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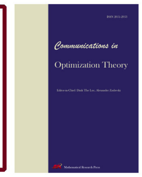

A NOTE ON GLOBAL STABILITY OF EQUILIBRIA FOR LOCALLY LIPSCHITZ MAPS

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Dedicated to Professor Francis Clarke on the occasion of his 75th birthday

Abstract. Let $F\colon \mathbb{R}^p \to \mathbb{R}^p$ be a locally Lipschitz map with $F(0) = 0$. Denote by $\partial F(x)$ the generalized Jacobian of F at x. We show that the following three statements hold true:

(1) Assume that for any $x \in \mathbb{R}^p$ and any $Z \in \partial F(x)$, the symmetric part of Z (i.e., the matrix $\frac{Z+Z^T}{2}$), is negative definite. Then the map F is injective and every solution of the autonomous system $\dot{x}(t) = F(x(t))$ goes to 0, as t tends to $+\infty$.

(2) Assume that for any $x \in \mathbb{R}^p$ and any $Z \in \partial F(x)$, the spectrum radius of the matrix $Z^T Z$ is less than 1. Then 0 is the unique fixed point of the map F and every orbit of the discrete dynamical system $x_{n+1} = F(x_n)$ goes to 0, as n tends to $+\infty$.

(3) Assume that for any $x \in \mathbb{R}^p$ and any $Z \in \partial F(x)$, the symmetric part of Z is negative definite. Then, for any $x_0 \in \mathbb{R}^p$, there exists a real number $h_0 > 0$ such that for every $h \in (0, h_0)$, the orbit of the discrete dynamical system $x_{n+1} = x_n + hF(x_n)$ converges to 0, as n goes to $+\infty$.

These results strengthen those obtained by Furi, Martelli, and O'Neill in [J. Difference Equ. Appl., 15(4):387–397, 2009], which requires further that the map F is Gateaux differentiable except possibly on a linearly countable set.

Keywords. Equilibria; Global stability; Generalized Jacobian; Locally Lipschitz maps.

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1. Introduction

Let $F\colon \mathbb{R}^p \to \mathbb{R}^p$ be a C^1 map with $F(0) = 0$. In a paper published in 1960, Markus and Yamabe [12] conjectured that *if all the eigenvalues of the Jacobian matrix of F at each point $x \in \mathbb{R}^p$ have negative real parts, then the origin $0 \in \mathbb{R}^p$ is a globally asymptotically solution of the autonomous system*

$$\dot{x}(t) \quad = \quad F(x(t))$$

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i.e., every solution $x(t)$ of the system exists for large t and $x(t) \to 0$ as $t \to +\infty$.

Nowadays, it is well-known that the Markus–Yamabe conjecture is true in $\mathbb{R}^2$ (see [5, 7, 8]), and false in $\mathbb{R}^p$ for $p \geq 3$ (see [2]).

In 1976, La Salle [11] proposed a similar result for discrete dynamical systems by conjecturing that *if all the eigenvalues of the Jacobian matrix of F at each point $x \in \mathbb{R}^p$ have modulus less than one, then every orbit of the discrete dynamical system*

$$x_{n+1} = F(x_n)$$

converges to the origin as n tends to $+\infty$, i.e., $x_n \to 0$ as $n \to +\infty$.

The conjecture of La Salle is false even in $\mathbb{R}^2$ (see [13]). However, it is true in $\mathbb{R}^2$ for polynomial maps (see [4]).

In 1961, Hartman [9] proved the global convergence conjectured by Markus–Yamabe, when all the eigenvalues of the symmetric part of the Jacobian matrix of F are negative. In 2009, Furi, Martelli, and O'Neill [6] showed that Hartman's condition can be considerably relaxed by assuming that F is locally Lipschitz and is Gateaux differentiable except possibly on a linearly countable set. Furthermore, they establish a companion result for discrete dynamical systems.

In this paper, we show that the results of Furi, Martelli, and O'Neill mentioned above can be extended for nondiferentiable locally Lipschitz maps.

The rest of this paper is organized as follows. In Section 2 we recall some definitions and preliminary results from nonsmooth analysis and the generalized Jacobian. The main results are given in Section 3.

2. Preliminaries

2.1. **Notation.** In this paper, we deal with the Euclidean space $\mathbb{R}^p$ equipped with the usual scalar product $\langle \cdot, \cdot \rangle$ and the corresponding Euclidean norm $\| \cdot \|$. We use $\mathbb{S}_r$ and $\mathbb{B}_r$ to denote the sphere and the closed ball, respectively, in $\mathbb{R}^p$ with center at the origin and radius r. The convex hull of a set $\Omega \subset \mathbb{R}^p$ will be written as $\mathrm{co}\Omega$. For two points $x, y \in \mathbb{R}^p$, the notation $[x, y]$ stands for the set of points $(1-t)x + ty$ with $t \in [0, 1]$. For a square matrix A, let A^T denote the transpose of A and call $\frac{A+A^T}{2}$ the *symmetric part* of A.

