Commun. Optim. Theory 2022 (2022) 24 https://doi.org/10.23952/cot.2022.24

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Communications in Optimization Theory

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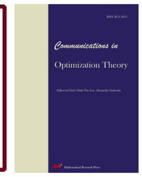

TURNPIKE PHENOMENON FOR SYMMETRIC VARIATIONAL PROBLEMS

ALEXANDER J. ZASLAVSKI

Department of Mathematics, The Technion – Israel Institute of Technology, Haifa, Israel

Dedicated to the memory of Professor Ram Verma

Abstract. In our recent research we studied the turnpike phenomenon for a class of symmetric variational problems. For this class of problems integrands possess two points of minimum and a certain well-posedness property holds. In this paper, we show that some versions of the turnpike property hold if a set of minimizers of an integrand is finite.

Keywords. Integrand; Symmetry; Turnpike; Variational problem.

1. Introduction

The study of the existence and the structure of solutions of variational problems, optimal control problems and dynamic games defined on infinite intervals and on sufficiently large intervals has been a rapidly growing area of research [3, 4, 10, 11, 16, 19, 21, 22, 24, 37, 39, 40, 47, 66, 69, 71, 74, 75, 76, 79] which has various applications in engineering [1, 16, 66], in models of economic growth [2, 15, 16, 20, 25, 35, 36, 38, 42, 46, 52, 60, 61, 64, 66, 76], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [5, 65], in model predictive control [18, 27] and in the theory of thermodynamical equilibrium for materials [17, 43, 49, 50, 51]. Discrete-time problems optimal control problems were considered in [3, 6, 7, 14, 23, 33], finite-dimensional continuous-time problems were analyzed in [10, 12, 13, 42, 45, 48, 56, 70, 77, 78], infinite-dimensional optimal control was studied in [16, 28, 29, 30, 54, 55, 57, 59, 62, 63, 80] while solutions of dynamic games were discussed in [9, 26, 31, 34, 41, 58, 68, 72, 73].

In this paper we study the turnpike phenomenon for symmetric variational problems in infinite dimensional spaces. To have the turnpike property means, roughly speaking, that the approximate solutions of the problems are determined mainly by the objective function and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints.

E-mail address: ajzasl@technion.ac.il.

Received: August 10, 2022; Accepted: September 23, 2022.

The turnpike property was discovered by P. Samuelson in 1948 when he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). It is well known in the economic literature, where it was studied for various models of economic growth. Usually for these models a turnpike is a singleton.

Now it is well-known that the turnpike property is a general phenomenon which holds for large classes of variational and optimal control problems. In our research, using the Baire category (generic) approach, it was shown that the turnpike property holds for a generic (typical) variational problem [66] and for a generic optimal control problem [70].

In this paper we are interested in individual (non-generic) turnpike results for symmetric variational problems. These problems have applications in crystallography [32, 53, 67]. In our recent research [81] we studied the turnpike phenomenon for a class of symmetric variational problems with integrands possessing two points of minimum and a certain well-posedness property. In this paper, we show that some versions of the turnpike property hold if a set of minimizers of an integrand is finite.

2. Banach space valued functions

In this section we present preliminaries which we need in order to study turnpike properties of infinite dimensional variational problems.

Let $(X, \|\cdot\|)$ be a Banach space and $a < b$ be real numbers. For any set $E \subset R^1$ define

$$\chi_E(t) = 1 \text{ for all } t \in E \text{ and } \chi_E(t) = 0 \text{ for all } t \in R^1 \setminus E.$$

If a set $E \subset R^1$ is Lebesgue measurable, then its Lebesgue measure is denoted by $|E|$ or by $\mathrm{mes}(E)$.

A function $f : [a,b] \to X$ is called a simple function if there exists a finite collection of Lebesgue measurable sets $E_i \subset [a,b]$, $i \in I$, mutually disjoint, and $x_i \in X$, $i \in I$ such that

$$f(t) = \sum_{i\in I} \chi_{E_i}(t)x_i,\ t \in [a,b].$$

A function $f : [a,b] \to X$ is strongly measurable if there exists a sequence of simple functions $\phi_k : [a,b] \to X$, $k = 1,2,\dots$ such that

$$\lim_{k\to\infty} \|\phi_k(t) - f(t)\| = 0,\ t \in [a,b] \text{ almost everywhere (a. e.).} \tag{2.1}$$

For every simple function $f(\cdot) = \sum_{i\in I} \chi_{E_i}(\cdot)x_i$, where the set I is finite, define its Bochner integral by

$$\int_a^b f(t)dt = \sum_{i\in I} |E_i|x_i.$$

Let $f : [a,b] \to X$ be a strongly measurable function. We say that f is Bochner integrable if there exists a sequence of simple functions $\phi_k : [a,b] \to X$, $k = 1,2,\dots$ such that (2.1) holds and the sequence $\{\int_a^b \phi_k(t)dt\}_{k=1}^\infty$ strongly converges in X. In this case we define the Bochner integral of the function f by

$$\int_a^b f(t)dt = \lim_{k\to\infty} \int_a^b \phi_k(t)dt.$$

It is known that the integral defined above is independent of the choice of the sequence $\{\phi_k\}_{k=1}^{\infty}$ [44]. Similar to the Lebesgue integral, for any measurable set $E \subset [a,b]$, the Bochner integral of f over E is defined by

$$\int_E f(t)dt = \int_a^b \chi_E(t) f(t)dt.$$

The following result is true (see Proposition 3.4, Chapter 2 of [44]).

Proposition 2.1. *Let $f : [a,b] \to X$ be a strongly measurable function. Then f is Bochner integrable if and only if the function $\|f(\cdot)\|$ is Lebesgue integrable. Moreover, in this case*

$$\|\int_a^b f(t)dt\| \le \int_a^b \|f(t)\|dt.$$

The Bochner integral possesses almost the same properties as the Lebesgue integral. If $f : [a,b] \to X$ is strongly measurable and $\|f(\cdot)\| \in L^p(a,b)$, for some $p \in [1,\infty)$, then we say that $f(\cdot)$ is L^p Bochner integrable. For every $p \ge 1$, the set of all L^p Bochner integrable functions is denoted by $L^p(a,b;X)$ and for every $f \in L^p(a,b;X)$,

$$\|f\|_{L^p(a,b;X)} = (\int_a^b \|f(t)\|^p dt)^{1/p}.$$

Clearly, the set of all Bochner integrable functions on $[a,b]$ is $L^1(a,b;X)$.

