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ON EXISTENCE OF AFFINE PARAMETER-DEPENDENT LYAPUNOV FUNCTIONS

TANER BÜYÜKKÖROĞLU*, BİRGÜL AKSOY, VAKIF DZHAFAROV

Department of Mathematics, Faculty of Science, Eskişehir Technical University, Eskişehir, Turkey

Abstract. In this paper, we consider the problem of existence of an affine parameter-dependent quadratic Lyapunov functions for a matrix segment. It is assumed that this segment is robustly Hurwitz stable. A sufficient condition for the existence of such a function is given. Concerning Barmish's conjecture, we give an explicit algorithm for the calculation of the matrices which play a key role in the counterexample. One conjecture on 2×2 matrix segments is provided.

Keywords. Linear matrix inequality; Matrix segment; Stability of linear systems.

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1. Introduction

Let A be an $n\times n$ real matrix ($A\in\mathbb{R}^{n\times n}$). A is called (Hurwitz) stable if all its eigenvalues lie in the open left-half of the complex plane. Given $A_1, A_2\in\mathbb{R}^{n\times n}$, the set of all convex combinations of the matrices A_1 and A_2

$$\{A(\alpha)=(1-\alpha)A_1+\alpha A_2:\ \alpha\in[0,1]\}=\{A_1+\alpha(A_2-A_1):\ \alpha\in[0,1]\}$$

is called a matrix segment and denote it by $[A_1,A_2]$.

By the Lyapunov theorem, the matrix segment $[A_1,A_2]$ is robustly stable (that is $A(\alpha)$ is stable for all $\alpha\in[0,1]$) if and only if there exist $n\times n$ positive definite matrices $P(\alpha)$ such that

$$A^T(\alpha)P(\alpha)+P(\alpha)A(\alpha)<0 \tag{1.1}$$

for all $\alpha\in[0,1]$ [1, p. 136]. The matrices $P(\alpha)$ are called the Lyapunov matrices.

Note that by the Lyapunov theorem if $[A_1,A_2]$ is robustly stable then the positive definiteness of $P(\alpha)$ can be replaced by symmetricity. In this paper we consider the problem of existence of an affine parameter-dependent Lyapunov matrices (APDLM) $P(\alpha)=(1-\alpha)P_1+\alpha P_2$ ($\alpha\in[0,1]$) for the segment $[A_1,A_2]$. A special case of APDLM is common Lyapunov matrices (see [2, 3] and references therein). Denote the set of real symmetric matrices with the symbol $\mathscr{S}$.

E-mail address: tbuyukkoroglu@eskisehir.edu.tr (T. Büyükköroğlu), b_aksoy@anadolu.edu.tr (B. Aksoy), vcaferov@eskisehir.edu.tr (V. Dzhafarov).

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In Section 2 we give a sufficient condition for the existence of $P(\alpha) = (1-\alpha)P_1 + \alpha P_2$. Section 3 is concerned to the Barmish's conjecture ([4, p. 346]) which conjectured that if $[A_1, A_2]$ is robustly stable then there exists $P(\alpha) = (1-\alpha)P_1 + \alpha P_2$ that satisfies (1.1). In [5] it is shown that this conjecture is false. In the section, using [5] we give an explicit algorithm for the existence problem of $P(\alpha) = (1-\alpha)P_1 + \alpha P_2$.

2. Existence Theorem

In this section we give a sufficient condition for the existence of one-parameter affine parameter-dependent quadratic Lyapunov functions for a matrix segment.

Proposition 2.1. *For given A_1, $A_2 \in \mathbb{R}^{n\times n}$ assume that $[A_1, A_2]$ is robustly stable. If there exist $P_1, P_2 \in \mathscr{S}$ such that*

$$\begin{gathered} A_1^T P_1 + P_1 A_1 < 0, \\ A_2^T P_2 + P_2 A_2 < 0, \\ (A_2^T - A_1^T)P_1 + P_1(A_2 - A_1) + A_1^T(P_2 - P_1) + (P_2 - P_1)A_1 = 0 \end{gathered} \tag{2.1}$$

are satisfied, then

$$A^T(\alpha)P(\alpha) + P(\alpha)A(\alpha) < 0$$

for all $\alpha \in [0,1]$, where $A(\alpha) = A_1 + \alpha(A_2 - A_1)$ and $P(\alpha) = P_1 + \alpha(P_2 - P_1)$.