2.2. **The generalized Jacobian.** Let $F\colon \Omega \to \mathbb{R}^q$ be a locally Lipschitz map, where Ω is an open subset of $\mathbb{R}^p$. By Rademacher's Theorem (see, for example, [15]), F is almost everywhere differentiable (in the sense of Lebesgue measure) on Ω. Let Ω_F be the set of points at which the map F fails to be differentiable. For each $x \in \Omega \setminus \Omega_F$, we shall write $JF(x)$ for the usual $p \times q$ Jacobian matrix of F. Clarke [3] defined the *generalized Jacobian* of F at $x \in \Omega$, denoted by $\partial F(x)$, is the convex hull of all matrices which are limits of Jacobian matrices $JF(x')$ as $x' \to x$ with F being differentiable at x'. Symbolically, then, one has

$$\partial F(x) \quad := \quad \mathrm{co}\{\lim JF(x') \mid x' \to x \text{ and } x' \not\in \Omega_F\}.$$

We have the following results (see [3, Propositions 2.6.2 and 2.6.5]).

Proposition 2.1. *Let $F\colon \Omega \to \mathbb{R}^q$ be a locally Lipschitz map, where Ω is an open subset of $\mathbb{R}^p$. Then the following statements hold:*

(i) *$\partial F(x)$ is a nonempty compact convex subset of $\mathbb{R}^{p \times q}$.*
(ii) *∂F is closed at x; that is, if $x_n \to x, Z_n \in \partial F(x_n)$ and $Z_n \to Z$, then $Z \in \partial F(x)$.*

(iii) *∂F is upper semicontinuous at x: for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $y \in \mathbb{R}^p$ with $\|y-x\| < \delta$,*

$$\partial F(y) \quad \subset \quad \partial F(x) + \mathscr{B}_\varepsilon,$$

where $\mathscr{B}_\varepsilon$ is the closed ball in $\mathbb{R}^{p\times q}$ with center at the origin and radius ε.

Proposition 2.2. *Let $F : \Omega \to \mathbb{R}^q$ be a locally Lipschitz map, where Ω is an open convex subset of $\mathbb{R}^p$. Then for any $x, y \in \Omega$, we have*

$$F(x) - F(y) \quad \in \quad \mathrm{co}\,\partial F([x,y])(x-y).$$

The right-hand side above denotes the convex hull of all points of the form $Z(x-y)$, where $Z \in \partial F(u)$ for some point u in $[x,y]$. Since

$$[\mathrm{co}\,\partial F([x,y])](x-y) \quad = \quad \mathrm{co}\,[\partial F([x,y])(x-y)],$$

there is no ambiguity.

3. Results

The following three theorems strengthen those obtained by Furi, Martelli, and O'Neill in [6], which requires further that the map in question is Gateaux differentiable except possibly on a linearly countable set.

Theorem 3.1. *Let $F : \mathbb{R}^p \to \mathbb{R}^p$ be a locally Lipschitz map with $F(0) = 0$. Assume that for any $x \in \mathbb{R}^p$ and any $Z \in \partial F(x)$, the symmetric part of Z is negative definite. Then the map F is injective and every solution of the autonomous system*

$$\dot{x}(t) \quad = \quad F(x(t)) \tag{3.1}$$

goes to 0, as t goes to $+\infty$.

Proof. Let $x, y \in \mathbb{R}^p$ with $x \neq y$. Combining Proposition 2.2 with Caratheodory's Theorem, we get nonnegative real numbers λ_k with $\sum_{k=1}^{p+1} \lambda_k = 1$, points $u_k \in [x,y]$ and matrices $Z_k \in \partial F(u_k)$ such that

$$F(x) - F(y) \quad = \quad \sum_{k=1}^{p+1} \lambda_k Z_k (x-y).$$

Observe that

$$\langle x-y, Z_k(x-y)\rangle \quad = \quad \frac{1}{2}\langle x-y, (Z_k + Z_k^T)(x-y)\rangle \;<\; 0,$$

where the inequality follows from the assumption that the symmetric part of Z_k is negative definite. Therefore,

$$\langle x-y, F(x)-F(y)\rangle \quad = \quad \sum_{k=1}^{p+1} \lambda_k \langle x-y, Z_k(x-y)\rangle \;<\; 0.$$