Let $a < b$ be real numbers. A function $x : [a,b] \to X$ is absolutely continuous (a. c.) on $[a,b]$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each pair of sequences $\{t_n\}_{n=1}^{q}$, $\{s_n\}_{n=1}^{q} \subset [a,b]$ satisfying

$$t_n < s_n,\ n = 1,\dots,q,\ \sum_{n=1}^{q}(s_n - t_n) \le \delta,$$

$$(t_n,s_n) \cap (t_m,s_m) = \emptyset \text{ for all } m,n \in \{1,\dots,q\} \text{ such that } m \ne n$$

we have

$$\sum_{n=1}^{q} \|x(t_n) - x(s_n)\| \le \varepsilon.$$

The following result is true (see Theorem 1.124 of [8]).

Proposition 2.2. *Let X be a reflexive Banach space. Then every a. c. function $x : [a,b] \to X$ is a. e. differentiable on $[a,b]$ and*

$$x(t) = x(a) + \int_a^t (dx/dt)(s)ds,\ t \in [a,b]$$

where $dx/dt \in L^1(a,b;X)$ is the strong derivative of x.

Let $-\infty < \tau_1 < \tau_2 < \infty$. Denote by $W^{1,1}(\tau_1,\tau_2;X)$ (or $W^{1,1}(\tau_1,\tau_2)$ if the space X is understood) the set of all functions $x : [\tau_1,\tau_2] \to X$ for which there exists a Bochner integrable function $u : [\tau_1,\tau_2] \to X$ such that for all $t \in (\tau_1,\tau_2]$,

$$x(t) = x(\tau_1) + \int_{\tau_1}^t u(s)ds.$$

3. Symmetric variational problems

In this section we begin to study the turnpike properties for symmetric variational problems in Banach spaces. To have the turnpike property means, roughly speaking, that the approximate solutions of the problems are determined mainly by the integrand and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints.

Assume that $(X, \|\cdot\|)$ is a Banach space. For each $x \in X$ and each $r > 0$ set

$$B(x,r) = \{y \in X : \ \|y-x\| \le r\}.$$

Suppose that the infimum over an empty set is ∞, the sum over an empty set is zero and denote by $\mathrm{Card}(C)$ the cardinality of a set C.

Assume that $f : X \times X \to R^1$ is a bounded from below borelian function such that

$$f(x,y) = f(x,-y) \text{ for all } x,y \in X, \tag{3.1}$$

m is a natural number and there exists $(\bar{x}_i, \bar{y}_i) \in X \times X$, $i = 1,\dots,m$ such that

$$\bar{x}_i \ne \bar{x}_j \text{ for each } i,j \in \{1,\dots,m\} \text{ satisfying } i \ne j,$$

$$\inf(f) := \inf\{f(\xi,\eta) : \ \xi,\eta \in X\}, \tag{3.2}$$

$$\{(x,y) \in X \times X : \ f(x,y) = \inf(f)\} = \{(\bar{x}_i,\bar{y}_i),\ (\bar{x}_i,-\bar{y}_i) : \ i = 1,\dots,m\}. \tag{3.3}$$

(Note that it is possible that $\bar{y}_i = 0$ for some $i \in \{1,\dots,m\}$.)

Assume that the following assumptions hold:

(A1) for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $(x,y) \in X \times X$ satisfying

$$f(x,y) \le \inf(f) + \delta$$

there exists $i \in \{1,\dots,m\}$ such that the inequalities

$$\|x - \bar{x}_i\| \le \varepsilon$$

and

$$\min\{\|y-\bar{y}_i\|,\ \|y+\bar{y}_i\|\} \le \varepsilon$$

hold;

(A2) for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $i \in \{1,\dots,m\}$ and each $(x,y) \in X \times X$ satisfying

$$\|x-\bar{x}_i\| \le \delta,\ \|y-\bar{y}_i\| \le \delta$$

the inequality

$$f(x,y) \le f(\bar{x}_i,\bar{y}_i) + \varepsilon$$

is true.

Assumption (A2) means that the function f is continuous at the points $(\bar{x}_i,\bar{y}_i)$, $i = 1,\dots,m$ while assumption (A1) means that the minimization problem

$$f(x,y) \to \min,\ x,y \in X$$

is well posed in a generalized sense.

Let $a > 0$ and $\psi : [0,\infty) \to [0,\infty)$ be an increasing function satisfying

$$\lim_{t\to\infty} \psi(t) = \infty. \tag{3.4}$$

Assume that the following assumption holds:

(A3) the function f is bounded on all bounded sets and for each $(x,u) \in X \times X$,

$$f(x,u) \geq \psi(\|u\|)\|u\| - a.$$

For each pair of nonnegative numbers $T_1 < T_2$ and each $y,z \in X$ we consider the problems

$$\int_{T_1}^{T_2} f(x(t),x'(t))dt \to \min, \tag{P_{T_1,T_2}}$$

$$x \in W^{1,1}(T_1,T_2),$$

$$\int_{T_1}^{T_2} f(x(t),x'(t))dt \to \min, \tag{$P_{T_1,T_2,y}$}$$

$$x \in W^{1,1}(T_1,T_2),\ x(T_1) = y,$$

$$\int_{T_1}^{T_2} f(x(t),x'(t))dt \to \min, \tag{$P_{T_1,T_2,y,z}$}$$

$$x \in W^{1,1}(T_1,T_2),\ x(T_1) = y,\ x(T_2) = z$$

and define

$$U(T_1,T_2) = \inf\{\int_{T_1}^{T_2} f(x(t),x'(t))dt :\ x \in W^{1,1}(T_1,T_2)\},$$

$$U(T_1,T_2,y) = \inf\{\int_{T_1}^{T_2} f(x(t),x'(t))dt :\ x \in W^{1,1}(T_1,T_2),\ x(T_1) = y\},$$

$$U(T_1,T_2,y,z) = \inf\{\int_{T_1}^{T_2} f(x(t),x'(t))dt :$$

$$x \in W^{1,1}(T_1,T_2),\ x(T_1) = y,\ x(T_2) = z\}.$$

Let $i \in \{1,\dots,m\}$. There are two cases: $\bar{y}_i = 0$; $\bar{y}_i \neq 0$. If $\bar{y}_i = 0$, then for each $T_2 > T_1 \geq 0$, the function $x(t) = \bar{x}_i$, $t \in [T_1,T_2]$ is a solution of the problems (P_{T_1,T_2}), $(P_{T_1,T_2,\bar{x}_i})$, $(P_{T_1,T_2,\bar{x}_i,\bar{x}_i})$.

