Commun. Optim. Theory 2024 (2024) 22 https://doi.org/10.23952/cot.2024.22

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org

# ON A CLASS OF MIXED RANDOM VARIATIONAL INEQUALITIES

JOACHIM GWINNER

Institute of Applied Mathematics, Department of Aerospace Engineering, Universität der Bundeswehr München, 85577 Neubiberg/München, Germany

Dedicated to Professor Elijah Lucien Polak on the occasion of his 90th birthday

**Abstract.** This paper is concerned with a class of mixed random variational inequalities that involve a $\omega$-dependent bilinear form and a $\omega$-dependent convex function on a convex constraint set. Measurable solvability results for variational inequalities in $\omega$-pointwise and in $\Omega$-integrated form are presented under coercivity assumptions. Moreover a stability result with respect to Mosco convergence is provided that is based on an abstract (deterministic) stability result of its own interest. For illustration of the presented theory a non-smooth random boundary value problem is considered that captures all the difficulties of Tresca frictional unilateral problems that result in unilateral boundary conditions and a non-smooth convex sublinear functional.

**Keywords.** Coercivity; Mosco convergence; Random elliptic boundary value problem; Stability; Signorini boundary condition.

**2020 Mathematics Subject Classification.** 49J40, 49J53, 49J55, 35R60.

## 1. Introduction

In this contribution we are concerned with the following class of random variational inequalities (VIs for short) in a separable Hilbert space $\mathbb{H}$: Find for every $\omega \in \Omega$, where $(\Omega, \mathbb{A}, \mu)$ is a complete $\sigma$-finite measure space, an element $\hat{y} = \hat{y}(\omega) \in \mathbb{K} \subset \mathbb{H}$ such that

$$\beta(\omega, \hat{y}, z - \hat{y}) + \varphi(\omega, z) - \varphi(\omega, \hat{y}) \geq 0 \quad \text{for all } z \in \mathbb{K}, \tag{1.1}$$

where $\mathbb{K}$ is closed convex, $\beta(\omega, \cdot, \cdot)$ is a continuous bilinear form and $\varphi(\omega, \cdot)$ is a convex lower semicontinuous real-valued functional. In addition to the $\omega$-pointwise formulation above we also treat a $\Omega$-integrated formulation.

Thus the main novelty of this paper is the analysis of mixed VIs under uncertainty in a random setting, while deterministic mixed VIs can be traced back to [27] and [9]. Following the terminology of [11, 22] we include both VIs of the first kind taking $\varphi(\omega, \cdot) := \lambda(\omega, \cdot)$ as a linear form and VIs of the second kind taking $\mathbb{K} := \mathbb{H}$ with generally non-smooth $\varphi(\omega, \cdot)$ as well. Here we present measurable solvability and stability results thus extending some of the

E-mail address: joachim.gwinner@unibw-muenchen.de
Received: September 20, 2023; Accepted: November 29, 2023.

results in [14, 15] and in [18, Section 6.1] to a more general class of problems. In addition we provide an illustration of the presented theory and consider a non-smooth random boundary value problem.

Let us note that the presented measurable solvability result can also be derived from the more general measurable solvability result [1, Theorem 9] for general elliptic random variational inequalities that may contain sums of set-valued pseudomonotone operators. However, the proofs of these results are completely different. Whereas [1, Theorem 9] relies on [1, Theorem 1] on the existence of a measurable limit function for a sequence of measurable functions that results from Tychonoff's theorem on a compact product space and arguments from the theory of measurable multifunctions given in [23, chapter 2], the presented measurability result, here Theorem 2.1, for the set-valued solution map is based on Minty's lemma, measurability of Carathéodory functions (see e.g. [4, Lemma 8.2.6]), and the Castaing characterization theorem (see [4, Theorem 8.1.4]). Let us also underline that this paper goes beyond measurable solvability and in addition provides results on existence in an appropriate Lebesgue-Bochner space and on stability with respect to perturbations in the convex closed set and in the convex lower semicontinuous function.

The outline of the paper is as follows. The subsequent section 2 is concerned with solvability in a general complete $\sigma$-finite measure space. In particular we show that under specific assumptions on the data the unique solution $\hat{y} : \omega \in \Omega \mapsto \hat{y}(\omega) \in \mathbb{K}$ lies in an appropriate Lebesgue-Bochner space [30, section 4.2], see Theorem 2.2. In section 3 we focus to the mean square case, consider the random VI in integrated form on a probability space $(\Omega, \mathbb{A}, P)$, and provide a stability result for perturbations in the convex closed set and in the convex lower semicontinuous function with respect to Mosco convergence, see Corollary 3.4. This latter stability result relies on a novel stability result for a general class of (deterministic) linear extended real-valued VIs in a real reflexive Banach space, which is of independent interest, see Theorem A.2 and Corollary A.3 in the appendix A. To illustrate the theory presented above section 4 describes a non-smooth random boundary value problem that captures all the difficulties of Tresca frictional unilateral problems that result in unilateral boundary conditions and a non-smooth convex sublinear functional. The paper ends in section 5 with an outlook to some open directions of research in the field of random VIs.

## 2. Mixed Random Variational Inequalities - Solvability in Measure Space

In this section we investigate a class of mixed random variational inequalities in the setting of a separable Hilbert space and a complete $\sigma$-finite measure space. We show the existence of a unique measurable solution which, moreover, lies in an appropriate Lebesgue-Bochner space [30, section 4.2] under reasonable assumptions on the data. To this end we use some concepts of measure theory, in particular measurability of set-valued maps (correspondences) what can be found with [4, chapter 8].

Let $(\Omega, \mathbb{A}, \mu)$ be a complete $\sigma$-finite measure space and $(\mathbb{H}, (\cdot,\cdot), \|\cdot\|)$ be a separable Hilbert space. Let $\mathbb{K}$ be a closed convex nonvoid subset of $\mathbb{H}$. We are given Carathéodory functions $\varphi : \Omega \times \mathbb{H} \to \mathbb{R}$ and $\beta : \Omega \times (\mathbb{H} \times \mathbb{H}) \to \mathbb{R}$, that is, for every $y, z \in \mathbb{H}$, $\varphi(\cdot, y)$ and $\beta(\cdot, y, z)$ are measurable and for every $\omega \in \Omega$, $\varphi(\omega, \cdot)$ and $\beta(\omega, \cdot, \cdot)$ are continuous. Moreover, for every $\omega \in \Omega$, $\varphi(\omega, \cdot)$ and $\beta(\omega, \cdot, \cdot)$ is assumed to be a convex function, respectively a bilinear form.

