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# CONVERGENCE RESULTS FOR SET-VALUED STRICT CONTRACTIONS

ALEXANDER J. ZASLAVSKI

Department of Mathematics, The Technion – Israel Institute of Technology, Haifa, Israel

Dedicated to Professor Elijah Lucien Polak

**Abstract.** In the present paper, we study an iterative process induced by a set-valued strict contraction in a complete metric space and show its convergence under the presence of computational errors.

**Keywords.** Complete metric space; Convergence analysis; Fixed point; Set-valued mapping; Strict contraction.

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## 1. Introduction and preliminaries

For more than sixty years, the fixed point theory of nonlinear operators is an important area of nonlinear analysis. One of its main topics is the study of the existence of fixed points of nonexpansive and contractive maps. See [1]-[24] and the references mentioned therein. Many existence results are collected in [8, 9, 19]. This topic is well developed for single-valued mappings [1, 3, 4, 6, 11, 12, 16] as well as for set-valued mappings [5, 14, 15, 18, 19, 20]. In the present paper we study an iterative process induced by a set-valued strict contraction in a complete metric space and show its convergence under the presence of computational errors.

Let $(X,\rho)$ be a complete metric space. For each $x \in X$ and each $r > 0$, set

$$B(x,r) = \{y \in X :\ \rho(x,y) \le r\},\ B^0(x,r) = \{y \in X :\ \rho(x,y) < r\}.$$

For each $x \in X$ and each $A \subset X$ set

$$\rho(x,A) = \inf\{\rho(x,y) :\ y \in A\}.$$

Note that we assume that the infimum over empty set is $\infty$. For each pair of nonempty sets $A,B \subset X$, set

$$H(A,B) = \max\{\sup\{\rho(x,B) :\ x \in A\},\ \sup\{\rho(y,A) :\ y \in B\}\}.$$

*Corresponding author.

E-mail address: ajzasl@technion.ac.il

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Assume that $T : X \to 2^X \setminus \{\emptyset\}$, $T(x)$ is closed for each $x \in X$, $c \in [0,1)$ and that for each $x, y \in X$,

$$H(T(x), T(y)) \leq c\rho(x,y).$$

We are interested to solve the problem

$$\text{Find } z \in X \text{ such that } z \in T(z).$$

By Nadler's classical theorem [13] the solution of this problem exists. In practice we can obtain only an approximate solution of this problem $z \in X$ such that $\rho(z, T(z))$ is small.

More precisely, in order to meet this goal we use the following algorithm

Initialization: select an arbitrary point $x_0 \in X$ and a small positive constant $\delta$.

Iterative step: given a current iteration point $x_n$ calculate the next iteration point $x_{n+1}$ such that

$$B(x_{n+1}, \delta) \cap \{\xi \in T(x_n) :$$
$$\rho(\xi, x_n) \leq \delta + \rho(x_n, T(x_n))\} \neq \emptyset.$$

Clearly, at the each iterative step $x_{n+1}$ is an approximate solution of the problem

$$\rho(\xi, x_n) \to \min, \ \xi \in T(x_n).$$

Since the space $X$ is not compact a solution of the problem above does not exist in general. Note that given current iteration point $x_n$ in order to obtain $x_{t+1}$ we should only know the set $T(x_t)$ or even its good approximation.

The following results were proved in [5] where iterative schemes for approximating fixed points of closed-valued strict contractions in metric spaces were considered.

**Theorem 1.1.** *Assume that $x_0 \in X$, $\{\varepsilon_i\}_{i=0}^{\infty} \subset (0,\infty)$, $\sum_{i=0}^{\infty} \varepsilon_i < \infty$, and that for each integer $i \geq 0$,*

$$x_{i+1} \in T(x_i), \ \rho(x_i, x_{i+1}) \leq \rho(x_i, T(x_i)) + \varepsilon_i.$$