For each pair of numbers $T_1 < T_2$ and each $x \in W^{1,1}(T_1,T_2)$ set

$$I(T_1,T_2,x) = \int_{T_1}^{T_2} f(x(t),x'(t))dt.$$

Analogously to Theorem 5.1 of [81] we can prove the following result.

Theorem 3.1. *Let $T > 0$ and $i \in \{1,\dots,m\}$. Then*

$$U(0,T) = U(0,T,\bar{x}_i) = U(0,T,\bar{x}_i,\bar{x}_I) = Tf(\bar{x}_i,\bar{y}_i).$$

Moreover, for each $\varepsilon > 0$ there exists $x \in W^{1,1}(0,T)$ such that

$$x(0) = x(T) = \bar{x}_i,$$

$$I(0,T,x) \leq Tf(\bar{x}_i,\bar{y}_i) + \varepsilon,$$

$$\|x(t) - \bar{x}_i\| \leq \varepsilon,\ t \in [0,T],$$

$$x'(t) \in \{\bar{y}_i,\ -\bar{y}_I\},\ t \in [0,T] \text{ a. e.}.$$

Analogously to Theorem 5.2 of [81] we can prove the following result.

Theorem 3.2. *Let $L_0, M_0 > 0$. Then there exist $M_1 > 0$ such that for each $T > L_0$ and each $y,z \in X$ satisfying $\|y\|, \|z\| \leq M_0$ the inequality*

$$U(0,T,y,z) \leq Tf(\bar{x}_1,\bar{y}_1) + M_1$$

holds.

4. THE FIRST WEAK TURNPIKE RESULT

In this section we prove our first turnpike result. It shows that for approximate solutions x of our variational problems on intervals $[0,T]$, where T is sufficiently large and given values $x(0), x(T)$ at the end points belong to a given bounded set C, the Lebesgue measure of all points $t \in [0,T]$ such that $(x(t), x'(t))$ does not belong to an ε-neighborhood of the set $\{(\bar{x}_i, \bar{y}_i),\ (\bar{x}_i, -\bar{y}_i) :\ i = 1, \dots, m\}$ does not exceed a constant L which depends only on ε and the set C and does not depend on $T, x(0), x(T)$. In the literature this property is known as the weak turnpike property.

Theorem 4.1. *Let $\varepsilon \in (0,1)$ and $L_0, M_0, M_1 > 0$. Then there exists $L_1 > L_0$ such that for each $T > L_1$ and each $x \in W^{1,1}(0,T)$ such that*

$$x(0) \in B(0, M_0) \tag{4.1}$$

and at least one of the following conditions holds:

(a)

$$x(T) \in B(0, M_0),\ I^f(0,T,x) \le U(0,T,x(0),x(T)) + M_1;$$

(b)

$$I^f(0,T,x) \le U(0,T,x(0)) + M_1$$

the inequality

$$mes(\{t \in [0,T] :\ \max\{\|x(t) - \bar{x}_i\|,$$
$$\min\{\|x'(t) - \bar{y}_i\|,\ \|x'(t) + \bar{y}_i\|\}\} > \varepsilon \textit{ for each } i \in \{1, \dots, m\}\}) \le L_1.$$

Proof. Theorem 3.2 implies that there exists $M_2 > 0$ such that for each $T > L_0$ and each $y, z \in B(0, M_0)$,

$$U(0,T,y,z) \le T f(\bar{x}_1, \bar{y}_1) + M_2. \tag{4.2}$$

Assumption (A1) implies that there exists $\delta \in (0, \varepsilon)$ such that for each $(x,y) \in X \times X$ satisfying for each $i \in \{1, \dots, m\}$,

$$\max\{\|x - \bar{x}_i\| + \min\{\|y - \bar{y}_i\|,\ \|y + \bar{y}_i\|\}\} > \varepsilon \tag{4.3}$$

we have

$$f(x,y) > f(\bar{x}_i, \bar{y}_i) + \delta. \tag{4.4}$$

Set

$$L_1 = \max\{L_0,\ \delta^{-1}(M_1 + M_2)\} + 1. \tag{4.5}$$

Assume that $T > L_1$, $x \in W^{1,1}(0,T)$, (4.1) is true and at least one of conditions (a) and (b) holds. Conditions (a) and (b) and (4.1), (4.2), (4.5) imply that

$$I^f(0,T,x) \le T f(\bar{x}_1, \bar{y}_1) + M_1 + M_2. \tag{4.6}$$

Set

$$E = \{t \in [0,T] :\ f(x(t), x'(t)) > f(\bar{x}_1, \bar{y}_1) + \delta\}. \tag{4.7}$$

Equations (4.6) and (4.7) imply that

$$M_1 + M_2 + T f(\bar{x}_1, \bar{y}_1) \ge I(0,T,x)$$
$$= \int_E f(x(t), x'(t))dt + \int_{[0,T]\setminus E} f(x(t), x'(t))dt$$
$$\ge T f(\bar{x}_1, \bar{y}_1) + \delta \mathrm{mes}(E)$$

and in view of (4.5),

$$\mathrm{mes}(E) \le \delta^{-1}(M_1 + M_2) \le L_1.$$

Assume that

$$t \in [T_1, T_2] \setminus E.$$

By (4.7),

$$f(x(t), x'(t)) \le f(\bar{x}_1, \bar{y}_1) + \delta.$$

Combined with the choice of δ (see (4.3) and (4.4)) this implies that there exists $i \in \{1, \dots, m\}$ such that

$$\|x(t) - \bar{x}_i\| \le \varepsilon,$$

$$\min\{\|x'(t) + \bar{y}_i\| \ \|x'(t) + \bar{y}_i\|\} \le \varepsilon.$$

Theorem 4.1 is proved. □

5. Auxiliary results

Analogously to Proposition 5.4 of [81] we can prove the following result.

