure Appl. Funct. Anal. 2 (2017) 685–699.

[27] W. Takahashi, A general iterative method for split common fixed point problems in Hilbert spaces and applications, Pure Appl. Funct. Anal. 3 (2018) 349–369.

[28] A. J. Zaslavski, Approximate solutions of common fixed point problems, Springer Optimization and Its Applications, Springer, Cham, 2016.

[29] A. J. Zaslavski, Algorithms for solving common fixed point problems, Springer Optimization and Its Applications, Springer, Cham, 2018.

[30] A. J. Zaslavski, Three convergence results for inexact orbits of nonexpansive mappings, J. Appl. Numer. Optim. 1 (2019) 157–165.