*Then $\{x_i\}_{i=0}^{\infty}$ converges to a fixed point of $T$.*

**Theorem 1.2.** *Let $\varepsilon > 0$ be given and let $\delta \in (0, 4^{-1}(1-c)\varepsilon]$. Then if $x \in X$ and $\rho(x, T(x)) < \delta$, then there is $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$ and $\rho(x, \bar{x}) \leq \varepsilon$.*

**Theorem 1.3.** *Fix $\theta \in X$. Let $\varepsilon \in (0,1]$ and $M > 0$ be given, a positive number $\delta$ satisfy $\delta < 2^{-1}\varepsilon(1-c]$ and an integer $n_0 \geq 1$ satisfy $c^{n_0}(2M + 1 + \rho(\theta, T(\theta)) < \varepsilon/2$. Then for each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $\rho(x_0, \theta) \leq M$ and such that for each integer $n \geq 0$,*

$$x_{n+1} \in T(x_n) \text{ and } \rho(x_{n+1}, x_n) \leq \delta + \rho(x_n, T(x_n)),$$

*we have $\rho(x_{n+1}, x_n) < \varepsilon$ for all integers $n \geq n_0$.*

Theorems 1.2 and 1.3 imply the following additional result.

**Theorem 1.4.** *Let $\varepsilon \in (0,1)$ and $M$ be given, a positive number $\delta$ satisfy $\delta \leq 8^{-1}(1-c)\varepsilon^2$ and a natural number $n$ satisfy $c^{n_0}(2M + 1 + \rho(\theta, T(\theta)) < 8^{-1}(1-c)\varepsilon$. Then for each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies*

$$\rho(x_0, \theta) \leq M, \ x_{n+1} \in T(x_n) \text{ and } \rho(x_n, x_{n+1}) \leq \rho(x_n, T(x_n)) + \delta$$

*for all integers $n \geq 0$, for each integer $n \geq n_0$, there is a point $y \in X$ such that $y \in T(y)$ and $\rho(y, x_n) < \varepsilon$.*

The following example shows that Theorem 1.4 cannot be improved in the sense that the fixed point $y$, the existence of which is guaranteed by the theorem, is not, in general, the same for all integers $n \geq n_0$.

**Example 1.5.** Let $X = [0,1]$, $\rho(x,y) = |x-y|$ and $T(x) = [0,1]$ for all $x \in [0,1]$. Let $\delta > 0$ be given. Choose a natural number $k$ such that $1/k < \delta$. Put

$$x_0 = 0,\ x_i = i/k,\ i = 1,\dots,k,$$

$$x_{i+k} = 1 - i/k,\ i = 0,\dots,k,$$

and for all integers $p \geq 0$ and any $i \in \{0,\dots,2k\}$, put $x_{2pk+i} = x_i$. Then $\{x_i\}_{i=0}^{\infty} \subset X$ and for any integer $i \geq 0$, we have

$$x_{i+1} \in T(x_i) \text{ and } |x_i - x_{i+1}| \leq k^{-1} < \delta.$$

On the other hand, for all $x \in X$ and any integer $p \geq 0$,

$$\max\{|x - x_i| :\ i = 2kp,\dots,2pk+2k\} \geq 1/2.$$

Note that in the results stated above at any iterative step $t$ we have $x_{t+1} \in T(x_t)$. In the present paper we generalize them in the case when $x_{t+1}$ is just sufficiently close to $T(x_t)$. These generalizations are interesting and important since computational errors always occur in practice.

## 2. Inexact iterates

Let $(X,\rho)$ be a complete metric space, $T : X \to 2^X \setminus \{\emptyset\}$, $T(x)$ be closed for each $x \in X$, $c \in [0,1)$ and that for each $x,y \in X$,

$$H(T(x),T(y)) \leq c\rho(x,y). \tag{1}$$

**Theorem 2.1.** *Let $\theta \in X$ and let $\varepsilon \in (0,1]$ and $M > 0$ be given, a positive number $\delta$ satisfy*

$$\delta < 9^{-1}\varepsilon(1-c) \tag{2}$$

*and a natural number $n_0$ satisfy*

$$c^{n_0}(2M + 1 + \rho(\theta,T(\theta)) < \varepsilon/2. \tag{3}$$

*Then for each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $\rho(x_0,\theta) \leq M$ and such that for each integer $n \geq 0$,*

$$B(x_{n+1},\delta) \cap \{\xi \in T(x_n) :\ \rho(\xi,x_n) \leq \delta + \rho(x_n,T(x_n))\} \tag{4}$$