Consequently, $F(x) \neq F(y)$, and so F is injective. Moreover, by letting $y = 0$ in the above inequality, we get

$$\langle x, F(x)\rangle \quad < \quad 0.$$

To show the second statement of the theorem, take any $x_0 \in \mathbb{R}^p$, and let $x(t)$ be the unique solution of (3.1) such that $x(0) = x_0$. The assumption that F is locally Lipschitz on $\mathbb{R}^p$ insures that the solution of the system (3.1) is uniquely determined by its initial value x_0 and varies continuously with it (see, for example, [1, 14]). In particular, if $x(t) = 0$ for some $t_* \geq 0$, then $x(t) = 0$ for all $t \geq t_*$ and so the second claim follows immediately. Assume that $x(t) \neq 0$ for all $t \geq 0$. We have

$$\frac{d\|x(t)\|^2}{dt} \ = \ 2\langle x(t), \dot{x}(t)\rangle \ = \ 2\langle x(t), F(x(t))\rangle \ < \ 0,$$

and so the function $t \mapsto \|x(t)\|$ is strictly decreasing. Thus, the following limit exists

$$r \ := \ \lim_{t \to +\infty} \|x(t)\| \ \geq \ 0.$$

We must show that $r = 0$. By contradiction, assume that $r > 0$. Let $\omega(x_0)$ be the (positive) limit set of the solution $x(\cdot)$, i.e.,

$$\omega(x_0) \ := \ \{y \in \mathbb{R}^p \mid \exists t_k \to +\infty \quad \text{such that} \quad x(t_k) \to y\}.$$

Then it is easy to see that $\omega(x_0)$ is nonempty and is contained in the sphere $\mathbb{S}_r$. Furthermore, it is well-known that the set $\omega(x_0)$ is *positively invariant*[1] (see, for example, [1, Theorem 17.2] or [10, Proposition, page 202]). Now pick a point $y \in \omega(x_0)$ and let $u(t)$ be the unique solution of (3.1) such that $u(0) = y$. Then for all $t \geq 0$ we have $u(t) \in \omega(x_0)$, and so $\|u(t)\| = r$. This, however, is impossible, since the function $t \mapsto \|u(t)\|$ is strictly decreasing. □

Theorem 3.2. *Let $F : \mathbb{R}^p \to \mathbb{R}^p$ be a locally Lipschitz map with $F(0) = 0$. Assume that for any $x \in \mathbb{R}^p$ and any $Z \in \partial F(x)$, the spectrum radius of $Z^T Z$ is less than 1. Then 0 is the unique fixed point of the map F and every orbit of the discrete dynamical system*

$$x_{n+1} \ = \ F(x_n)$$

goes to 0, as n goes to $+\infty$.

Proof. Take any $x \in \mathbb{R}^p \setminus \{0\}$. In view of Proposition 2.2, $F(x) \in \mathrm{co}\,\partial F([0,x])x$. According to Caratheodory's Theorem, there exist nonnegative real numbers λ_k with $\sum_{k=1}^{p+1} \lambda_k = 1$, points $u_k \in [0,x]$ and matrices $Z_k \in \partial F(u_k)$ such that

$$F(x) \ = \ \sum_{k=1}^{p+1} \lambda_k Z_k x.$$

The assumption that the spectrum radius of $Z_k^T Z_k$ is less than 1 implies that

$$\|Z_k x\|^2 \ = \ \langle Z_k x, Z_k x\rangle \ = \ \langle x, Z_k^T Z_k x\rangle \ < \|x\|^2.$$

Therefore,

$$\|F(x)\| \ \leq \ \sum_{k=1}^{p+1} \lambda_k \|Z_k x\| \ < \ \|x\|. \tag{3.2}$$

Consequently, 0 is the unique fixed point of F.

[1]Recall that a set $\Omega \subset \mathbb{R}^n$ is called *positively invariant* if for each $y \in \Omega$, the unique solution of (3.1) that passes through y at $t = 0$ is defined and in Ω for all $t \geq 0$.

To show the second of the theorem, take any $x_0 \in \mathbb{R}^p$ and consider the orbit $O(x_0)$ defined by $x_{n+1} := F(x_n)$ for $n = 0, 1, \ldots$. If $x_n = 0$ for some integer $n_* \geq 0$, then $x_n = 0$ for all $n \geq n_*$ and so the desired conclusion holds. Assume that $x_n \neq 0$ for all n. The inequality (3.2) implies that

$$\|x_{n+1}\| \quad < \quad \|x_n\|.$$

Hence, the sequence $\{\|x_n\|\}_{n\geq 0}$ is strictly decreasing, and so it converges to a value $r \geq 0$. We must show that $r = 0$.