Proposition 5.1. *Let $\varepsilon, \Delta \in (0,1]$. Then there exist $\gamma \in (0, 2^{-1}\Delta)$ and $\delta > 0$ such that for each $k \in \{1, \dots, m\}$ and each*

$$y, z \in B(\bar{x}_k, \delta)$$

there exists $\xi \in W^{1,1}(0, \gamma)$ such that

$$\xi(0) = y, \ \xi(\gamma) = z,$$

$$I(0, \gamma, \xi) \le \gamma f(\bar{x}_k, \bar{y}_k) + \varepsilon,$$

$$\|\xi(t) - \bar{x}_k\| \le \varepsilon, \ t \in [0, \gamma],$$

$$B(\xi'(t), \varepsilon) \cap \{\bar{y}_k, \ -\bar{y}_k\} \ne \emptyset, \ t \in [0, \gamma] \ a.\ e..$$

Analogously to Proposition 5.5 of [81] we can prove the following result.

Proposition 5.2. *Let $\varepsilon, \Delta \in (0,1]$. Then there exist $\gamma \in (0, \Delta)$ and $\delta > 0$ such that for each $T > \gamma$, each $i \in \{1, \dots, m\}$ and each*

$$y, z \in B(\bar{x}_i, \delta)$$

there exists $\xi \in W^{1,1}(0, T)$ such that

$$\xi(0) = y, \ \xi(T) = z,$$

$$I(0, T, \xi) \le T f(\bar{x}_i, \bar{y}_i) + \varepsilon,$$

$$\|\xi(t) - \bar{x}_t\| \le \varepsilon, \ t \in [0, T],$$

$$B(\xi'(t), \varepsilon) \cap \{\bar{y}_i, \ -\bar{y}_i\} \ne \emptyset, \ t \in [0, T] \ a.\ e..$$

Denote by $\mathfrak{M}$ the set of all borelian functions $g : X \times X \to R^1$ such that

$$g(x, u) \ge \psi(\|u\|)\|u\| - a \tag{5.1}$$

(see (3.4) and (A3)) for each $(x, u) \in X \times X$.

The following result was obtained in [81] (Proposition 5.6).

Proposition 5.3. *Let $M_1, \varepsilon > 0$ and $0 < \tau_0 < \tau_1$. Then there exists $\delta > 0$ such that for each pair of numbers T_1, T_2 satisfying*

$$0 \le T_1,\ T_2 \in [T_1 + \tau_0, T_1 + \tau_1],$$

each $g \in \mathfrak{M}$, each $x \in W^{1,1}(T_1, T_2)$ satisfying

$$\int_{T_1}^{T_2} g(x(t), x'(t))dt \le M_1$$

and each $t_1, t_2 \in [T_1, T_2]$ satisfying $|t_1 - t_2| \le \delta$ the inequality

$$\|x(t_1) - x(t_2)\| \le \varepsilon$$

holds.

Proposition 5.4. *Let $\varepsilon, \Delta \in (0, 1]$. Then there exist $\delta > 0$ such that for each $k \in \{1, \dots, m\}$, each $T \ge \Delta$ and each $\xi \in W^{1,1}(0, T)$ satisfying*

$$\|\xi(0) - \bar{x}_k\| \le \delta,\ \|\xi(T) - \bar{x}_k\| \le \delta, \tag{5.2}$$

$$I(0, T, \xi) \le U(0, T, \xi(0), \xi(T)) + \delta; \tag{5.3}$$

the inequality

$$\|\xi(t) - \bar{x}_k\| \le \varepsilon$$

holds for all $t \in [0, T]$.

Proof. We may assume without loss of generality that

$$\varepsilon < \min\{\|\bar{x}_i - \bar{x}_j\| :\ i, j \in \{1, \dots, m\},\ i < j\}/8. \tag{5.4}$$

Proposition 5.3 implies that there exists

$$\varepsilon_0 \in (0, \min\{\varepsilon/4,\ \Delta/8\})$$

such that the following property holds:

(a) for each pair of numbers S_1, S_2 satisfying

$$0 \le S_1,\ S_2 \in [S_1 + \Delta/8, S_1 + \Delta/4],$$

each $g \in \mathfrak{M}$, each $x \in W^{1,1}(S_1, S_2)$ satisfying

$$\int_{S_1}^{S_2} g(x(t), x'(t))dt \le 4^{-1}\Delta|f(\bar{x}_1, \bar{y}_1)| + 1$$

and each $t_1, t_2 \in [S_1, S_2]$ satisfying $|t_1 - t_2| \le \varepsilon_0$ we have

$$\|x(t_1) - x(t_2)\| \le \varepsilon/4.$$

By (A1) there exists $\varepsilon_1 \in (0, \varepsilon_0/4)$ such that the following property holds:

(b) for each $(x, y) \in X \times X$ satisfying

$$f(x, y) \le \inf(f) + \varepsilon_1$$

there exists $i \in \{1, \dots, m\}$ such that $\|x - \bar{x}_i\| \le \varepsilon/4$.

Proposition 5.2 implies that there exists

$$\delta \in (0, \varepsilon_1^2/4) \tag{5.5}$$

such that the following property holds:

(c) for each $T \geq \Delta$, each $i \in \{1,\dots,m\}$ and each

$$y, z \in B(\bar{x}_i, \delta)$$

there exists $\xi \in W^{1,1}(0,T)$ such that

$$\xi(0) = y,\ \xi(T) = z,$$
$$I(0,T,\xi) \leq Tf(\bar{x}_i, \bar{y}_i) + \varepsilon_1^2/4.$$

Assume that $T \geq \Delta$, $\xi \in W^{1,1}(0,T)$, $k \in \{1,\dots,m\}$ and (64) holds, Property (c) and (5.3) imply that

$$U(0,T,\xi(0),\xi(T)) \leq Tf(\bar{x}_k, \bar{y}_k) + \varepsilon_1^2/4. \tag{5.6}$$

Equations (5.3)-(5.6) imply that

$$I(0,T,\xi) \leq Tf(\bar{x}_k, \bar{y}_k) + \delta + \varepsilon_1^2/4 \leq Tf(\bar{x}_k, \bar{y}_k) + \varepsilon_1^2/2. \tag{5.7}$$