*we have, for each integer $n \geq n_0$,*

$$\rho(x_{n+1},x_n) < \varepsilon \text{ and } B^0(x_n,\varepsilon) \cap T(x_n) \neq \emptyset.$$

*Proof.* Let

$$\{x_n\}_{n=0}^{\infty} \subset X,\ \rho(x_0,\theta) \leq M, \tag{5}$$

and assume that (4) holds for each integer $n \geq 0$. We now estimate $\rho(x_0,T(x_0))$. In view of (1) and (5),

$$\begin{aligned}\rho(x_0,T(x_0)) &\leq \rho(x_0,\theta) + \rho(\theta,T(\theta)) + H(T(\theta),T(x_0)) \\ &\leq \rho(x_0,\theta) + \rho(\theta,T(\theta)) + \rho(\theta,x_0) \leq 2M + \rho(\theta,T(\theta)).\end{aligned} \tag{6}$$

In view of (2), (4) and (6),

$$\rho(x_0,x_1) \leq \rho(x_0,T(x_0)) + 2\delta \leq 2M + 1 + \rho(\theta,T(\theta)). \tag{7}$$

By (4), for each integer $n \geq 0$, there exists $\xi_n \in B(x_{n+1}, \delta)$ such that

$$\rho(\xi, x_n) \leq \delta + \rho(x_n, T(x_n)).$$

By (1) and the relations above, we have, for each integer $n \geq 0$,

$$\begin{aligned} \rho(x_{n+2}, x_{n+1}) &\leq \rho(x_{n+1}, \xi_{n+1}) + \rho(\xi_{n+1}, x_{n+2}) \\ &\leq \rho(x_{n+1}, \xi_{n+1}) + \delta \\ &\leq \rho(x_{n+1}, T(x_{n+1})) + 2\delta \\ &\leq \rho(x_{n+1}, \xi_n) + \rho(\xi_n, T(x_{n+1})) + 2\delta \\ &\leq \rho(\xi_n, T(x_{n+1})) + 3\delta \\ &\leq 3\delta + H(T(x_n), T(x_{n+1})) \leq 3\delta + c\rho(x_n, x_{n+1}). \end{aligned} \tag{8}$$

Next, we show by induction that for each integer $n \geq 1$,

$$\rho(x_{n+1}, x_n) \leq 3\delta \sum_{i=0}^{n-1} c^i + c^n \rho(x_0, x_1). \tag{9}$$

By (8), inequality (9) holds for $n = 1$. Assume that $k \geq 1$ is an integer and that (9) holds with $n = k$. Then by (8),

$$\rho(x_{k+2}, x_{k+1}) \leq 3\delta + c\rho(x_k, x_{k+1}) + \leq 3\delta \sum_{i=0}^{k} c^i + c^{k+1} \rho(x_0, x_1).$$

Thus (9) holds with $n = k+1$ and therefore it holds for all integers $n \geq 1$. By (7) and (9), for all natural numbers $n$,

$$\rho(x_{n+1}, x_n) \leq 3\delta(1-c)^{-1} + c^n(2M + 1 + \rho(\theta, T(\theta))). \tag{10}$$

Assume that $n \geq n_0$ is an integer. It follows from (2), (3) and (10) that

$$\rho(x_n, x_{n+1}) \leq 3\delta(1-c)^{-1} + c^{n_0}(2M + 1 + \rho(\theta, T(\theta))) < \varepsilon/3 + \varepsilon/2$$

and in view of (2) and (4),

$$\rho(x_n, T(x_n)) \leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, T(x_n))$$

$$< \varepsilon/3 + \varepsilon/2 + \delta < \varepsilon/3 + \varepsilon/2 + \varepsilon/9 < \varepsilon.$$

Theorem 2.1 is proved. □

Theorems 1.2 and Theorem 2.1 imply the following result.