Proof. By the Lyapunov theorem $P_1 > 0$, $P_2 > 0$. We have

$$A_2^T P_1 + P_1 A_2 + A_1^T P_2 + P_2 A_1 = 2(A_1^T P_1 + P_1 A_1),$$

$$\begin{aligned} C(\alpha) &= A^T(\alpha)P(\alpha) + P(\alpha)A(\alpha) \\ &= \alpha^2[(A_2 - A_1)^T(P_2 - P_1) + (P_2 - P_1)(A_2 - A_1)] + A_1^T P_1 + P_1 A_1 \\ &= \alpha^2[A_2^T P_2 + P_2 A_2 + A_1^T P_1 + P_1 A_1 - (A_2^T P_1 + P_1 A_2 + A_1^T P_2 + P_2 A_1)] \\ &\quad + A_1^T P_1 + P_1 A_1 \\ &= \alpha^2[(A_2^T P_2 + P_2 A_2) - (A_1^T P_1 + P_1 A_1)] + A_1^T P_1 + P_1 A_1. \end{aligned}$$

Define

$$\begin{aligned} Q_1 &:= -(A_1^T P_1 + P_1 A_1) > 0, \\ Q_2 &:= -(A_2^T P_2 + P_2 A_2) > 0. \end{aligned}$$

Therefore

$$C(\alpha) = \alpha^2(Q_1 - Q_2) - Q_1 = -[(1-\alpha^2)Q_1 + \alpha^2 Q_2].$$

Since $Q_1 > 0$, $Q_2 > 0$ and $0 \le \alpha^2 \le 1$ for $\alpha \in [0,1]$, then $(1-\alpha^2)Q_1 + \alpha^2 Q_2 > 0$. Hence $C(\alpha) < 0$ for all $\alpha \in [0,1]$. □

Note that robust stability of $[A_1, A_2]$ can be easily checked by known criteria.

System (2.1) is not a system of Linear Matrix Inequalities (LMI) due to third equality. Therefore the known LMI solution methods can not be applied. To overcome this, (2.1) can be approximated by the following LMI's

$$\begin{gathered} A_1^T P_1 + P_1 A_1 < 0, \quad A_2^T P_2 + P_2 A_2 < 0, \\ -\varepsilon I < (A_2^T - A_1^T)P_1 + P_1(A_2 - A_1) + A_1^T(P_2 - P_1) + (P_2 - P_1)A_1 < \varepsilon I, \end{gathered} \tag{2.2}$$

where $\varepsilon > 0$ is sufficiently small and I is the identity matrix. If $P_1, P_2 \in \mathscr{S}$ are solutions of (2.2) the verification of (1.1) is an easy problem.

Example 2.2. Consider robustly stable $[A_1, A_2]$, where

$$A_1 = \begin{bmatrix} -2 & -2 \\ 1 & -1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}.$$

The matrices

$$P_1 = \begin{bmatrix} 0.289 & 0.365 \\ 0.365 & 0.685 \end{bmatrix} > 0, \quad P_2 = \begin{bmatrix} 0.508 & 0.668 \\ 0.668 & 1.223 \end{bmatrix} > 0.$$

are the solutions of LMI (2.2) with $\varepsilon = 0.009$.

For all $\alpha \in [0,1]$,

$$\begin{aligned} C(\alpha) &= A^T(\alpha)P(\alpha) + P(\alpha)A(\alpha) \\ &= \begin{bmatrix} -0.426 & -0.988 \\ -0.988 & -2.830 \end{bmatrix} + \begin{bmatrix} 0.004 & -0.004 \\ -0.004 & -0.008 \end{bmatrix}\alpha + \begin{bmatrix} 0.102 & 0.625 \\ 0.625 & 1.954 \end{bmatrix}\alpha^2. \end{aligned}$$

The leading principal minors $\Delta_1(\alpha)$ and $\Delta_2(\alpha)$ of $C(\alpha)$ are polynomials in α and $\Delta_1(\alpha) < 0$, $\Delta_2(\alpha) > 0$ for all $\alpha \in [0,1]$. Therefore $C(\alpha) < 0$ for all $\alpha \in [0,1]$ and

$$P(\alpha) = (1-\alpha)\begin{bmatrix} 0.289 & 0.365 \\ 0.365 & 0.685 \end{bmatrix} + \alpha \begin{bmatrix} 0.508 & 0.668 \\ 0.668 & 1.223 \end{bmatrix}$$

is APDLM for the segment $[A_1, A_2]$.

Example 2.3. Consider robustly stable $[A_1, A_2]$, where

$$A_1 = \begin{bmatrix} -3 & 3 & 1 \\ -5 & 1 & -12 \\ -2 & 7 & -4 \end{bmatrix}, \; A_2 = \begin{bmatrix} -1 & 3 & -1 \\ -1 & -2 & -2 \\ 5 & 3 & -2 \end{bmatrix}.$$

The matrices

$$P_1 = \begin{bmatrix} 1.733 & 0.009 & -0.152 \\ 0.009 & 0.583 & -0.197 \\ -0.152 & -0.197 & 0.770 \end{bmatrix} > 0, \quad P_2 = \begin{bmatrix} 2.274 & 0.282 & -0.427 \\ 0.282 & 0.722 & -0.197 \\ -0.427 & -0.197 & 0.669 \end{bmatrix} > 0.$$

are the solutions of LMI (2.2) with $\varepsilon = 0.02$.

$$\begin{aligned} C(\alpha) &= A^T(\alpha)P(\alpha) + P(\alpha)A(\alpha) \\ &= \begin{bmatrix} -9.880 & 1.596 & 2.134 \\ 1.596 & -1.538 & -1.462 \\ 2.134 & -1.462 & -1.736 \end{bmatrix} + \begin{bmatrix} 0 & 0.009 & 0.01 \\ 0.009 & -0.006 & 0.002 \\ 0.01 & 0.002 & 0.006 \end{bmatrix}\alpha \\ &\quad + \begin{bmatrix} 0.498 & 1.383 & -0.159 \\ 1.383 & -0.834 & 1.248 \\ -0.159 & 1.248 & 0.696 \end{bmatrix}\alpha^2. \end{aligned}$$