Via

$$\beta(\omega,y,z) = \langle \beta(\omega)y,z\rangle_{\mathbb{H}^*\times\mathbb{H}} \quad (\forall y,z\in\mathbb{H}),$$

where $\mathbb{H}^* := \mathbb{L}(\mathbb{H},\mathbb{R})$, we find that $\beta(\omega) := \beta(\omega,\cdot,\cdot) \in \mathbb{L}(\mathbb{H}\times\mathbb{H},\mathbb{R}) \cong \mathbb{L}(\mathbb{H},\mathbb{H}^*)$. Furthermore we assume that, for every $\omega\in\Omega$, $\beta(\omega,\cdot,\cdot)$ is nonnegative, that is, that for every $y\in\mathbb{H}$, $\beta(\omega,y,y)\geq 0$ holds.

In this setting we study the following mixed random variational inequality: Find for every $\omega\in\Omega$, an element $\hat{y}\in\mathbb{K}$ depending on $\omega\in\Omega$ such that

$$\beta(\omega,\hat{y},z-\hat{y}) + \varphi(\omega,z) - \varphi(\omega,\hat{y}) \geq 0 \quad \text{for all } z\in\mathbb{K}. \tag{2.1}$$

Our first result concerns the measurability of the solution (set-valued) mapping $\Sigma : \Omega \rightsquigarrow \mathbb{H}$ given by

$$\Omega \ni \omega \rightsquigarrow \Sigma(\omega) = \{\hat{y}\in\mathbb{K} : \hat{y} \text{ solves } (2.1)\}$$

with respect to the $\sigma$-algebra $\mathbb{B}(\mathbb{H})$ of the Borel subsets of $\mathbb{H}$. Here and later on we use Minty's lemma (see e.g. [17, Prop. 3.2]) which states in our context that $\hat{y}\in\Sigma(\omega)$, if and only if

$$\hat{y}\in\mathbb{K},\ \beta(\omega,z,\hat{y}-z) + \varphi(\omega,\hat{y}) - \varphi(\omega,z) \leq 0 \quad \text{for all } z\in\mathbb{K}.$$

**Theorem 2.1.** *Suppose that for every $\omega\in\Omega$, (2.1) possesses a solution. Then the set-valued map $\Sigma$ is measurable.*

PROOF. With $\mathbb{H}$ separable, the metric subspace $\mathbb{K}$ is separable, too. Let $\{z_\nu\}_{\nu\in\mathbb{N}}$ be dense in $\mathbb{K}$. Then by Minty's lemma and by continuity, $\hat{y}\in\Sigma(\omega)$, if and only if

$$\hat{y}\in\mathbb{K},\ \beta(\omega,z_\nu,\hat{y}-z_\nu) + \varphi(\omega,\hat{y}) - \varphi(\omega,z_\nu) \leq 0 \quad \text{for all } \nu\in\mathbb{N}.$$

Therefore $\Sigma = \bigcap_{\nu\in\mathbb{N}} \Sigma_\nu$, where for any $\nu\in\mathbb{N}$, $\Sigma_\nu : \Omega \rightsquigarrow \mathbb{H}$ is given by

$$\omega \rightsquigarrow \Sigma_\nu(\omega) := \{\hat{y}\in\mathbb{K} : \beta(\omega,z_\nu,\hat{y}) + \varphi(\omega,\hat{y}) \leq \beta(\omega,z_\nu,z_\nu) + \varphi(\omega,z_\nu)\}.$$

Then $\Sigma_\nu(\omega)$ is closed by continuity, and by $\mathbb{A}\otimes\mathbb{B}(\mathbb{H}) - \mathbb{B}(\mathbb{R})$ measurability of Carathéodory functions (see e.g. [4, Lemma 8.2.6] ) the graph of the set valued map $\Sigma_\nu$ belongs to $\mathbb{A}\otimes\mathbb{B}(\mathbb{H})$. Since

$$\operatorname{graph}\Sigma = \bigcap_{\nu\in\mathbb{N}} \operatorname{graph}\Sigma_\nu \in \mathbb{A}\otimes\mathbb{B}(\mathbb{H}),$$

by the Castaing characterization theorem (see [4, Theorem 8.1.4]) the claimed measurability of $\Sigma$ follows. □

REMARKS. - If for any $\omega\in\Omega, \beta(\omega,\cdot,\cdot)$ is a symmetric bilinear form, then it is well-known that $\hat{y}\in\mathbb{K}$ solves (2.1) if and only if, $\hat{y}\in\mathbb{K}$ depending on $\omega\in\Omega$ minimizes the convex function

$$\frac{1}{2}\beta(\omega,y,y) + \varphi(\omega,y), \quad y\in\mathbb{K}.$$

In this case, Theorem 2.1 above follows directly from the measurability of the marginal map in stochastic optimization (see e.g. [4, Theorem 8.2.1]). - With $\{x_\nu\}_{\nu\in\mathbb{N}}$ a given dense sequence in $\mathbb{H}$, the sequence $\{z_\nu\}_{\nu\in\mathbb{N}} := \{\operatorname{proj}_{\mathbb{K}} x_\nu\}_{\nu\in\mathbb{N}}$ is easily constructed as a dense subset of $\mathbb{K}$, since the projection operator $\operatorname{proj}_{\mathbb{K}} : \mathbb{H}\to\mathbb{K}$ is nonexpansive.

The measurability result above can be combined with any existence result for deterministic mixed VIs (see [27] and [17] for VIs with more general monotone bifunctions). In this paper we concentrate on the coercive case, where we do not have only existence, but also uniqueness. To this end we extend the notion of coercivity in the deterministic theory.

DEFINITION. - We call $\beta$ *coercive*, if there is some constant $c_0 > 0$ that may depend on $\omega \in \Omega$ such that

$$c_0 \,\|y\|^2 \le \beta(\omega, y, y) \quad (\forall y \in \mathbb{H})\,. \tag{2.2}$$

Further we call $\beta$ $\mu$-*coercive*, if this constant $c_0$ is independent of $\omega \in \Omega$, more precisely, if there holds

$$c_0 \,\|y\|^2 \le \beta(\omega, y, y) \quad \text{for } \mu - \text{almost all } \omega \in \Omega, \forall y \in \mathbb{H}\,. \tag{2.3}$$

Under the assumption of coercivity, the unique solvability of (2.1) for every $\omega \in \Omega$ can be derived from the extension [27] of the Lions-Stampacchia theorem to mixed VIs in Hilbert space or from existence results for extended real-valued equilibrium problems with monotone bifunctions under asymptotic coercivity condition [17, Theorem 5.2].