**Theorem 2.2.** *Fix $\theta \in X$. Let $\varepsilon \in (0,1)$ and $M > 0$ be given, a positive number $\delta$ satisfy $\delta < 36^{-1}\varepsilon(1-c)^2$ and a natural number $n_0$ satisfy $c^{n_0}(2M + 1 + \rho(\theta, T(\theta))) < 8^{-1}(1-c)\varepsilon$. Then, for each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $\rho(x_0, \theta) \leq M$ and such that for each integer $n \geq 0$,*

$$B(x_{n+1}, \delta) \cap \{\xi \in T(x_n) : \rho(\xi, x_n) \leq \delta + \rho(x_n, T(x_n))\} \neq \emptyset$$

*we have that, for each integer $n \geq n_0$, there exist $y \in B(x_n, \varepsilon) \cap T(y)$.*

**Theorem 2.3.** *Assume that* $x_0 \in X$,

$$\{\varepsilon_i\}_{i=0}^{\infty} \subset (0,\infty),\ \sum_{i=0}^{\infty} \varepsilon_i < \infty,\ \{r_i\}_{i=0}^{\infty} \subset (0,\infty),\ \sum_{i=0}^{\infty} r_i < \infty \tag{11}$$

*and that for each integer* $i \geq 0$, $x_i \in X$,

$$B(x_{i+1}, r_i) \cap \{\xi \in T(x_i) :\ \rho(\xi, x_i) \leq \rho(x_i, T(x_i)) + \varepsilon_i\} \neq \emptyset. \tag{12}$$

*Then* $\{x_i\}_{i=0}^{\infty}$ *converges to a fixed point of* $T$.

*Proof.* We show that $\{x_i\}_{i=0}^{\infty}$ is a Cauchy sequence. By (12), for each integer $i \geq 0$, there exists

$$\xi_i \in B(x_{i+1}, r_i) \cap T(x_i) \tag{13}$$

such that

$$\rho(\xi_i, x_i) \leq \rho(x_i, T(x_i)) + \varepsilon_i. \tag{14}$$

Let $i \geq 0$ be an integer. By (1), (13), and (14), we have

$$\begin{aligned} \rho(x_{i+1}, x_{i+2}) &\leq \rho(x_{i+1}, \xi_{i+1}) + \rho(\xi_{i+1}, x_{i+2}) \\ &\leq \rho(x_{i+1}, \xi_{i+1}) + r_{i+1} \\ &\leq \rho(x_{i+1}, T(x_{i+1})) + \varepsilon_{i+1} + r_{i+1} \\ &\leq \rho(x_{i+1}, \xi_i) + \rho(\xi_i, T(x_{i+1})) + \varepsilon_{i+1} + r_{i+1} \\ &\leq H(T(x_i), T(x_{i+1})) + r_i + r_{i+1} + \varepsilon_{i+1} \\ &\leq c\rho(x_i, x_{i+1}) + r_i + r_{i+1} + \varepsilon_{i+1}. \end{aligned} \tag{15}$$

By (15), we have $\rho(x_1, x_2) \leq c\rho(x_0, x_1) + r_0 + r_1 + \varepsilon_1$ and

$$\begin{aligned} \rho(x_2, x_3) &\leq c\rho(x_1, x_2) + r_1 + r_2 + \varepsilon_2 \\ &\leq c^2\rho(x_0, x_1) + cr_0 + cr_1 + c\varepsilon_1 + r_1 + r_2 + \varepsilon_2. \end{aligned} \tag{16}$$

Now we use induction to show that for each integer $n \geq 1$,

$$\rho(x_n, x_{n+1}) \leq c^n \rho(x_0, x_1) + \sum_{i=0}^{n-1} c^i \varepsilon_{n-i} + \sum_{i=0}^{n-1} c^i r_{n-i} + + \sum_{i=0}^{n-1} c^i r_{n-i-1}. \tag{17}$$