Leading principal minors $\Delta_1(\alpha) < 0$, $\Delta_2(\alpha) > 0$, $\Delta_3(\alpha) < 0$ for all $\alpha \in [0,1]$. Therefore $C(\alpha) < 0$ for all $\alpha \in [0,1]$ and

$$P(\alpha) = (1-\alpha)\begin{bmatrix} 1.733 & 0.009 & -0.152 \\ 0.009 & 0.583 & -0.197 \\ -0.152 & -0.197 & 0.770 \end{bmatrix} + \alpha \begin{bmatrix} 2.274 & 0.282 & -0.427 \\ 0.282 & 0.722 & -0.197 \\ -0.427 & -0.197 & 0.669 \end{bmatrix}$$

is APDLM.

3. Affine Lyapunov Functions and Barmish's Conjecture

R. Barmish suggest the following conjecture.

Conjecture 3.1 ([4]). Consider stable matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$. Suppose $[A_1, A_2]$ is robustly stable. Then there exist $P_1, P_2 \in \mathscr{S}$ such that $P(\alpha) = (1-\alpha)P_1 + \alpha P_2$ satisfies

$$A^T(\alpha)P(\alpha) + P(\alpha)A(\alpha) < 0$$

for all $\alpha \in [0,1]$.

An example that this conjecture is not valid for $n \geq 3$ is given in [5]. This example is based on the following propositions.

Proposition 3.2 ([5]). *For given matrices $B_1, B_2 \in \mathbb{R}^{n\times n}$, there exist $Q_1, Q_2 \in \mathscr{S}$ that satisfy*

$$B^T(\theta)Q(\theta) + Q(\theta)B(\theta) < 0 \tag{3.1}$$

for all $\theta \in [-1,1]$ with $B(\theta) = B_1 + \theta B_2$ and $Q(\theta) = Q_1 + \theta Q_2$ if and only if there exist $Q_1, Q_2 \in \mathscr{S}$, $D > 0$ and $G = -G^T$ such that

$$\mathrm{He}\left\{\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}\begin{bmatrix} B_1 & B_2 \end{bmatrix}\right\} + \begin{bmatrix} D & G \\ G^T & -D \end{bmatrix} < 0,$$

where $\mathrm{He}\{A\} = A + A^T$.

A necessary and sufficient condition based on Proposition 3.2 for the non-existence of the Lyapunov matrix of the form $Q(\theta) = Q_1 + \theta Q_2$ is the following

Proposition 3.3 ([5]). *For given matrices $B_1, B_2 \in \mathbb{R}^{n\times n}$, let us consider the parametrized inequality (3.1). Then inequality condition (3.1) does not hold for any $Q_1, Q_2 \in \mathscr{S}$ if and only if there exists $X, Y, Z \in \mathscr{S}$ such that*

$$\begin{gathered} \mathrm{He}\{B_1X + B_2Z\} = 0,\ \mathrm{He}\{B_1Z + B_2Y\} = 0, \\ X - Y \geq 0,\ \begin{bmatrix} X & Z \\ Z & Y \end{bmatrix} \geq 0,\ \begin{bmatrix} X & Z \\ Z & Y \end{bmatrix} \neq 0. \end{gathered} \tag{3.2}$$

Let us reformulate conditions (3.1), (3.2) for the segment $[A_1, A_2]$. The existence of APDLM

$$(1-\alpha)P_1 + \alpha P_2 = P_1 + \alpha(P_2 - P_1)$$

for the segment $[A_1, A_2]$ is equivalent to the following inequality

$$\begin{aligned} \inf_{P_1,P_2\in\mathscr{S}} \max_{\alpha\in[0,1]} \lambda_{\max}\big((A_1 + \alpha(A_2 - A_1))^T(P_1 + \alpha(P_2 - P_1)) + \\ (P_1 + \alpha(P_2 - P_1))(A_1 + \alpha(A_2 - A_1))\big) < 0, \end{aligned} \tag{3.3}$$

where $\lambda_{\max}(C)$ denotes the maximum eigenvalue of the symmetric matrix C.
Take $\alpha = \frac{\theta+1}{2}, P_1 = Q_1 + Q_2, P_2 = Q_2 - Q_1$ then $\theta \in [-1,1]$ and (3.3) is equivalent to

$$\begin{aligned} \inf_{P_1,P_2\in\mathscr{S}} \max_{\theta\in[-1,1]} \lambda_{\max}\big\{[(A_1 + A_2) + \theta(A_2 - A_1)]^T[P_1 + \theta P_2] + \\ [P_1 + \theta P_2][(A_1 + A_2) + \theta(A_2 - A_1)]\big\} < 0 \end{aligned} \tag{3.4}$$

with $B_1 = A_1 + A_2$, $B_2 = A_2 - A_1$. Propositions 3.2 and 3.3 take the following forms.