To proceed further towards regularity for the solution $\hat{u}(\omega) := \hat{y}, \omega \in \Omega$, we need a growth condition to hold for the convex function $\varphi$. Note by the separation theorem it can be shown that any convex continuous function $\phi : \mathbb{H} \to \mathbb{R}$ is *conically minorized*, that is, it enjoys the estimate

$$\phi(z) \ge -c_\phi\,(1 + \|z\|)\,, z \in \mathbb{H}$$

with some $c_\phi > 0$. This suggests to require that $\varphi$ is $\mu$-*conically minorized*, that is, there exists a constant $c_\varphi > 0$ such that

$$\varphi(\omega, z) \ge -c_\varphi\,(1 + \|z\|) \quad \text{for } \mu - \text{almost all } \omega \in \Omega, \forall z \in \mathbb{H}\,. \tag{2.4}$$

Under these conditions we can derive the following a priori estimate:

$$c_0 \,\|\hat{u}(\omega)\|^2 \le \|\beta(\omega)\|_{\mathbb{L}(\mathbb{H},\mathbb{H}^*)} \,\|z_0\| \,\|\hat{u}(\omega)\| + c_\varphi(1 + \|\hat{u}(\omega)\|) + \varphi(\omega, z_0)$$

with some arbitrary fixed $z_0 \in \mathbb{K}$, hence

$$\|\hat{u}(\omega)\| \le \tilde{c}\left(1 + |\varphi(\omega, z_0)| + \|\beta(\omega)\|_{\mathbb{L}(\mathbb{H},\mathbb{H}^*)}\right) \tag{2.5}$$

where $\tilde{c} > 0$ depends on $\|z_0\|$, $c_0$ and $c_\varphi$. Thus the solution $\hat{u}$ shows the same $\omega$ – regularity as the data $\beta$ and $\varphi(\cdot, z_0)$.

Note that in the separable space $\mathbb{H}$, the measurability of $\hat{u}$ with respect to $\mathbb{B}(\mathbb{H})$ implies the measurability of the real valued function $\|\hat{u}(\cdot)\|$ (see e.g. [6, Lemma 1.5]). However, in contrary to $\mathbb{H}^* \cong \mathbb{H}$ by the Riesz isomorphism, the space $\mathbb{L}(\mathbb{H} \times \mathbb{H}, \mathbb{R}) \cong \mathbb{L}(\mathbb{H}, \mathbb{H})$ does not need to be separable (see the counterexample in [6, Section 1.2]).

Therefore in the following we introduce assumptions on the function $\omega \mapsto \|\beta(\omega)\|$ involved in (2.5), instead on the mapping $\omega \mapsto \beta(\omega)$. Then we can exploit (2.5) and can conclude that the solution $\hat{u}$ belongs to the Bochner-Lebesgue space $L^q(\Omega, \mu, \mathbb{H})$, the Banach space of (classes of) measurable maps $v : \Omega \to \mathbb{H}$ such that $\int_\Omega \|v(\omega)\|^q \, d\mu(\omega) < \infty$ (for $1 \le q < \infty$), respectively in the Banach space of (classes of) measurable, $\mu$-essentially bounded maps $v : \Omega \to \mathbb{H}$ (for $q = \infty$). To sum up, we arrive at the following result.

**Theorem 2.2.** *Let $(\Omega, \mathbb{A}, \mu)$ be a complete $\sigma$-finite measure space, $\mathbb{H}$ a separable Hilbert space and $\beta$ a coercive bilinear form. Then the random variational inequality (2.1) admits a unique solution $\hat{u} : \omega \in \Omega \mapsto \hat{u}(\omega) \in \mathbb{K}$.*
*Suppose in addition, that $\mu$ is a finite measure, $\beta$ is $\mu$ - coercive, and $\varphi$ is $\mu$ - conically*

*minorized. Moreover suppose, that the real valued functions* $\omega \mapsto \|\beta(\omega)\|_{\mathbb{L}(\mathbb{H},\mathbb{H}^*)}$ *and* $\omega \mapsto \varphi(\omega, z_0)$ *belong to* $L^q(\Omega,\mu)$ *for some* $z_0 \in \mathbb{K}, 1 \le q \le \infty$. *Then we have* $\hat{u} \in L^q(\Omega,\mu,\mathbb{H})$.

In the following sections we concentrate on the mean square case $q = 2$ and not only give existence results, but also a stability result.

## 3. Mixed Random Variational Inequalities - Well-Posedness in Probability Space

Let us focus to the mean square case $q = 2$ and start from a probability space $(\Omega, \mathbb{A}, P)$. Then we can consider the above random variational problem in the Hilbert space $X =: L^2(\Omega, P, \mathbb{H})$ of all $\mathbb{H}$ - valued $P$ - measurable random variables $V$ such that

$$E^P\|V\|^2 = \int_\Omega \|V(\omega)\|^2 \, dP(\omega) < \infty.$$

Clearly the set

$$K =: L^2(\Omega, P, \mathbb{K}) = \{V \in L^2(\Omega, P, \mathbb{H}) : V \in \mathbb{K}\ P\text{-almost everywhere}\}.$$

is convex. We have also

**Lemma 3.1.** *The set* $K$ *is closed in* $L^2(\Omega, P, \mathbb{H})$.

Here for convenience of the reader we reproduce the proof from [15, Lemma 2.3]. - Without any restriction of generality we can assume that $0 \in \mathbb{K}$, since addition with the constant random variable $\omega \in \Omega \mapsto z_0$ with some fixed $z_0 \in \mathbb{K}$ is a topological isomorphism in $L^2(\Omega, P, \mathbb{H})$. Thus the bipolar theorem applies, and we have

$$\mathbb{K} = \{z \in \mathbb{H} : (z, \zeta) \le 1, \ \forall\, \zeta \in \mathbb{K}^0\},$$

where

$$\mathbb{K}^0 = \{\zeta \in \mathbb{H}^* : (z, \zeta) \le 1, \ \forall z \in \mathbb{K}\}.$$