In view of (16), we see that inequality (17) is valid for $n = 1, 2$. Assume that $k \geq 1$ is an integer and that (17) holds for $n = k$. This combined with (15) implies that

$$\begin{aligned} &\rho(x_{k+1}, x_{k+2}) \\ &\leq c\rho(x_k, x_{k+1}) + r_k + r_{k+1} + \varepsilon_{k+1} \\ &\leq c^{k+1}\rho(x_0, x_1) + \sum_{i=0}^{k-1} c^{i+1}\varepsilon_{k-i} + \sum_{i=0}^{k-1} c^{i+1} r_{k-i} + \sum_{i=0}^{k-1} c^{i+1} r_{k-i-1} + \varepsilon_{k+1} + r_k + r_{k+1} \\ &= c^{k+1}\rho(x_0, x_1) + \sum_{i=0}^{k} c^i \varepsilon_{k+1-i} + \sum_{i=0}^{k} c^i r_{k+1-i} + \sum_{i=0}^{k} c^i r_{k-i}. \end{aligned}$$

Thus (17) holds with $n = k+1$ and therefore (17) holds for all integers $n \geq 1$. By (17),

$$\sum_{n=1}^{\infty} \rho(x_n, x_{n+1}) \leq \sum_{n=1}^{\infty} (c^n \rho(x_0, x_1) + \sum_{i=0}^{n} c^{n-i}(c_i + \varepsilon_i) + \sum_{i=0}^{n-1} c^{n-i-i} r_i)$$

$$\leq \rho(x_0,x_1)\sum_{n=1}^{\infty} c^n + 2\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} c^j)(\varepsilon_i + r_i) < \infty.$$

Thus $\{x_n\}_{n=0}^{\infty}$ is indeed a Cauchy sequence and there exists

$$x_* = \lim_{n\to\infty} x_n. \tag{18}$$

We claim that $x_* \in T(x_*)$. Let $\varepsilon > 0$ be given. By (18), there is an integer $n_0 \geq 1$ such that for each integer $n \geq n_0$,

$$r_n < \varepsilon/8, \tag{19}$$

$$\rho(x_n, x_*) \leq \varepsilon/8. \tag{20}$$

Let $n \geq n_0$ be an integer. By (1) and (20),

$$H(T(x_n), T(x_*)) \leq c\rho(x_n, x_*) \leq c\varepsilon/8. \tag{21}$$

By (13), we have

$$\xi_n \in T(x_n). \tag{22}$$

By (13), (19), (21), and (22), we obtain

$$\begin{aligned}\rho(x_{n+1}, T(x_*)) &\leq \rho(x_{n+1}, \xi_n) + \rho(\xi_n, T(x_*))\\ &\leq r_n + H(T(x_n, T(x_*))\\ &\leq r_n + c\varepsilon/8 < \varepsilon/8 + c\varepsilon/8.\end{aligned}$$

This implies that there exists $y \in T(x_*)$ such that $\rho(x_{n+1}, y) \leq \varepsilon/4$. This together with (14) implies that

$$\rho(x_*, T(x_*)) \leq \rho(x_*, y) \leq \rho(x_*, x_{n+1}) + \rho(x_{n+1}, y) \leq \varepsilon/8 + \varepsilon/4.$$

Since $\varepsilon$ is an arbitrary positive number, we conclude that $x_* \in T(x_*)$, as claimed. Theorem 2.3 is proved. □