Proposition 3.4. *Let $[A_1, A_2]$ be robustly stable. There exists APDLM for this segment if and only if (3.4) is satisfied.*

Proposition 3.5. *Let $[A_1, A_2]$ be robustly stable. There exists APDLM $(1-\alpha)P_1 + \alpha P_2$ for this segment if and only if there exist $P_1, P_2 \in \mathscr{S}$, $D > 0$ and skew-symmetric G such that*

$$\mathrm{He}\left\{ \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \begin{bmatrix} A_1 + A_2 & A_2 - A_1 \end{bmatrix} \right\} + \begin{bmatrix} D & G \\ G^T & -D \end{bmatrix} < 0.$$

Proposition 3.6. *Let $[A_1, A_2]$ be robustly stable. Then (3.4) does not hold if and only if there exists $X, Y, Z \in \mathscr{S}$ such that*

$$\mathrm{He}\{(A_1 + A_2)X + (A_2 - A_1)Z\} = 0, \tag{3.5}$$

$$\mathrm{He}\{(A_1 + A_2)Z + (A_2 - A_1)Y\} = 0, \tag{3.6}$$

$$X - Y \geq 0, \begin{bmatrix} X & Z \\ Z & Y \end{bmatrix} \geq 0, \begin{bmatrix} X & Z \\ Z & Y \end{bmatrix} \neq 0. \tag{3.7}$$

In the counterexample given in [5]

$$B_1 = \begin{bmatrix} -4 & 2 & -2 \\ 5 & -6 & 1 \\ -2 & 2 & -7 \end{bmatrix}, B_2 = \begin{bmatrix} -5 & -3 & -13 \\ -5 & 0 & 0 \\ 10 & 13 & 16 \end{bmatrix} \tag{3.8}$$

and

$$X = \begin{bmatrix} 27923000270 & 1613316900 & -18116392840 \\ 1613316900 & 2449636443 & -373270542 \\ -18116392840 & -373270542 & 15686330948 \end{bmatrix},$$

$$Y = \begin{bmatrix} 18406827180 & -567157140 & -11343142800 \\ -567157140 & 979635060 & 412477920 \\ -11343142800 & 412477920 & 10002589560 \end{bmatrix},$$

$$Z = \begin{bmatrix} 20054829240 & -1400900940 & -12946463460 \\ -1400900940 & -450021420 & 1468590480 \\ -12946463460 & 1468590480 & 11543189400 \end{bmatrix}$$

which have been calculated implicitly by using softwares MATLAB, SeDuMi.

The conditions (3.5)-(3.7) are not LMI and the techniques of LMIs can not be applied. Therefore an explicit algorithm for the existence-nonexistence of an APDLM is needed.

Let

$$X = \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ x_2 & x_{n+1} & \dots & x_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_{2n-1} & \dots & x_d \end{bmatrix}, Y = \begin{bmatrix} y_1 & y_2 & \dots & y_n \\ y_2 & y_{n+1} & \dots & y_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ y_n & y_{2n-1} & \dots & y_d \end{bmatrix}, Z = \begin{bmatrix} z_1 & z_2 & \dots & z_n \\ z_2 & z_{n+1} & \dots & z_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_n & z_{2n-1} & \dots & z_d \end{bmatrix},$$

$x = (x_1, x_2, \dots, x_d)^T \in \mathbb{R}^d$, $y = (y_1, y_2, \dots, y_d)^T \in \mathbb{R}^d$, $z = (z_1, z_2, \dots, z_d)^T \in \mathbb{R}^d$, where $d = n(n+1)/2$. The matrix equations (3.5), (3.6) can be written in vector forms

$$C_1 x = D_1 z, \quad C_2 y = D_2 z.$$

Assume that

$$\det(C_1) \neq 0, \quad \det(C_2) \neq 0. \tag{3.9}$$

Then

$$x = C_1^{-1} D_1 z, \quad y = C_2^{-1} D_2 z. \tag{3.10}$$

To every vector $a = (a_1, a_2, \dots, a_d)^T \in \mathbb{R}^d$ corresponds unique symmetric matrix

$$(a_1, a_2, \dots, a_d)^T \to \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ a_2 & a_{n+1} & \dots & a_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{2n-1} & \dots & a_d \end{bmatrix}.$$

Therefore for every vector $z \in \mathbb{R}^d$, relations (3.10) define the symmetric matrices $X(z)$, $Y(z)$ and $Z(z)$ uniquely.