In view of (5.7), for each set $E \subset [0,T]$,

$$\int_E f(\xi(t), \xi'(t))dt \leq \mathrm{mes}(E) f(\bar{x}_k, \bar{y}_k) + \varepsilon_1^2/2. \tag{5.8}$$

We show that for each $t \in [0,T]$,

$$\min\{\|\xi(t) - \bar{x}_i\| :\ i = 1,\dots,m\} \leq \varepsilon. \tag{5.9}$$

Assume the contrary. Then there exists

$$t_0 \in [0,T]$$

such that

$$\|\xi(t_0) - \bar{x}_i\| > \varepsilon,\ i = 1,\dots,m. \tag{5.10}$$

Clearly, there exists $a \in R^1$ satisfying

$$[a, a+\Delta/4] \subset [0,T],\ t_0 \in [a, a+\Delta/4]. \tag{5.11}$$

In view of (5.8),

$$I(a, a+\Delta/4, \xi) \leq 4^{-1}\Delta f(\bar{x}_k, \bar{y}_k) + 1. \tag{5.12}$$

Property (a) and (5.12) imply that for each

$$t \in [a, a+\Delta/4] \cap [t_0 - \varepsilon, t_0 + \varepsilon] \tag{5.13}$$

and each $i \in \{1,\dots,m\}$,

$$\|\xi(t) - \xi(t_0)\| \leq \varepsilon/4$$

and

$$\|\xi(t) - \bar{x}_i\| \geq \|\xi(t_0) - \bar{x}_i\| - \|\xi(t) - \xi(t_0)\| \geq \varepsilon - \varepsilon/4. \tag{5.14}$$

Property (b), (5.13) and (5.14) imply that for each $t \in [a, a+\Delta/4] \cap [t_0 - \varepsilon, t_0 + \varepsilon]$,

$$f(\xi(t), \xi'(t)) > \inf(f) + \varepsilon_1. \tag{5.15}$$

By (5.71) and the inequality, $\varepsilon_0 < \Delta/8$,

$$\mathrm{mes}([a, a+\Delta/4] \cap [t_0 - \varepsilon, t_0 + \varepsilon]) \geq \varepsilon_0. \tag{5.16}$$

It follows from (5.11), (5.15) and (5.16) that

$$I(0,T,\xi) \geq (\inf(f) + \varepsilon_1)\mathrm{mes}([a, a+\Delta/4] \cap [t_0 - \varepsilon, t_0 + \varepsilon])$$
$$+ \inf(f)\mathrm{mes}([0,T] \setminus ([a, a+\Delta/4] \cap [t_0 - \varepsilon, t_0 + \varepsilon]))$$

$$\geq \inf(f)T + \varepsilon_0\varepsilon_1 \geq T\inf(f) + \varepsilon_1^2.$$

This contradicts (5.7). The contradiction we have reached proves that for each $t \in [0,T]$ (5.9) holds. We show that for each $t \in [0,T]$,

$$\|\xi(t) - \bar{x}_k\| \leq \varepsilon.$$

Assume the contrary. Then there exists $S_0 \in [0,T]$ such that

$$\|\xi(S_0) - \bar{x}_k\| > \varepsilon. \tag{5.17}$$

In view of (5.3), (5.4) and (5.17),

$$S_0 \in (0,T).$$

Set

$$S_1 = \sup\{\tau \in (0,T] : \ \|\xi(t) - \bar{x}_k\| \leq \varepsilon, \ t \in [0,T]\}. \tag{5.18}$$

Clearly, S_1 is well-defined,

$$S_1 > 0, \ S_1 < S_0, \tag{5.19}$$

$$\|\xi(t) - \bar{x}_k\| \leq \varepsilon, \ t \in [0,S_1]. \tag{5.20}$$

By (5.18)-(5.20), there exists a strictly decreasing sequence $\{\tau_j\}_{j=1}^{\infty}$ such that

$$\tau_j \in (S_1,T], \ j = 1,2,\ldots, \ \lim_{j\to\infty} \tau_j = S_1, \tag{5.21}$$

$$\|\xi(\tau_j) - \bar{x}_k\| > \varepsilon, \ j = 1,2,\ldots. \tag{5.22}$$

By (5.9), extracting a subsequence and re-indexing, we may assume without loss of generality that there exists $p \in \{1,\ldots,m\}$ such that

$$\|\xi(\tau_j) - \bar{x}_p\| \leq \varepsilon, \ p = 1,2,\ldots. \tag{5.23}$$

In view of (5.22) and (5.23),

$$p \neq k. \tag{5.24}$$

Equations (5.21) and (5.23) imply that

$$\|\xi(S_1) - \bar{x}_p\| \leq \varepsilon.$$

Together with (5.20) this implies that

$$\|\bar{x}_p - \bar{x}_k\| \leq 2\varepsilon.$$

This contradicts (5.5) (see (5.24)). The contradiction we have reached proves that $\|\xi(t) - \bar{x}_k\| \leq \varepsilon$, $t \in [0,T]$. Proposition 5.4 is proved. □

6. A TURNPIKE RESULT

Now we state and prove our main result which give the full description of the structure of approximate solutions u of our variational problems on an interval $[0,T]$ where T is sufficiently large. It is shown that there are mutually disjoint subintervals E_i, $i = 1,\dots,q$ of $[0,T]$ where $q \le m$ and an injective mapping $p : \{1,\dots,q\} \to \{1,\dots,m\}$ such that the measure of the complement $[0,T] \setminus \cup_{i=1}^{q} E_i$ does not exceed a constant which does not depend on T and for each $i \in \{1,\dots,q\}$ the set $u(E_i)$ is contained in a small neighborhood of $\bar{x}_{p(i)}$.