Since $\mathbb{K}^0$ as a metric subspace of the separable Hilbert space $\mathbb{H}^* \cong \mathbb{H}$ is separable, too, we find a sequence $\{\zeta_\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{K}^0$ such that

$$\mathbb{K} = \{z \in \mathbb{H} : (z, \zeta_\nu) \le 1, \ \forall\, \nu \in \mathbb{N}\}. \tag{3.1}$$

Now let $\{V_n\}_{n \in \mathbb{N}} \subset K$ such that $V_n \to V\,(n \to \infty)$ strongly in $L^2(\Omega, P, \mathbb{H})$. This implies $\Big(V_n(\cdot), \zeta_\nu\Big) \le 1$ a.s. and $\zeta_\nu \circ V_n \to \zeta_\nu \circ V\,(n \to \infty)$ strongly in $L^2(\Omega, P)$ for any $\nu \in \mathbb{N}$. Therefore by well-known results in measure theory (see e.g. [5, Propositions 3.1.4, 3.1.2]), for any $\nu \in \mathbb{N}$, there exists a subsequence converging a.s. on $\Omega$ and hence $\Big(V\,(\cdot), \zeta_\nu\Big) \le 1$ a.s.. By the representation (3.1) we conclude that $V(\cdot) \in \mathbb{K}$ a.s. and the closedness of $K$ follows. □

Next the given Carathéodory functions $\varphi : \Omega \times \mathbb{H} \to \mathbb{R}$ and $\beta : \Omega \times (\mathbb{H} \times \mathbb{H}) \to \mathbb{R}$, under appropriate growth assumptions, give rise to Nemitskii operators

$$V \mapsto \varphi(\cdot, V(\cdot)) \in L^1(\Omega, P, \mathbb{R}), (V, W) \mapsto \beta(\cdot, V(\cdot), W(\cdot)) \in L^1(\Omega, P, \mathbb{R})$$

such that

$$\begin{aligned} f : V \ &\mapsto \ E^P\{\varphi(\cdot, V(\cdot))\} \\ b : (V, W) \ &\mapsto \ E^P\{\beta(\cdot, V(\cdot), W(\cdot))\} \end{aligned}$$

is a continuous convex functional on $X$, respectively a coercive bilinear form in $\mathbb{L}(X \times X, \mathbb{R}) \cong \mathbb{L}(X, X)$. Therefore immediately by the extension [27] of the Lions-Stampacchia theorem to mixed VIs in Hilbert space, the following variational inequality in integrated form: Find $U \in K := L^2(\Omega, P, \mathbb{K})$ such that for all $V \in K$

$$b(U, V - U) + f(V) \geq f(U) \tag{3.2}$$

has a unique solution $\hat{U}$.

By uniqueness, both formulations (2.1) and (3.2) are equivalent.

Let us now discuss stability with respect to the function $\varphi$ and the set $\mathbb{K}$. Thus in addition, consider a given sequence $\{\varphi^{(\nu)}\}_{\nu \in \mathbb{N}}$, where $\varphi^{(\nu)} : \Omega \times \mathbb{H} \to \mathbb{R}$ are Carathéodory functions with $\varphi^{(\nu)}(\omega, \cdot)$ convex continuous, and a given sequence $\{\mathbb{K}^{(\nu)}\}_{\nu \in \mathbb{N}}$ of closed convex nonempty subsets of $\mathbb{H}$. This gives rise to the continuous convex functionals on $X$, respectively convex closed subsets of $X$,

$$f^{(\nu)} : V \mapsto E^P\{\varphi^{(\nu)}(\cdot, V(\cdot))\}$$
$$K^{(\nu)} := L^2(\Omega, P, \mathbb{K}^{(\nu)}) .$$

Thus we are led to the perturbed variational inequality: Find $U^{(\nu)} \in K^{(\nu)}$ such that for all $V \in K^{(\nu)}$,

$$b(U^{(\nu)}, V - U^{(\nu)}) + f^{(\nu)}(V) \geq f^{(\nu)}(U^{(\nu)}) . \tag{3.3}$$

Here we employ Mosco convergence as concept of set convergence and the concept of epi-convergence for convex functions in the sense of Mosco also called Mosco convergence for short, both written $\xrightarrow{\text{M}}$; see the appendix for the definitions or see [3].

Then abstracting from the special structure we obtain from Corollary A.4 the following stability result.

**Corollary 3.2.** *Let $B$, $f, f^{(\nu)}$, $K, K^{(\nu)}$ be given as above. Suppose that $K^{(\nu)} \xrightarrow{\text{M}} K$ and $f^{(\nu)} \xrightarrow{\text{M}} f$ for $\nu \to \infty$. Assume that $W^{(\nu)} \to W$ in $X$ implies $f^{(\nu)}(W^{(\nu)}) \to f(W)$ for $\nu \to \infty$. Then $lim_{\nu \to \infty} \|U^{(\nu)} - \hat{U}\|_X = 0$ holds.*

## 4. A MIXED RANDOM VARIATIONAL INEQUALITY MODELLING FRICTIONAL UNILATERAL CONTACT MECHANICS

For illustration of the previous theory we consider a non-smooth random boundary value problem that captures all the difficulties of Tresca frictional unilateral problems that result in unilateral boundary conditions and a non-smooth convex sublinear functional. We take the deterministic description from [16].

Let $Q, R_b, R_d, S, T$ be real-valued random variables on a probability space $(\Omega, \mathbb{A}, P)$. Further let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded smooth domain with outer unit normal $n$ and with its boundary $\Gamma := \partial D$ which splits in mutually disjoint open parts, namely the Dirichlet part $\Gamma_D$, the Neumann part $\Gamma_N$, the Signorini part $\Gamma_S$, and the Tresca part $\Gamma_T$, such that $\partial D = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_S \cup \overline{\Gamma}_T$ with meas $(\Gamma_S) > 0$ and meas $(\Gamma_T) > 0$.

With given data $f, g \geq 0, h, \chi$ specified later, we consider the following non-smooth boundary value problem in its strong form: There holds almost surely (a.s.)

$$(BVP) \quad \begin{cases} \quad -\nabla \cdot (S\nabla u) = R_d f & \text{in } D, \\ \quad q_\nu := (-S\nabla u)\cdot \nu & \text{on } \partial D \\ \quad u = T\chi & \text{on } \Gamma_D, \\ \quad q_\nu = R_b h & \text{on } \Gamma_N, \\ \quad u \leq T\chi, q_\nu \leq 0, (u - T\chi)q_\nu = 0 & \text{on } \Gamma_S, \\ (FC) \quad |q_\nu| \leq Qg, u\, q_\nu + Qg|u| = 0 & \text{on } \Gamma_T. \end{cases}$$

Note that without any restriction of generality, we can assume in $(FC)$ that $g > 0$ on $\Gamma_T$; otherwise the part where $g$ vanishes and hence $q_\nu$ is required to vanish can be subsumed to $\Gamma_N$ with an obvious modification of $h$.