Proposition 3.7. *For given matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$ assume that $[A_1, A_2]$ is robustly stable and (3.9) is satisfied. Then there is no APDLM if and only if there exists $z \in \mathbb{R}^d$, $z \neq 0$ such that*

$$C(z) = \begin{bmatrix} X(z) - Y(z) & 0 & 0 \\ 0 & X(z) & Z(z) \\ 0 & Z(z) & Y(z) \end{bmatrix} \geq 0.$$

Proof. The vector z defines the matrices $X = X(z)$, $Y = Y(z)$, $Z = Z(z)$. $z \neq 0$ implies that $\begin{bmatrix} X & Z \\ Z & Y \end{bmatrix} \neq 0$. By construction X, Y and Z satisfy (3.5), (3.6). From $C(z) \geq 0$ it follows that ([6, p. 233])

$$X - Y \geq 0, \quad \begin{bmatrix} X & Z \\ Z & Y \end{bmatrix} \geq 0.$$

Then by Proposition 3.6 there is no APDLM. □

Define scalar function

$$h(z) = \lambda_{\min}(C(z)) = \min_{\|v\|=1} v^T C(z) v \tag{3.11}$$

and the box $\mathscr{Z} = [-1,1] \times \cdots \times [-1,1] = [-1,1]^d$, where $\lambda_{\min}(C)$ denotes the minimum eigenvalue of C. The function $h(z)$ is positive homogeneous ($h(\alpha z) = \alpha h(z)$ for $\alpha > 0$). Therefore there exists $z \in \mathbb{R}^d$ such that $C(z) \geq 0$ if and only if there exists $z \in \mathscr{Z}$ such that $h(z) \geq 0$.

Proposition 3.8. *Let $\hat{z} \in \mathscr{Z}$ and $\hat{v} \in \mathbb{R}^{3n}$ be a unit vector corresponding to $\lambda_{\min}(C(\hat{z}))$, that is $h(\hat{z}) = \hat{v}^T C(\hat{z}) \hat{v}$, and $\hat{g}$ be the gradient vector of the linear map $z \mapsto \hat{v}^T C(z) \hat{v}$. Then*

i) $h(\hat{z}) = \langle \hat{z}, \hat{g} \rangle$

ii) $h(z) \leq \langle z, \hat{g} \rangle$ *for all* $z \in \mathscr{Z}$,

where $\langle \cdot, \cdot \rangle$ denote the scalar product in $\mathbb{R}^d$.

Proposition 3.8 can be proven as in [7, 8] ([7] Proposition 1, [8] Proposition 2).

Recall that the convex hull of a finite set of vectors is called a polytope, and the point

$$S = \frac{1}{N} \sum_{i=1}^{N} q^i$$

is called the analytic center of the polytope $\mathrm{co}\{q^1, q^2, \dots, q^N\}$.

Let $\hat{z}$ be a nonzero vector in $\mathscr{Z}$. If $h(\hat{z}) \geq 0$ then $C(\hat{z}) \geq 0$ and $X(\hat{z})$, $Y(\hat{z})$, $Z(\hat{z})$ satisfy (3.5)-(3.7) and therefore there is no APDLM. If $h(\hat{z}) < 0$ we must to move to another point z.

If z satisfies the inequality $\langle z,\hat{g}\rangle < 0$ then $h(z) \le \langle z,\hat{g}\rangle < 0$ and $C(z) < 0$ and this z is not a required point. Therefore the polytope

$$\{z \in \mathscr{Z} : \langle z,\hat{g}\rangle \ge 0\}$$

is a new set of feasible z.

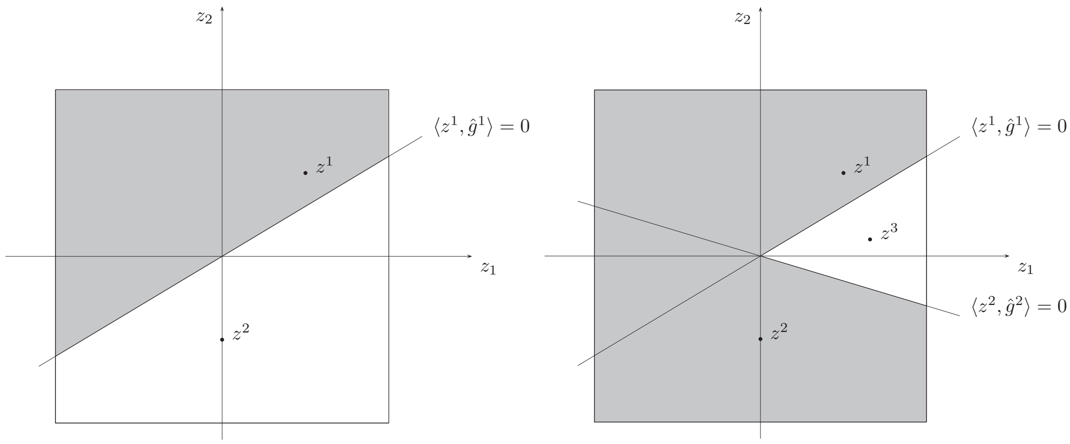

FIGURE 1. The procedure of choosing z^k where nonzero $z^1 \in \mathscr{Z}$ is arbitrary.

Combining the aboves, the following algorithm can be proposed for the existence-nonexistence of APDLM.

Algorithm 3.9.