Theorem 6.1. *Let $M > 0$ and $\varepsilon \in (0,1)$ satisfy*

$$\varepsilon < \min\{\|\bar{x}_i - \bar{x}_j\| :\ i,j \in \{1,\dots,m\},\ i < j\}/4. \tag{6.1}$$

Then there exist $L > 0$, $\varepsilon_1, \delta \in (0,\varepsilon)$ such that for each $T > 2L$ and each $u \in W^{1,1}(0,T)$ such that

$$u(0) \in B(0,M) \tag{6.2}$$

and at least of the following conditions holds:

(i)

$$u(T) \in B(0,M),\ I(0,T,u) \le U(0,T,u(0),u(T)) + \delta; \tag{6.3}$$

(ii)

$$I(0,T,u) \le U(0,T,u(0)) + \delta \tag{6.4}$$

there exist an integer $q \ge 1$ and numbers $S_i, \tilde{S}_i \in [0,T]$, $i = 1,\dots,q$ such that

$$S_i \le \tilde{S}_i,\ i = 1,\dots,q,\ S_{i+1} > \tilde{S}_i,\ i \in \{1,\dots,q\} \setminus \{q\},$$

$$S_i < \tilde{S}_i \text{ if } i \in \{1,\dots,q\} \text{ and } S_i < T \tag{6.5}$$

and there exist $j_1,\dots,j_q \in \{1,\dots,m\}$ such that

$$j_{p_1} \ne j_{p_2} \text{ for each } p_{,}p_2 \in \{1,\dots,q\} \text{ satisfying } p_1 \ne p_2, \tag{6.6}$$

$$S_1 \in [0,L],\ \min\{\|u(S_1) - \bar{x}_i\| :\ i = 1,\dots,m\} \le \delta,$$

for each $t \in [0,S_1)$,

$$\min\{\|u(t) - \bar{x}_i\| :\ i = 1,\dots,m\} > \delta,$$

for each $p \in \{1,\dots,q\}$,

$$\|u(S_p) - \bar{x}_{j_p}\| \le \delta,$$

$$\|u(t) - \bar{x}_{j_p}\| \le \varepsilon,\ t \in [S_p,\tilde{S}_p],$$

$$\|u(t) - \bar{x}_{j_p}\| > \delta,\ t \in [\tilde{S}_p,S_p) \text{ if } p > 1,$$

$$\varepsilon_1 \le S_p - \tilde{S}_{p-1} \le L \text{ if } p > 1,$$

$$\|u(t) - \bar{x}_{j_p}\| > \delta,\ t \in [\tilde{S}_p,T] \setminus \{\tilde{S}_p\},$$

$$\text{if } S_p + \varepsilon_1 \le T \text{ then } \tilde{S}_p \ge S_p + \varepsilon_1,$$

$$\text{if } S_p + \varepsilon_1 > T \text{ then } \tilde{S}_p = T,\ p = q,$$

$$\text{if } \tilde{S}_p < T \text{ then } \|u(\tilde{S}_p) - \bar{x}_{j_p}\| = \varepsilon,$$

$$\tilde{S}_q \ge T - L,$$

$$\{t \in [\tilde{S}_q,T] \setminus \{\tilde{S}_q\} :\ \min\{\|u(t) - \bar{x}_i\| :\ i = 1,\dots,m\} \le \delta\} = \emptyset.$$

Proof. Theorem 3.2 implies that there exist $M_0 > M$ such that

$$U(0,T,y,z) \leq T\inf(f) + M_0 \tag{6.7}$$

for each $T \geq 1$ and each $y,z \in B(0,M)$.

By Proposition 5.3 there exists

$$\varepsilon_1 \in (0, \varepsilon/8)$$

such that the following property holds:

(a) for each pair of numbers S_1, S_2 satisfying

$$0 \leq S_1,\ S_2 \in [S_1 + 4^{-1}, S_1 + 4],$$

each $g \in \mathfrak{M}$, each $x \in W^{1,1}(S_1,S_2)$ satisfying

$$\int_{S_1}^{S_2} g(x(t),x'(t))dt \leq 4|\inf(f)| + M_0 + 4$$

and each $t_1,t_2 \in [S_1,S_2]$ satisfying $|t_1 - t_2| \leq \varepsilon_1$ the inequality

$$\|x(t_1) - x(t_2)\| \leq \varepsilon/8$$

holds.

Proposition 5.4 implies that there exists $\delta \in (0, \varepsilon_1/8)$ such that the following property holds:

(b) for each $k \in \{1,\dots,m\}$, each $T \geq \varepsilon_1/8$ and each $\xi \in W^{1,1}(0,T)$ satisfying

$$\|\xi(0) - \bar{x}_k\| \leq \delta,\ \|\xi(T) - \bar{x}_k\| \leq \delta$$

and

$$I(0,T,\xi) \leq U(0,T,\xi(0),\xi(T)) + \delta$$

the inequality

$$\|\xi(t) - \bar{x}_k\| \leq \varepsilon$$

holds for all $t \in [0,T]$.

Theorem 4.1 implies that there exists $L > 1$ such that the following property holds:

(c) for each $T > L$ and each $x \in W^{1,1}(0,T)$ such that

$$x(0) \in B(0,M)$$

and at least one of the following conditions holds:

$$x(T) \in B(0,M),\ I^f(0,T,x) \leq U(0,T,x(0),x(T)) + 1;$$

$$I^f(0,T,x) \leq U(0,T,x(0)) + 1$$

the inequality

$$\text{mes}(\{t \in [0,T] :\ \|x(t) - \bar{x}_i\| > \delta \text{ for each } i \in \{1,\dots,m\}\}) < L$$

is true.

Assume that $T \geq 2L$, $u \in W^{1,1}(0,T)$, (6.2) is true and at least of conditions (i) and (ii) hold. Conditions (i), (ii) and (6.7) imply that

$$I(0,T,u) \leq T\inf(f) + M_0 + 1. \tag{6.8}$$

In view of (6.8) for each measurable set $J \subset [0,T]$,

$$\int_J f(u(t),u'(t))dt \le \text{mes}(J)\inf(f) + M_0 + 1. \tag{6.9}$$

Property (a) and (6.9) imply that for each $t_1,t_2 \in [0,T]$ satisfying $|t_1 - t_2| \le \varepsilon_1$ we have

$$\|u(t_1) - u(t_2)\| \le \varepsilon/8. \tag{6.10}$$

Property (c), conditions (i), (ii) and (6.2) imply that there exists a number S_1 such that

$$S_1 \in [0,L],\ \min\{\|u(S_1) - \bar{x}_i\| :\ i = 1,\dots,m\} \le \delta \tag{6.11}$$

and that for each $t \in [0,S_1] \setminus \{S_1\}$,

$$\min\{\|u(t) - \bar{x}_i\| :\ i = 1,\dots,m\} > \delta. \tag{6.12}$$

By (6.1) and (6.11), there exists a unique integer $j_1 \in \{1,\dots,m\}$ such that

$$\|u(S_1) - \bar{x}_{j_1}\| \le \delta. \tag{6.13}$$

In view of (6.11),

$$S_1 + \varepsilon_1 \le 2L \le T. \tag{6.14}$$

It follows from (6.10), (6.11), (6.13) and (6.14) that for each $t \in [S_1, S_1 + \varepsilon_1]$,

$$\|u(t) - u(S_1)\| \le \varepsilon/8,$$

$$\|u(t) - \bar{x}_{j_1}\| \le \|u(t) - u(S_1)\| + \|u(S_1) - \bar{x}_{j_1}\| \le \varepsilon/8 + \delta \le \varepsilon$$

and

$$\|u(t) - \bar{x}_{j_1}\| \le \varepsilon,\ t \in [S_1, S_1 + \varepsilon_1]. \tag{6.15}$$