## 3. AN EXTENSION OF THEOREM 2.3

Let $(X,\rho)$ be a complete metric space, $T : X \to 2^X \setminus \{\emptyset\}$, $T(x)$ be closed for each $x \in X$, $c \in [0,1)$, $X_0 \subset X$ is a nonempty closed set, $T(X_0) \subset X_0$ and let for each $x \in X_0$ and each $y \in X$,

$$H(T(x), T(y)) \leq c\rho(x,y). \tag{23}$$

Theorem 1.1 implies that there exists $x_* \in X_0$ such that

$$x_* \in T(x_*). \tag{24}$$

**Theorem 3.1.** *Assume that* $\{x_i\}_{i=0}^{\infty} \subset X$, $\{r_i\}_{i=0}^{\infty} \subset (0,\infty)$,

$$\sum_{i=0}^{\infty} r_i < \infty \tag{25}$$

*and that for each integer* $i \geq 0$,

$$B(x_{i+1}, r_i) \cap \{\xi \in T(x_i) :\ \rho(\xi, x_i) \leq r_i + \rho(x_i, T(x_i))\} \neq \emptyset. \tag{26}$$

*Then* $\{x_i\}_{i=0}^{\infty}$ *converges to a fixed point* $x_*$ *of* $T$ *such that* $x_* \in T(x_*)$.

*Proof.* By (26), for each integer $i \geq 0$, there exists

$$\xi_i \in B(x_{i+1}, r_i) \cap T(x_i) \tag{27}$$

such that

$$\rho(x_i, \xi_i) \leq \rho(x_i, T(x_i)) + r_i. \tag{28}$$

For each integer $i \geq 0$, there exists

$$u_i \in X_0 \tag{29}$$

such that

$$\rho(u_i, x_i) \leq \rho(x_i, X_0) + r_i. \tag{30}$$

Let $i \geq 0$ be an integer. By (23), (27), (30) and the inclusion

$$T(X_0) \subset X_0, \tag{31}$$

we obtain

$$\begin{aligned} \rho(x_{i+1}, u_{i+1}) &\leq \rho(x_{i+1}, X_0) + r_{i+1} \leq \rho(x_{i+1}, \xi_i) + \rho(\xi_i, X_0) + r_{i+1} \\ &\leq \rho(\xi_i, X_0) + r_{i+1} + r_i \leq \rho(\xi_i, T(u_i)) + r_{i+1} + r_i \\ &\leq H(T(x_i), T(u_i)) + r_{i+1} + r_i \\ &\leq c\rho(x_i, u_i) + r_{i+1} + r_i. \end{aligned} \tag{32}$$

By (23), (27), and (29), we have

$$\xi \in T(x_i),\ \rho(T(u_i), T(x_i)) \leq c\rho(x_i, u_i). \tag{33}$$

In view of (33), there exists

$$v_i \in T(u_i) \tag{34}$$

such that

$$\rho(v_i, \xi_i) \leq c\rho(x_i, u_i) + r_i. \tag{35}$$

It follows from (27), (32), and (35) that

$$\begin{aligned} \rho(v_i, u_{i+1}) &\leq \rho(v_i, \xi_i) + \rho(\xi_i, u_{i+1}) \\ &\leq c\rho(x_i, u_i) + r_i + \rho(\xi_i, x_{i+1}) + \rho(x_{i+1}, u_{i+1}) \\ &\leq c\rho(x_i, u_i) + r_i + r_i + c\rho(x_i, u_i) + r_{i+1} + r_i \\ &= 2c\rho(x_i, u_i) + r_{i+1} + 3r_i. \end{aligned} \tag{36}$$

Equations (28) and (35) imply that

$$\begin{aligned} \rho(u_i, v_i) &\leq \rho(u_i, x_i) + \rho(x_i, \xi_i) + \rho(\xi_i, v_i) \\ &\leq \rho(x_i, u_i) + \rho(x_i, T(x_i)) + r_i + c\rho(x_i, u_i) + r_i. \end{aligned} \tag{37}$$

By (23), (26), and (29), we have

$$\begin{aligned} \rho(x_i, T(x_i)) &\leq \rho(x_i, u_i) + \rho(u_i, T(u_i)) + H(T(u_i), T(x_i)) \\ &\leq \rho(u_i, T(u_i)) + (1+c)\rho(x_i, u_i). \end{aligned} \tag{38}$$