1) *For given matrices* $A_1, A_2 \in \mathbb{R}^{n\times n}$ *assume that* $[A_1,A_2]$ *is robustly stable and* (3.9) *is satisfied. Construct* $X(z)$, $Y(z)$ *and* $Z(z)$ *where* $z \in \mathbb{R}^d$, $d = \frac{n(n+1)}{2}$.
2) *Take* $z^1 = (1/2,1/2,\ldots,1/2)^T \in \mathscr{Z} = [-1,1]^d$, *compute* $\lambda_{\min}\left(C(z^1)\right)$ *and a unit eigenvector* v^1 *corresponding to this minimum eigenvalue. If* $h(z^1) \ge 0$ *stop;* $X(z^1)$, $Y(z^1)$ *and* $Z(z^1)$ *satisfy the Proposition 3.6 and there is no APDLM, otherwise continue, compute* g^1 *as the gradient vector of the linear map* $z \mapsto {v^1}^T C(z) v^1$.
3) *Determine the vertices of the polytope* $\mathscr{P}_1 = \{z \in \mathscr{Z} : \langle z, g^1\rangle \ge 0\}$ *by solving an appropriate linear program* ([9, p. 329-337]) *and compute the analytical center of* $\mathscr{P}_1$: $z^2 = \frac{1}{N_1}\sum_{i=1}^{N_1} q^i$.
4) *If* $h(z^2) \ge 0$ *stop;* $X(z^2)$, $Y(z^2)$ *and* $Z(z^2)$ *satisfy the Proposition 3.6, otherwise continue, compute* g^2. *Determine the vertices of* $\mathscr{P}_2 = \{z \in \mathscr{Z} : \langle z, g^1\rangle \ge 0,\ \langle z, g^2\rangle \ge 0\}$ *and compute the analytical center of* $\mathscr{P}_2$: $z^3 = \frac{1}{N_2}\sum_{i=1}^{N_2} q^i$ *(see Figure 1).*
5) *At the kth step determine the vertices of the polytope* $\mathscr{P}_k$ *and compute* $z^{k+1} \in \mathscr{Z}$.
6) *If* $h(z^{\hat{k}}) \ge 0$ *for some* $\hat{k}$ *then there is no APDLM for the segment* $[A_1,A_2]$. *If* $\mathscr{P}_k = \{0\}$ *at some k then there is APDLM and this Lyapunov matrix can be obtained by solving appropriate LMI.*

Example 3.10. Consider the counterexample from [5]. The segment $\{B_1+\theta B_2 : \theta \in [-1,1]\}$ is transforming to the segment $[A_1, A_2]$, where

$$A_1 = B_1 - B_2 = \begin{bmatrix} 1 & 5 & 11 \\ 10 & -6 & 1 \\ -12 & -11 & -23 \end{bmatrix}, \quad A_2 = B_1 + B_2 = \begin{bmatrix} -9 & -1 & -15 \\ 0 & -6 & 1 \\ 8 & 15 & 9 \end{bmatrix}$$

Define

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}, \; Y = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_2 & y_4 & y_5 \\ y_3 & y_5 & y_6 \end{bmatrix}, \; Z = \begin{bmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{bmatrix}.$$

From the equation (3.5) the following is obtained

$$\begin{aligned}
x_1 &= -\tfrac{325}{108}z_1 - \tfrac{11}{4}z_2 - \tfrac{197}{36}z_3 - \tfrac{25}{54}z_4 - \tfrac{43}{54}z_5 + \tfrac{34}{27}z_6, \\
x_2 &= -\tfrac{2195}{927}z_1 - \tfrac{261}{103}z_2 - \tfrac{985}{309}z_3 - \tfrac{787}{927}z_4 - \tfrac{2200}{927}z_5 + \tfrac{593}{927}z_6, \\
x_3 &= \tfrac{3200}{2781}z_1 + \tfrac{151}{103}z_2 + \tfrac{1165}{927}z_3 + \tfrac{214}{2781}z_4 - \tfrac{2171}{2781}z_5 - \tfrac{5225}{2781}z_6, \\
x_4 &= -\tfrac{1557}{824}z_1 - \tfrac{11241}{4120}z_2 - \tfrac{2151}{824}z_3 - \tfrac{1079}{2060}z_4 - \tfrac{3659}{2060}z_5 - \tfrac{91}{206}z_6, \\
x_5 &= \tfrac{1861}{3708}z_1 + \tfrac{2677}{2060}z_2 + \tfrac{341}{1236}z_3 + \tfrac{10217}{9270}z_4 + \tfrac{11207}{9270}z_5 - \tfrac{508}{927}z_6, \\
x_6 &= -\tfrac{1031}{5562}z_1 - \tfrac{49}{1030}z_2 + \tfrac{2129}{1854}z_3 + \tfrac{4073}{13905}z_4 + \tfrac{33728}{13905}z_5 + \tfrac{7414}{2781}z_6.
\end{aligned}$$