Define

$$\tilde{S}_1 = \sup\{\tau \in (S_1,T] :\ \|u(t) - \bar{x}_{j_1}\| \le \varepsilon,\ t \in [0,\tau]\}. \tag{6.16}$$

By (6.15) and (6.16),

$$\tilde{S}_1 \ge S_1 + \varepsilon_1,\ \|u(\tilde{S}_1) - \bar{x}_{j_1}\| \le \varepsilon. \tag{6.17}$$

If $\tilde{S}_1 = T$, then our construction is completed.

Assume that $\tilde{S}_1 < T$. In view of (6.16),

$$\|u(\tilde{S}_1) - \bar{x}_{j_1}\| = \varepsilon. \tag{6.18}$$

Property (b), conditions (i), (ii) and equations (6.13) and (6.17) imply that for each $t \in (\tilde{S}_1, T]$,

$$\|u(t) - \bar{x}_{j_1}\| > \delta. \tag{6.19}$$

There are two cases:

$$\{t \in [\tilde{S}_1,T] :\ \min\{\|u(t) - \bar{x}_i\| :\ i = 1,\dots,m\} \le \delta\} = \emptyset; \tag{6.20}$$

$$\{t \in [\tilde{S}_1,T] :\ \min\{\|u(t) - \bar{x}_i\| :\ i = 1,\dots,m\} \le \delta\} \ne \emptyset. \tag{6.21}$$

If (6.20) holds, then property (c) and (6.17) imply that

$$\tilde{S}_1 + L \ge T$$

and our construction is completed.

Assume that (6.21) holds. Set

$$S_2 = \inf\{t \in [\tilde{S}_1,T] :\ \min\{\|u(t) - \bar{x}_i\| :\ i = 1,\dots,m\} \le \delta\}. \tag{6.22}$$

Equations (6.1), (6.18) and (6.22) imply that

$$S_2 > \tilde{S}_1$$

and there exists a unique $j_2 \in \{1, \dots, m\}$ such that

$$\|u(S_2) - \bar{x}_{j_2}\| \le \delta. \tag{6.23}$$

By (6.19) and (6.23),

$$j_2 \ne j_1,\ \|u(S_2) - \bar{x}_{j_2}\| = \delta.$$

It follows from (6.10), (6.16), (6.17), (6.22) and (6.23) that for every

$$t \in [0,T] \cap [S_2 - \varepsilon_1, S_2 + \varepsilon_1]$$

we have

$$\begin{aligned} \|u(t) - \bar{x}_{j_2}\| &\le \|u(t) - u(S_2)\| + \|u(S_2) - \bar{x}_{j_2}\| \\ &\le \delta + \varepsilon/8 \le \varepsilon/4. \end{aligned} \tag{6.24}$$

Equations (6.1), (6.17) and (6.24) imply that

$$\tilde{S}_1 \le S_2 - \varepsilon_1. \tag{6.25}$$

If $S_2 + \varepsilon_1 \ge T$, then $\tilde{S}_2 = T$ and our construction is completed. Otherwise we set

$$\tilde{S}_2 = \inf\{S \in [S_2, T] :\ \|u(t) - \bar{x}_{j_2}\| \le \varepsilon,\ t \in [S_2, S]\}. \tag{6.26}$$

Property (c), (6.22) and (6.26) imply that

$$S_2 - \tilde{S}_2 \le L,\ S_2 - \tilde{S}_1 \le L. \tag{6.27}$$

Assume that k is an integer and we defined $S_i, \tilde{S}_i \in [0,T]$, $i = 1, \dots, k$ such that $S_i \le \tilde{S}_i$, $i = 1, \dots, k$, if $S_i < T$, then $S_i < \tilde{S}_i$ for each $i \in \{1, \dots, k\}$,

$$S_{i+1} > \tilde{S}_i,\ i \in \{1, \dots, k\} \setminus \{k\}$$

and we defined $j_1, \dots, j_k \in \{1, \dots, m\}$ for which

$$j_{p_1} \ne j_{p_2} \text{ for each } p_1, p_2 \in \{1, \dots, k\} \text{ satisfying } p_1 \ne p_2$$

and that (91)-(96) are true, for each $p \in \{1, \dots, k\}$,

$$\|u(S_p) - \bar{x}_{j_p}\| \le \delta, \tag{6.28}$$

$$\|u(t) - \bar{x}_{j_p}\| \le \varepsilon,\ t \in [S_p, \tilde{S}_p], \tag{6.29}$$

$$\|u(t) - \bar{x}_{j_p}\| > \delta,\ t \in [\tilde{S}_{p-1}, S_p) \text{ if } p > 1, \tag{6.30}$$

$$\varepsilon_1 \le S_p - \tilde{S}_{p-1} \le L \text{ if } p > 1, \tag{6.31}$$

$$\|u(t) - \bar{x}_{j_p}\| > \delta,\ t \in [\tilde{S}_p, T] \setminus \{\tilde{S}_p\}, \tag{6.32}$$

$$\text{if } S_p + \varepsilon_1 \le T \text{ then } \tilde{S}_p \ge S_p + \varepsilon_1, \tag{6.33}$$

$$\text{if } S_p + \varepsilon_1 > T \text{ then } \tilde{S}_p = T,\ p = k, \tag{6.34}$$

$$\text{if } S_p < T \text{ then } \|u(\tilde{S}_p) - \bar{x}_{j_p}\| = \varepsilon. \tag{6.35}$$

(It is not difficult to see that for $k = 1$ our assumption holds.)