In accordance with the standard case of uniformly strongly elliptic operators, we assume that $P$-almost surely, $S$ is contained in a compact interval that is included in $(0, +\infty)$. Hence $S$ belongs to $L^\infty(\Omega)$. Similarly we assume that the random variables $Q, T$ are nonnegeative a.s. and in $L^\infty(\Omega)$. For the right hand side, we only assume that $R$ is in $L^2(\Omega)$.

In literature one encounters different, but equivalent friction conditions. Here dropping the random variable $Q$ for simplicity, we list from [24] the conditions

$$(FC') \quad |q_\nu| \leq g, (g - |q_\nu|)u = 0, u\, q_\nu \leq 0,$$

$$(FC'') \quad \begin{cases} |q_\nu| \leq g, \\ \text{if } |q_\nu| < g \text{ then } u = 0, \\ \text{if } |q_\nu| = g \text{ then } \exists \kappa \geq 0 : u = -\kappa q_\nu, \end{cases}$$

which are equivalent to $(FC)$.

As weak formulation of the above random nonsmooth boundary value problem $(BVP)$ we are going to derive a variational inequality with the data $f \in L^2(D), \chi \in H^{1/2}(\Gamma_D \cup \Gamma_S), h \in H^{-1/2}(\Gamma_N), g \in L^\infty(\Gamma_T)$, where $g > 0$.

Fix first $\omega \in \Omega$. Multiplying the pde above, $-\nabla \cdot (S\nabla u) = R_d f$ by a test function $w \in H^1_{\Gamma_D}(D) = \{w \in H^1(D) \,:\, w|_{\Gamma_D} = 0\}$ and integrating by parts yields

$$\begin{aligned} R_d(\omega) \int_D f\, w\, dx &= S(\omega) \int_D \nabla u \cdot \nabla w\, dx - \int_\Gamma q_\nu(\omega)\, w\, ds \\ = \quad & R_d(\omega) \int_D \nabla u \cdot \nabla w\, dx - R_b(\omega) \int_{\Gamma_N} h\, w\, ds - \int_{\Gamma_S \cup \Gamma_T} q_\nu(\omega)\, w\, ds. \end{aligned}$$

Furthermore introduce the convex closed set

$$\mathbb{K}(\omega) := \{v \in H^1(D) : v|\Gamma_D = T(\omega)\chi \text{ and } v|\Gamma_S \leq T(\omega)\chi\},$$

the bilinear form associated to the Laplacian

$$a(u,v) := \int_D \nabla u \cdot \nabla v\, dx,$$

the linear forms

$$l_d(v) := \int_D f\, v\, dx;\; l_b(v) := \int_{\Gamma_N} h\, v\, ds\,,$$

and the continuous, positively homogeneous and sublinear, hence convex functional

$$\varphi(v) := \int_{\Gamma_T} g|v|\, ds\,.$$

For $u \in \mathbb{K}(\omega)$ and $w = v - u$ for arbitrary $v \in \mathbb{K}(\omega)$ the previous equality writes

$$S(\omega)\, a(u, v-u) = R_d(\omega) l_d(v-u) + R_b(\omega) l_b(v-u) + \int_{\Gamma_S \cup \Gamma_T} q_\nu(\omega)(v-u)\, ds\,.$$

On $\Gamma_S$ we have

$$q_\nu(\omega)(w-u) = q_\nu(\omega)(w - T(\omega)\chi) - q_\nu(\omega)(u - T(\omega)\chi) = q_\nu(\omega)(w - T(\omega)\chi) \geq 0\,,$$

on $\Gamma_T$ we have

$$-q_\nu(\omega)(v-u) \leq |q_\nu(\omega)||v| - Q(\omega) g|u| \leq Q(\omega) g(|v| - |u|)\,.$$

Thus we arrive at the concrete variational inequality problem $(\pi)$ which is of the form (1.1): For any fixed $\omega \in \Omega$ find $u = u(\omega) \in \mathbb{K}(\omega)$ such that for all $v \in \mathbb{K}(\omega)$,

$$S(\omega)\, a(u, v-u) + Q(\omega)\, g(|v| - |u|) \geq R_d\, l_d(v-u) + R_b\, l_b(v-u)\,. \tag{4.1}$$

By the Poincaré inequality the bilinear form is coercive, provided $\Gamma_D$ has positive measure - what we assume in what follows -, hence (4.1) has a unique solution. Further to (4.1) there corresponds the VI in integrated form (3.2): Find $U \in K := L^2(\Omega, P, \mathbb{K})$ such that for all $V \in K$

$$b(U, V-U) + f(V) \geq f(U)\,, \tag{4.2}$$

where now in the concrete case

$$b(U,V) := \int_\Omega S(\omega)\, a(U(\omega,\cdot), V(\omega,\cdot))\, dP\,,$$

$$f(V) := \int_\Omega [Q\, g\, |V| - R_d\, l_d(V) - R_b\, l_b(V)]\, dP\,.$$

Since by assumption, $P$-almost surely, the random variable $S$ is contained in a compact interval that is included in $(0, +\infty)$, (4.2) has a unique solution. Again by uniqueness, both formulations (4.1) and (4.2) are equivalent.

To exhibit the relation of (4.2) to unilateral contact with friction in linear elasticity we insert the following remark.

**Remark 4.1.** In linear elasticity, instead of the unknown scalar field $u$, there is the displacement field $u$ which decomposes in its normal component $u_n = u \cdot n$ and its tangential component $u_t = u - u_n n$. Similarly as dual variable, the flux $q_\nu = \dfrac{\partial u}{\partial n}$ is to be replaced by the stress field $T$ with its normal component $T_n$ and its tangential component $T_t$. The boundary parts $\Gamma_S$ and $\Gamma_T$ from above collapse to $\Gamma_c$. Then unilateral contact with a rigid foundation together with friction according to Coulomb's law requires the following conditions (see [24, 22]) on the contact surface $\Gamma_C$:

$$u_n \leq \chi, T_n \leq 0, (u_n - \chi) T_n = 0$$

and

$$|T_t| \leq \mathbb{F}|T_n|, \Big(\mathbb{F}|T_n| - |T_t|\Big) u_t = 0, u_t \cdot T_t \leq 0,$$

where $\mathbb{F} \geq 0$ is the friction coefficient. The latter condition expresses the obvious law that the modulus of the tangential component is limited by a multiple of the modulus of the normal component; if it is attained, then the body can slip off in the direction opposite to $T_t$; otherwise, the body sticks.