Similarly, from the equation (3.6) we obtain

$$\begin{aligned}
y_1 &= -\tfrac{147}{44}z_1 + \tfrac{303}{110}z_2 - \tfrac{711}{55}z_3 - \tfrac{1544}{275}z_4 + \tfrac{921}{1100}z_5 - \tfrac{78}{11}z_6, \\
y_2 &= z_2 - \tfrac{6}{5}z_4 + \tfrac{1}{5}z_5, \\
y_3 &= \tfrac{43}{44}z_1 - \tfrac{25}{22}z_2 + \tfrac{53}{11}z_3 + \tfrac{134}{55}z_4 - \tfrac{81}{220}z_5 + \tfrac{30}{11}z_6, \\
y_4 &= -\tfrac{1135}{484}z_1 + \tfrac{565}{242}z_2 - \tfrac{802}{121}z_3 - \tfrac{1444}{605}z_4 + \tfrac{44681}{2420}z_5 - \tfrac{403}{121}z_6, \\
y_5 &= \tfrac{535}{242}z_1 - \tfrac{333}{121}z_2 + \tfrac{796}{121}z_3 + \tfrac{2012}{605}z_4 - \tfrac{1209}{1210}z_5 + \tfrac{423}{121}z_6, \\
y_6 &= -\tfrac{1165}{484}z_1 + \tfrac{713}{242}z_2 - \tfrac{996}{121}z_3 - \tfrac{2556}{605}z_4 + \tfrac{2219}{2420}z_5 - \tfrac{497}{121}z_6.
\end{aligned}$$

When we apply Algorithm 3.9 in the 17th step we obtained

$$z^{17} = (0.658, -0.037, -0.420, -0.004, 0.037, 0.365)^T$$

and $h(z^{17}) = \lambda_{\min}(C(z^{17})) = 0.000739 > 0$, and

$$\begin{aligned}
X &= \begin{bmatrix} 0.8520092585 & 0.0236202805 & -0.5398982380 \\ 0.0236202805 & 0.0516148058 & 0.0065874326 \\ -0.5398982380 & 0.0065874326 & 0.4591387630 \end{bmatrix}, \\
Y &= \begin{bmatrix} 0.594473636 & -0.02480000000 & -0.3664590915 \\ -0.02480000000 & 0.019835123 & 0.019240497 \\ -0.3664590915 & 0.019240497 & 0.315966530 \end{bmatrix}, \\
Z &= \begin{bmatrix} 0.658 & -0.037 & -0.420 \\ -0.037 & -0.004 & 0.037 \\ -0.420 & 0.037 & 0.365 \end{bmatrix}
\end{aligned}$$

and consequently there is no APDLM by Proposition 3.6. As a result we have obtained an another X, Y and Z (than X, Y and Z from [5]) satisfying (3.2).

Example 3.11. Consider the following matrices whose convex hull is robustly stable

$$A_1 = \begin{bmatrix} -1 & -1 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -1 & 2 \\ 20 & -1 & 2 \\ 1 & 0 & -2 \end{bmatrix}.$$

As in Example 3.10, the following are obtained

$$\begin{aligned}
x_1 &= -\tfrac{1587}{3451}z_1 + \tfrac{904}{3451}z_2 + \tfrac{1966}{3451}z_3 + \tfrac{6}{3451}z_4 - \tfrac{18}{493}z_5 + \tfrac{200}{3451}z_6,\\
x_2 &= \tfrac{9266}{44863}z_1 - \tfrac{23129}{44863}z_2 + \tfrac{11306}{44863}z_3 + \tfrac{526}{44863}z_4 + \tfrac{394}{6409}z_5 - \tfrac{872}{44863}z_6,\\
x_3 &= \tfrac{1502}{44863}z_1 - \tfrac{11002}{44863}z_2 - \tfrac{35303}{44863}z_3 + \tfrac{1286}{44863}z_4 + \tfrac{86}{6409}z_5 + \tfrac{6056}{44863}z_6,\\
x_4 &= \tfrac{182908}{44863}z_1 + \tfrac{147566}{44863}z_2 + \tfrac{271342}{44863}z_3 - \tfrac{685}{44863}z_4 + \tfrac{5506}{6409}z_5 + \tfrac{64592}{44863}z_6,\\
x_5 &= \tfrac{53988}{44863}z_1 - \tfrac{1188}{44863}z_2 + \tfrac{97984}{44863}z_3 - \tfrac{4314}{44863}z_4 - \tfrac{1355}{6409}z_5 + \tfrac{49456}{44863}z_6,\\
x_6 &= \tfrac{17996}{44863}z_1 - \tfrac{396}{44863}z_2 + \tfrac{62570}{44863}z_3 - \tfrac{1438}{44863}z_4 - \tfrac{2588}{6409}z_5 + \tfrac{1531}{44863}z_6,
\end{aligned}$$