Suppose to the contrary that $r > 0$. Let $L(x_0)$ be the set of limit points of the orbit $O(x_0)$, i.e.,

$$L(x_0) \quad := \quad \{y \in \mathbb{R}^p \mid \exists n_k \to +\infty \quad \text{such that} \quad x_{n_k} \to y\}.$$

Then it is easy to see that $L(x_0)$ is a nonempty compact subset of the sphere $\mathbb{S}_r$. Take any $x_* \in L(x_0)$. By definition, there is a sequence n_k tending to infinity such that x_{n_k} converges to x_*. Then $x_{n_k+1} = F(x_{n_k})$ converges to $F(x_*)$, which yields $F(x_*) \in L(x_0)$. (In fact, we can show that $F(L(x_0)) = L(x_0)$. Since we do not use this fact, we leave the proof to the reader.) Observe from (3.2) that $\|F(x_*)\| < \|x_*\| = r$. This, however, contradicts $F(x_*) \in L(x_0) \subset \mathbb{S}_r$. Therefore, $r = 0$, which yields the origin is the only limit point of the orbit $O(x_0)$, i.e. $L(x_0) = \{0\}$. □

Theorem 3.3. *Let $F : \mathbb{R}^p \to \mathbb{R}^p$ be a locally Lipschitz map with $F(0) = 0$. Assume that for any $x \in \mathbb{R}^p$ and any $Z \in \partial F(x)$, the symmetric part of Z is negative definite. Then, given an initial point $x_0 \in \mathbb{R}^p$, there exists a real number $h_0 := h(x_0) > 0$ such that for any $h \in (0, h_0)$, the orbit of the discrete dynamical system*

$$x_{n+1} \quad = \quad x_n + hF(x_n)$$

converges to 0, as n goes to $+\infty$.

Proof. Take any $x_0 \in \mathbb{R}^p$. Obviously, the conclusion is trivial if $x_0 = 0$. Thus, without loss of generality, we assume that $x_0 \neq 0$. Let $r := \|x_0\| > 0$ and

$$M_1 \quad := \quad \sup\{\langle x, Zx\rangle \mid \|x\| = r \text{ and } Z \in \operatorname{co}\partial F([0,x])\}.$$

Since the symmetric part of Z is negative definite for all $Z \in \partial F(x)$ and all $x \in \mathbb{R}^p$, it follows easily from Proposition 2.1 that $M_1 < 0$. Observe that if $x \in \mathbb{B}_r \setminus \{0\}$, then $[0,x] \subset [0, \frac{r}{\|x\|}x]$. By definition, therefore

$$\langle x, Zx\rangle \quad \leq \quad \frac{M_1}{r^2}\|x\|^2 \tag{3.3}$$

for all $x \in \mathbb{B}_r$ and all $Z \in \operatorname{co}\partial F([0,x])$.

Since the map F is locally Lipschitz, it is globally Lipschitz on every compact subset of $\mathbb{R}^p$. (This fact may follows from Propositions 2.1 and 2.2.) In particular, we can find a constant $M_2 > 0$ such that

$$\|F(x)\| \quad = \quad \|F(x) - F(0)\| \;\leq\; M_2\|x\| \tag{3.4}$$

for all $x \in \mathbb{B}_r$. Let

$$h_0 := -\frac{2M_1}{r^2(M_2^2+1)}.$$

Take any $h \in (0, h_0)$ and consider the dynamical system

$$x_{n+1} \quad = \quad x_n + hF(x_n).$$

In order to prove the theorem, it suffices to consider the case $x_n \neq 0$ for all $n \geq 0$.

According to Proposition 2.2, there exists $Z \in \operatorname{co} \partial F([0, x_n])$ such that $F(x_n) = Z x_n$. Hence

$$\begin{aligned} \|x_{n+1}\|^2 &= \|x_n\|^2 + 2h\langle x_n, F(x_n)\rangle + h^2\|F(x_n)\|^2 \\ &= \|x_n\|^2 + 2h\langle x_n, Zx_n\rangle + h^2\|F(x_n)\|^2. \end{aligned}$$

From the inequalities (3.3) and (3.4), we know that if $x_n \in \mathbb{B}_r$, then

$$\langle x_n, Zx_n\rangle \leq \frac{M_1}{r^2}\|x_n\|^2 \quad \text{and} \quad \|F(x_n)\| \leq M_2\|x_n\|,$$

which yield

$$\|x_{n+1}\|^2 \leq \|x_n\|^2 + 2h\frac{M_1}{r^2}\|x_n\|^2 + h^2M_2^2\|x_n\|^2 < \|x_n\|^2.$$

Since $x_0 \in \mathbb{B}_r$, it follows that $\|x_1\| < \|x_0\| = r$. By induction, then $\|x_{n+1}\| < \|x_n\|$, i.e., the sequence $\{\|x_n\|\}_{n\geq 0}$ is strictly decreasing. Finally, the reasoning of Theorem 3.2 can be applied to complete the proof of the theorem. □

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