The fixed point approach to unilateral frictional contact as proposed by Panagiotopoulos [29], employed in the existence proof (see [10]) and numerically realized in [8] leads to a approximating sequence of unilateral problems with given friction, also known as contact problems with Tresca friction. In these approximations the unknown normal component is replaced by a given slip stress $g_n \geq 0$, such that the latter condition above reduces to

$$|T_t| \leq \mathbb{F} g_n, \Big(\mathbb{F} g_n - |T_t|\Big) u_t = 0, u_t \cdot T_t \leq 0\,,$$

which compares to the friction condition $(FC')$ above. The weak formulation of the Tresca frictional unilateral contact problem is the following VI (see [22, section 7] for the proof of the formal equivalence of the classical and weak formulation): Find $u \in K$ such that for all $v \in K$

$$a(u, v-u) + \int_{\Gamma_C} \mathbb{F} g_n \Big(|v_t| - |u_t|\Big)\, ds \geq \int_{\Gamma_N} f \cdot (v-u)\, ds\,,$$

where $f$ is the surface force, $a(\cdot,\cdot)$ is the bilinear form of strain energy in linear elasticity, and $K$ is the appropriately defined convex set. In this sense, (4.2) gives a simplified (scalar) model of the unilateral contact problem with given friction.

## 5. Some Concluding Remarks - An Outlook

In this paper we focused to linear random variational inequalities. By more involved arguments one can generalize some of the presented results to more general classes of random nonlinear variational inequalities and hemivariational inequalities, (see e.g. [12, 13, 28] for such nonlinear problems in the deterministic case). In particular, using monotonicity methods the results above can be further extended to variational problems involving nonlinear monotone operators with random coefficients, see [19] albeit in the case of variational inequalities of the first kind. Moreover, for the existence analysis of considerably more general variational-hemivariational inequalities by an equilibrium approach we can refer to [17] albeit in the deterministic case. One can expect that this theory can help in the analysis of more involved problems in stochastic continuum mechanics; see [2, 33].for the treatment of stochastic contact and plasticity problems in the framework of VI of the first kind. We left out the discussion of semicoercive random variational problems; such an extension is not obvious, since a compact imbedding $\mathbb{X} \subset \mathbb{Y}$ does not necessarily lead to the compact imbedding $L^2(\Omega, P, \mathbb{X}) \subset L^2(\Omega, P, \curvearrowright \mathbb{Y})$.

In this paper we dealt with the issues of measurability, existence, and stability, but did not touch on numerical aspects. The numerical approach in the 2000 paper [14] that combines finite element discretization in the deterministic variable with piecewise constant approximation by averaging in the random variable mimicing the constructive definition of the Lebesgue integral via elementary functions can in principle be extended to the more general mixed VIs considered in this paper. But then the nonsmooth convex functional has to be regularized in addtion what leads to a multi-level approximation similar to [31]. The piecewise constant approximation in the random variable by averaging leads to a comparative numerical procedure in comparison with other solution methods for finite dimensional stochastic VIs with few random varables, see [21] and [18, chapter 12]. On the other hand, since 2000 there appeared many works that endeavour to overcome the curse of dimensions in the numerical treatment of stochastic elliptic VIs (of first kind) and stochastic PDEs; let us mention here (adaptive) multi-level Monte Carlo finite element methods [25, 26].

Finally let us point out that this paper is devoted to the *direct problem* of random/stochastic VIs while the *inverse problem* of parameter identification in random/stochastic VIs is a largely unexplored field. In this context we can only refer to [7] for the identification of the random coefficients in elliptic stochastic pdes up to order four. The efficient identification of a random friction parameter and random Lamé coefficients in frictional contact problems using boundary element methods [20] for discretization would be a challenging research topic.

**Acknowledgements**
The author would like to thank the referee for his suggestions, in particular for calling attention to the reference [1].

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## A. Stability of Linear Extended Real-Valued Variational Inequalities

In this appendix we establish a general stability result for the following class of (deterministic) linear extended real-valued VIs in a real reflexive Banach space $V$ with norm $\|\cdot\|$ and dual $V^*$: Find an element $\hat{v}$ such that

$$\langle B\hat{v}, v-\hat{v}\rangle_{V^*\times V} + F(v) - F(\hat{v}) \geq 0 \qquad \forall v \in V\,. \tag{A.1}$$

where $B : V \mapsto V^*$ is bounded, linear, and monotone, and further $F : V \to \mathbb{R}\cup\{+\infty\}$ is a convex lower semicontinuous function that is supposed to be proper (i.e. $F \not\equiv \infty$ on $V$). This means that the effective domain of $F$ in the sense of convex analysis ([32]),

$$\operatorname{dom} F := \{v \in V : F(v) < +\infty\}$$

is nonempty, closed and convex.

Here we assume that the operator $B$ is strongly monotone, that is, there exists some $b_0 > 0$ such that

$$\langle Bv, v\rangle_{V^*\times V} \geq b_0\,\|v\|^2 \qquad \forall v \in V\,. \tag{A.2}$$

Clearly strong monotonicity implies uniqueness of the solution $\hat{u}$ of (A.1). Note by the separation theorem it can be shown that $F$ is conically minorized, that is, it enjoys the estimate

$$F(v) \geq -c_F\,(1+\|v\|)\,, v \in V$$

with some $c_F > 0$. Hence strong monotonicity implies the asymptotic coercivity condition in [17], too. Thus the existence result [17, Theorem 5.2] applies to the bifunction $\varphi_B(u,v) := \langle Bu, v-u\rangle_{V^*\times V}$ to conclude the following