$$\begin{aligned}
y_1 &= -\tfrac{1439}{1155}z_1 - \tfrac{104}{1155}z_2 + \tfrac{262}{1155}z_3 + \tfrac{118}{1155}z_4 - \tfrac{6}{55}z_5 + \tfrac{8}{21}z_6,\\
y_2 &= \tfrac{2}{693}z_1 - \tfrac{907}{693}z_2 + \tfrac{86}{693}z_3 + \tfrac{74}{693}z_4 - \tfrac{2}{11}z_5 + \tfrac{8}{63}z_6,\\
y_3 &= \tfrac{2026}{3465}z_1 + \tfrac{2206}{3465}z_2 - \tfrac{893}{3465}z_3 + \tfrac{118}{3465}z_4 - \tfrac{2}{55}z_5 + \tfrac{8}{63}z_6,\\
y_4 &= \tfrac{8180}{693}z_1 + \tfrac{4850}{693}z_2 + \tfrac{14942}{693}z_3 + \tfrac{1205}{693}z_4 - \tfrac{62}{11}z_5 + \tfrac{464}{63}z_6,\\
y_5 &= -\tfrac{4}{77}z_1 + \tfrac{120}{77}z_2 - \tfrac{172}{77}z_3 + \tfrac{6}{77}z_4 + \tfrac{3}{11}z_5 - \tfrac{16}{7}z_6,\\
y_6 &= \tfrac{4232}{3465}z_1 - \tfrac{988}{3465}z_2 + \tfrac{5954}{3465}z_3 - \tfrac{34}{3465}z_4 + \tfrac{36}{55}z_5 - \tfrac{29}{63}z_6.
\end{aligned}$$

For $z^1 = (1/2, \ldots, 1/2)^T \in [-1,1]^6$, the minimum eigenvalue of $C(z^1)$ is calculated as $h(z^1) = \lambda_{\min}(C(z^1)) = -14.925 < 0$. The unit eigenvector corresponding to this eigenvalue and g^1 are

$$v^1 = (-0.049, 0.970, -0.236, 0, 0, 0, 0, 0, 0)^T$$

and

$$g^1 = (-7.92, -2.84, -16.67, -1.56, 6.25, -7.09)^T,$$

respectively. The analytical center of the polytope

$$\mathscr{P}_1 = \{x \in [-1,1]^6 : \langle z, g^1 \rangle \geq 0\}$$

is: $z^2 = (-0.1, 0, -0.3, 0, 0.1, -0.01)^T$. Continuing in this way the following are obtained

$$\begin{aligned}
k = 2,\ & h(z^2) = -9.052,\\
& v^2 = (0, 0, 0, 0.003, 0.0001, 0.012, -0.0070, -0.9930, 0.108)^T,\\
& g^2 = (11.68, 6.54, 21.80, 1.70, -5.64, 7.76)^T,\\
& z^3 = (0.084, 0.409, 0.030, 0.034, 0.292, -0.206)^T,\\
k = 3,\ & h(z^3) = -1.300,\\
& v^3 = (0.047, 0.581, 0.812, 0, 0, 0, 0, 0, 0)^T,\\
& g^3 = (-1.99, -2.59, -1.31, -0.77, 1.05, 1.51)^T,\\
& z^4 = (-0.027, -0.090, 0.279, -0.122, 0.659, -0.064)^T,
\end{aligned}$$

$$
\begin{aligned}
k=4,\ & h(z^4)=-1.130,\\
& v^4=(0.053,-0.349,0.935,0,0,0,0,0,0)^T,\\
& g^4=(-2.54,0.70,-5.12,-0.11,0.18,-2.50)^T,\\
& z^5=(-0.378,0.604,0.077,-0.333,0.475,0.342)^T,\\
k=5,\ & h(z^5)=-0.831,\\
& v^5=(0,0,0,0.295,0.125,0.682,-0.200,-0.376,-0.500)^T,\\
& g^5=(2.35,1.09,2.92,0.18,-1.41,-0.26)^T,\\
& z^6=(-0.327,0.157,0.327,-0.902,0.021,-0.317)^T,\\
k=6,\ & h(z^6)=-0.512,\\
& v^6=(0,0,0,0.035,-0.532,0.225,-0.209,-0.760,0.206)^T,\\
& g^6=(7.65,4.19,14.46,1.84,-3.66,5.24)^T,\\
& z^7=(-0.235,0.244,0.197,-0.625,0.107,-0.087)^T,\\
k=7,\ & h(z^7)=-0.213,\\
& v^7=(0.755,-0.496,0.428,0,0,0,0,0,0)^T,\\
& g^7=(-2.64,-1.15,-5.99,-0.35,1.50,-2.87)^T,\\
& z^8=(0,0,0,0,0,0)^T,
\end{aligned}
$$

For $k=8$, $\mathscr{P}_8=\{(0,0,\dots,0)^T\}$. This means that there are no matrices X, Y and Z that satisfy the conditions in Proposition 3.6. So there exists APDLM. It has the form $P(\alpha)=(1-\alpha)P_1+\alpha P_2$, where

$$
P_1=\begin{bmatrix} 0.222 & -0.003 & 0.184\\ -0.003 & 0.089 & 0.053\\ 0.184 & 0.053 & 0.293 \end{bmatrix},\quad P_2=\begin{bmatrix} 0.999 & 0.057 & 0.329\\ 0.057 & 0.087 & -0.101\\ 0.329 & -0.101 & 0.524 \end{bmatrix}
$$

and the matrices P_1 and P_2 have been calculated by using MATLAB LMI Control Toolbox.

We have solved a great number of examples on existence APDLM for 2×2 matrix segments. Based on this observation we state the following conjecture

Conjecture 3.12. Let A_1, A_2 be any 2×2 matrices and $[A_1, A_2]$ be robustly stable. Then there exists APDLM for this segment.

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