In view of (37) and (38), we obtain

$$\begin{aligned} \rho(u_i, v_i) &\leq (1+c)\rho(u_i, x_i) + 2r_i + \rho(x_i, T(x_i)) \\ &\leq (1+c)\rho(u_i, x_i) + 2r_i + \rho(u_i, T(u_i)) + (1+c)\rho(x_i, u_i) \\ &\leq 2(1+c)\rho(u_i, x_i) + 2r_i + \rho(u_i, T(u_i)). \end{aligned}$$

By the relation above, (34) and (36),

$$v_i \in T(u_i),\ \rho(v_i,u_{i+1}) \le 2c\rho(x_i,u_i) + r_{i+1} + 3r_i,$$
$$\rho(u_i,v_i) \le 2(1+c)\rho(u_i,x_i) + 2r_i + \rho(u_i,T(u_i)). \tag{39}$$

We show that $\sum_{i=0}^{\infty} \rho(u_i,v_i) < \infty$. For each integer $i \ge 0$, set

$$\Delta_i = r_{i+1} + r_i. \tag{40}$$

By (25) and (40), we have $\sum_{i=0}^{\infty} \Delta_i < \infty$. In view of (32) and (40), we obtain , for each integer $i \ge 0$,

$$\rho(x_{i+1},u_{i+1}) \le c\rho(x_i,u_i) + \Delta_i. \tag{41}$$

By (41), we obtain $\rho(x_1,u_1) \le c\rho(x_0,u_0) + \Delta_0$ and

$$\rho(x_2,u_2) \le c\rho(x_1,u_1) + \Delta_1 \le c^2\rho(x_0,u_0) + c\Delta_0 + \Delta_1. \tag{42}$$

By induction we show that for each integer $n \ge 1$,

$$\rho(x_n,u_n) \le c^n\rho(x_0,u_0) + \sum_{i=0}^{n-1} c^i \Delta_{n-i-1}. \tag{43}$$

In view of (42), inequality (43) is valid for $n = 1,2$. Assume that $k \ge 1$ is an integer and that (43) holds for $n = k$. This combined with (41) implies that

$$\begin{aligned}\rho(x_{k+1},u_{k+1}) &\le c\rho(x_k,u_k) + \Delta_k\\ &\le c^{k+1}\rho(x_0,u_0) + \sum_{i=0}^{k-1} c^{i+1}\Delta_{k-i-1} + \Delta_k\\ &= c^{k+1}\rho(x_0,u_0) + \sum_{i=0}^{k} c^i\Delta_{k-i}.\end{aligned}$$

Thus (43) holds with $n = k+1$ and therefore (43) holds for all integers $n \ge 1$. By (43), we have

$$\begin{aligned}\sum_{n=1}^{\infty} \rho(x_n,u_n) &\le \sum_{n=1}^{\infty} (c^n\rho(x_0,u_0) + \sum_{i=0}^{n-1} c^i\Delta_{n-1-i})\\ &\le \rho(x_0,u_0)\sum_{n=1}^{\infty} c^n + \sum_{j=0}^{\infty} c^j \sum_{j=0}^{\infty} \Delta_j < \infty.\end{aligned} \tag{44}$$

For each integer $i \ge 0$ set

$$\gamma_i = 4\rho(x_i,u_i) + 3r_i + r_{i+1}. \tag{45}$$

In view of (25), (44), and (45), we obtain $\sum_{i=0}^{\infty} \gamma_i < \infty$. By (29) and (39), we have $\{u_n\}_{n=0}^{\infty} \subset X_0$ and for each integer $i \ge 0$,

$$v_i \in B(u_{i+1},\gamma_i) \cap T(u_i),$$
$$\rho(u_i,v_i) \le \rho(u_i,T(u_i)) + \gamma_i.$$

Applying Theorem 2.3 for the space $X_0$ and the restriction of $T$ to $X_0$ with $\varepsilon_i = r_i = \gamma_i$, $i = 0,1,\ldots$ we obtain that $\{u_i\}_{i=0}^{\infty}$ converges to a point $x_* \in X_0$ such that $x_* \in T(x_*)$. By (44), we have $x_* = \lim_{n\to\infty} x_n$. Theorem 3.1 is proved.

□

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