**Theorem A.1.** *Suppose (A.2). Then the VI (A.1) is uniquely solvable.*

Next we investigate the stability of the solution $\hat{v}$, the solution of (A.1) with respect to the extendend real-valued function $F$. So we are led to introduce the solution map $\mathbb{S}$ by $\mathbb{S}(F) := \hat{v}$, the solution of (A.1). Here we follow the concept of epi-convergence in the sense of Mosco [3] ("Mosco convergence"). Let $F_n$ $(n \in \mathbb{N}), F : V \to \mathbb{R}\cup\{+\infty\}$ be convex lower semicontinuous proper functions. Then $F_n$ are called to converge to $F$ in the Mosco sense, written $F_n \xrightarrow{\mathrm{M}} F$, if and only if the subsequent two hypotheses hold:

(M1) If $v_n \in V$ $(n \in \mathbb{N})$ weakly converge to $v$ for $n \to \infty$, then

$$F(v) \leq \liminf_{n\to\infty} F_n(v_n)\,.$$

(M2) For any $v \in V$ with $F(v) < \infty$ there exist $v_n \in V$ $(n \in \mathbb{N})$ strongly converging to $v$ for $n \to \infty$ such that

$$F(v) = \lim_{n\to\infty} F_n(v_n)\,.$$

In view of our applications it is not hard to require that the functions $F_n$ are uniformly conically minorized, that is, there holds the estimate

$$F_n(v) \geq -d_0\,(1+\|v\|),\ \forall n \in \mathbb{N}, v \in V \tag{A.3}$$

with some $d_0 \geq 0$.

**Theorem A.2.** *Suppose that the bounded linear operator $B$ is strongly monotone with constant $b_0 > 0$. Let $F, F_n : V \to \mathbb{R} \cup \{+\infty\}$ $(n \in \mathbb{N})$ be convex lower semicontinuous proper functions that are uniformly conically minorized; let $F_n \xrightarrow{\mathrm{M}} F$. Then strong convergence $\mathbb{S}(F_n) \to \mathbb{S}(F)$ holds.*

**Proof.** We divide the proof in three parts. We first show that the $\hat{u}_n = \mathbb{S}(F_n)$ are bounded, before we can establish the convergence result. In the following $c_0, c_1, \dots$ are generic positive constants.

**(1)** *The sequence $\{\hat{u}_n\} \subset V$ is bounded..*

By definition, $\hat{u}_n$ satisfies for all $v \in V$,

$$\langle B\hat{u}_n, v - \hat{u}_n\rangle + F_n(v) - F_n(\hat{u}_n) \geq 0\,. \tag{A.4}$$

Now let $v_0$ be an arbitrary element of dom $F$. Then by Mosco convergence, (M2), there exist $v_n \in \text{dom } F_n$ $(n \in \mathbb{N})$ such that for $n \to \infty$ the (strong) convergences hold

$$v_n \to v_0;\, F_n(v_n) \to F(v_0)\,. \tag{A.5}$$

Let $n \in \mathbb{N}$. Then insert $v = v_n$ in (A.4) and obtain

$$\langle B\hat{u}_n, \hat{u}_n - v_n\rangle \leq F_n(v_n) - F_n(\hat{u}_n)\,.$$

By the strong monotonicity of the operator $B$ and the estimate (A.3) we get

$$\begin{aligned} &b_0\|\hat{u}_n - v_n\|_V^2 \\ &\leq \|Bv_n\|_{V^*}\|\hat{u}_n - v_n\|_V + F_n(v_n) + d_0(1 + \|\hat{u}_n\|_V)\,. \end{aligned} \tag{A.6}$$

By the convergences (A.5), $|F_n(u_n)| \leq c_0, \|Bv_n\|_{V^*} \leq c_1, \|v_n\|_V \leq c_2$. Thus (A.6) results in

$$b_0\|\hat{u}_n - v_n\|_V^2 \leq c_0 + c_1\|\hat{u}_n - v_n\|_V + d_0(1 + \|\hat{u}_n\|_V)\,.$$

Hence the elementary chain of inequalities,

$$x^2 \leq ax + b \Rightarrow x \leq a + \sqrt{b}, \quad \forall x, a, b \geq 0$$

proves the claimed boundedness of $\{\hat{u}_n\}$.

**(2)** *$\hat{u}_n = \mathbb{S}(F_n)$ converges weakly to $\hat{u} = \mathbb{S}(F)$ for $n \to \infty$ .*

To prove this claim we employ a "Minty trick" similar to the proof of [17, Prop.3.2] using the monotonicity of the operator $B$.

Take $v \in V$ arbitrarily. By $(M2)$ there exist $v_n \in V$ $(n \in \mathbb{N})$ such that

$$\lim_{n\to\infty} v_n = v;\, \lim_{n\to\infty} F_n(v_n) = F(v) \tag{A.7}$$

We test the inequality (A.4) with $v_n$, use the monotonicity of the operator $B$, and obtain

$$\langle Bv_n, v_n - \hat{u}_n\rangle \geq F_n(\hat{u}_n) - F_n(v_n)\,. \tag{A.8}$$

On the other hand, by the previous step, there exists a subsequence $\{\hat{u}_{n_k}\}_{k\in\mathbb{N}}$ that converges weakly to some $\tilde{u} \in \text{dom } F \subset V$. Thus the continuity of $B$, $(M1)$, and (A.7) entail together with

(A.8)

$$\begin{aligned}&\langle Bv, v-\tilde{u}\rangle \\ &= \lim_{k\to\infty} \langle Bv_{n_k}, v_{n_k} - \hat{u}_{n_k}\rangle \\ &\geq \liminf_{k\to\infty} F_n(\hat{u}_n) - \lim_{k\to\infty} F_n(v_n) \\ &\geq F(\tilde{u}) - F(v)\,.\end{aligned}$$

Hence for $v \in \operatorname{dom} F$ fixed, for arbitrary $s \in [0,1)$ and $w_s := v + s(\tilde{u} - v) \in \operatorname{dom} F$ inserted above, the convexity of $F$ implies after division by the factor $(1-s) > 0$

$$\langle Bw_s, v-\tilde{u}\rangle + F(v) \geq F(\tilde{u})\,.$$

Letting $s \to 1$, hence $w_s \to \tilde{u}$, $Bw_s \to B\tilde{u}$ results in

$$\langle B\tilde{u}, v-\tilde{u}\rangle + F(v) \geq F(\tilde{u}) \quad \forall v \in \operatorname{dom} F\,.$$

This shows by uniqueness that $\tilde{u} = \mathbb{S}(F)$ and the entire sequence $\{\hat{u}_n\}$ converges weakly to $\hat{u} = \mathbb{S}(F)$.

**(3)** *$\hat{u}_n = \mathbb{S}(F_n)$ converges strongly to $\hat{u} = \mathbb{S}(F)$ for $n \to \infty$.*

By $(M2)$ there exist $u_n \in V$ $(n \in \mathbb{N})$ such that

$$(i)\ \lim_{n\to\infty} u_n = \hat{u}; (ii)\ \lim_{n\to\infty} F_n(u_n) = F(\hat{u})\,. \tag{A.9}$$

Test the inequality (A.4) with $u_n$, use the strong monotonicity of the operator $B$, and obtain

$$\langle Bu_n, u_n - \hat{u}_n\rangle + F_n(u_n) - F_n(\hat{u}_n) \geq b_0 \|u_n - \hat{u}_n\|^2\,. \tag{A.10}$$

Analyze the summands in (A.10) separately: By (A.9) (i), $Bu_n \to B\hat{u}$, hence

$$\lim_{n\to\infty} \langle Bu_n, u_n - \hat{u}_n\rangle = 0\,.$$

By (A.9) (ii) and by $(M1)$,

$$\limsup_{n\to\infty} [F_n(u_n) - F_n(\hat{u}_n)] \leq 0\,.$$

Thus from (A.10) finally by the triangle inequality,

$$0 \leq \|\hat{u}_n - \hat{u}\| \leq \|\hat{u}_n - u_n\| + \|u_n - \hat{u}\| \to 0$$

and the theorem is proved. □

We also need the following result on Mosco convergence of the sum of two convex lower semicontinuous proper functions.

**Lemma A.3.** *Let $F_{i,n}$ $(n \in \mathbb{N}), F_i : V \to \mathbb{R} \cup \{+\infty\}$ $(i = 1,2)$ be convex lower semicontinuous proper functions. Suppose that for $n \to \infty$, $F_{1,n} \xrightarrow{\mathrm{M}} F_1$; $(F_{2,n}; F_2)$ satisfies $(M1)$ and there holds*

*(C) $F_2(v) < \infty$ and $w_n \to w$ in $V$ for $n \to \infty$ imply $F_2(w) = \lim_{n\to\infty} F_{2,n}(w_n)$.*

*Then $F_n := F_{1,n} + F_{2,n} \xrightarrow{\mathrm{M}} F := F_1 + F_2$.*

*Proof.* To show (M1) for $(F_n;F)$ let $v_n \rightharpoonup v$ (weak convergence) in $V$. Then by (M1) for $(F_{1,n};F_1)$ and $(F_{2,n};F_2)$,

$$\begin{aligned} F(v) &= F_1(v) + F_2(v) \\ &\leq \liminf_{n\to\infty} F_{1,n}(v_n) + \liminf_{n\to\infty} F_{2,n}(v_n) \\ &\leq \liminf_{n\to\infty} [F_{1,n}(v_n) + F_{2,n}(v_n)] \\ &= \liminf_{n\to\infty} F_n(v_n)\,. \end{aligned}$$

To show (M2) for $(F_n;F)$ let $v \in V$ with $F(v) < \infty$. Then $F_i(v) < \infty; i = 1,2$. Since $(F_{1,n};F_1)$ satisfies (M2), there exist $v_n \in V$ such that $v_n \to v$ and $F_1(v) = \lim_{n\to\infty} F_{1,n}(v_n)$. By (C), also $F_2(v) = \lim_{n\to\infty} F_{2,n}(v_n)$ and the conclusion follows. □

We can include in the above stability Theorem A.2 also the Mosco convergence [3] of a sequence $\{K_n\}_{n\in\mathbb{N}}$ of nonvoid convex closed subsets of $V$ towards a nonvoid convex closed subset $K$ of $V$, written $K_n \xrightarrow{\text{M}} K$, satisfying the subsequent two hypotheses:

(m1) If $\{v_n\}_{n\in\mathbb{N}} \subset V$ weakly converges to $v$ and for all $k \in \mathbb{N}$, $v_{n_k}$ lies in $K_{n_k}$ for a subsequence $\{K_{n_k}\}_{k\in\mathbb{N}}$ of the sequence $\{K_n\}_{n\in\mathbb{N}}$, then the weak limit $v$ belongs to $K$.

(m2) For any $v \in K$ there exists a sequence $\{v_n\}_{n\in\mathbb{N}}$ strongly converging to $v$ such that $v_n$ lies in $K_n$ for all large $n$.

To obtain a stability result applicable to the mixed random VIs treated in the main part of the paper we introduce the indicator function on $K$ in the sense of convex analysis ([32]):

$$\chi_K(v) := \begin{cases} 0 & \text{if } v \in K \\ +\infty & \text{elsewhere} \end{cases}$$

Now we set

$$F(v) := f(v) + \chi_K(v)$$

where $f : V \to \mathbb{R}$ is a convex continuous function, and the VI (A.1) becomes: Find an element $\hat{v} \in K$ such that

$$\langle B\hat{v}, v - \hat{v}\rangle_{V^*\times V} + f(v) - f(\hat{v}) \geq 0 \qquad \forall v \in K\,. \tag{A.11}$$

Likewise unique solvability of the VI (A.11) leads to consider the solution map $\tilde{\mathbb{S}}$ by $\tilde{\mathbb{S}}(f,K) := \hat{v}$, the solution of (A.11). Now we are in the position to state the following consequence.

**Corollary A.4.** *Suppose that the bounded linear operator $B$ is strongly monotone with constant $b_0 > 0$. Let $K, K_n$ be convex closed subsets of $V$ and let $f, f_n : V \to \mathbb{R}$ $(n \in \mathbb{N})$ be convex continuous functions that are uniformly conically minorized. Suppose $K_n \xrightarrow{\text{M}} K$ and $f_n \xrightarrow{\text{M}} f$. Moreover assume that $(f_n; f)$ satisfies (C), that is, $w_n \to w$ in $V$ for $n \to \infty$ implies $f(w) = \lim_{n\to\infty} f_n(w_n)$. Then strong convergence $\tilde{\mathbb{S}}(f_n,K_n) \to \tilde{\mathbb{S}}(f,K)$ holds.*

**Proof.** Set

$$F_n(v) := f_n(v) + \chi_{K_n}(v)\,.$$

Then by Lemma A.3, $F_n \xrightarrow{\text{M}} F := f + chi_K$. Finally apply Theorem A.2. □