If $\tilde{S}_k = T$, then our construction is completed. Assume that $\tilde{S}_k < T$. There are two cases:

$$\{t \in [\tilde{S}_k, T] :\ \min\{\|u(t) - \bar{x}_i\| :\ i = 1, \dots, m\} \le \delta\} = \emptyset; \tag{6.36}$$

$$\{t \in [\tilde{S}_k, T] :\ \min\{\|u(t) - \bar{x}_i\| :\ i = 1, \dots, m\} \le \delta\} \ne \emptyset. \tag{6.37}$$

If (6.36) holds, then property (c), conditions (i), (ii) and (6.29) imply that

$$\tilde{S}_k + L \geq T$$

and our construction is completed.

Assume that (6.37) holds. Set

$$S_{k+1} = \inf\{t \in [\tilde{S}_k, T] : \min\{\|u(t) - \bar{x}_i\| : i = 1, \dots, m\} \leq \delta\}. \tag{6.38}$$

Equations (6.35), (6.37) and (6.38) imply that

$$S_{k+1} > \tilde{S}_k$$

and there exists a unique $j_{k+1} \in \{1, \dots, m\}$ such that

$$\|u(S_{k+1}) - \bar{x}_{j_{k+1}}\| \leq \delta. \tag{6.39}$$

By (6.32) and (6.39),

$$j_{k+1} \neq j_p,\ p = 1, \dots, k.$$

By (6.10) and (6.38), for every

$$t \in [0, T] \cap [S_{k+1} - \varepsilon_1, S_{k+1} + \varepsilon_1]$$

we have

$$\begin{aligned} \|u(t) - \bar{x}_{j_{k+1}}\| &\leq \|u(t) - u(S_{k+1})\| + \|u(S_{k+1}) - \bar{x}_{j_{k+1}}\| \\ &\leq \delta + \varepsilon/8 \leq \varepsilon/4. \end{aligned} \tag{6.40}$$

It follows from (6.1), (6.35) and (6.40),

$$\tilde{S}_k \leq S_{k+1} - \varepsilon_1.$$

If $S_{k+1} + \varepsilon_1 \geq T$, then $\tilde{S}_{k+1} = T$ and our construction is completed. Otherwise we set

$$\tilde{S}_{k+1} = \inf\{S \in [\tilde{S}_{k+1}, T] : \|u(t) - \bar{x}_{j_{k+1}}\| \leq \varepsilon,\ t \in [S_{k+1}, S]\}. \tag{6.41}$$

Property (b), (6.37)-(6.41), (6.41) and equations above imply that the assumption made for k also holds for $k+1$. Therefore by induction we constructed an integer $q \geq 1$, numbers $S_i, \tilde{S}_i \in [0, T]$, $i = 1, \dots, q$ such that $S_i \leq \tilde{S}_i$, $i = 1, \dots, q$, if $S_i < T$, then $S_i < \tilde{S}_i$ for each $i \in \{1, \dots, q\}$,

$$S_{i+1} > \tilde{S}_i,\ i \in \{1, \dots, q\} \setminus \{q\}$$

and we defined $j_1, \dots, j_q \in \{1, \dots, m\}$ for which

$$j_{p_1} \neq j_{p_2} \text{ for each } p_1, p_2 \in \{1, \dots, q\} \text{ satisfying } p_1 \neq p_2$$

and that (6.11)-(6.16) are true, for each $p \in \{1, \dots, q\}$, (6.28)-(6.35) hold (with $p = q$),

$$\tilde{S}_q \geq T - L,$$

$$\{t \in [\tilde{S}_q, T] : \min\{\|u(t) - \bar{x}_i\| : i = 1, \dots, m\} \leq \delta\} = \emptyset.$$

Theorem 6.1 is proved. □

7. THE SECOND TURNPIKE RESULT

Theorem 10 describes the structure of M-approximate solutions u of our variational problems on an interval $[0,T]$ where T is sufficiently large and $M>0$ is sufficiently small. Our next solutions is related to the case when M is fixed but not necessarily small. It is shown that there are mutually disjoint subintervals E_i, $i=1,\dots,q$ of $[0,T]$ where q does not exceed a constant which does not depend on T and a mapping $p:\{1,\dots,q\}\to\{1,\dots,m\}$ (not necessarily injective) such that the measure of the complement $[0,T]\setminus\cup_{i=1}^{q}E_i$ does not exceed a constant which does not depend on T and for each $i\in\{1,\dots,q\}$ the set $u(E_i)$ is contained in a small neighborhood of $\bar{x}_{p(i)}$.

Theorem 7.1. *Let $\varepsilon\in(0,1]$, $M_1>0$ and $M_0>\|\bar{x}_i\|+\|\bar{y}_i\|+1$, $i=1,\dots,m$. Then there exist $l>0$ and a natural number Q such that for each $T>lQ$ and each $x\in W^{1,1}(0,T)$ such that*

$$\|x(0)\|\le M_0 \tag{7.1}$$

and at least one of the following conditions holds:

(a)

$$\|x(T)\|\le M_0,\ I(0,T,x)\le U(0,T,x(0),x(T))+M_1;$$

(b)

$$I(0,T,x)\le U(0,T,x(0))+M_1$$

there exist an integer $q\in[1,Q]$ and intervals $[a_i,b_i]\subset[0,T]$, $i=1,\dots,q$ such that

$$a_{i+1}\ge b_i,\ i\in\{1,\dots,q\}\setminus\{q\},$$

for each $i\in\{1,\dots,q\}$, there exists $p_i\in\{1,\dots,m\}$, such that

$$\|x(t)-\bar{x}_{p_i}\|\le\varepsilon,\ t\in[a_i,b_i]$$

and

$$mes([0,T]\setminus\cup_{i=1}^{q}[a_i,b_i])\le l.$$

Proof. Theorem 6.1 implies that there exist $L_0>0$, $\delta>0$ such that the following property holds:

(c) for each $T>2L_0$ and each $u\in W^{1,1}(0,T)$ such that

$$u(0),u(T)\in B(0,M_0)$$

$$I(0,T,u)\le U(0,T,u(0),u(T))+\delta$$

there exist an integer $q\in\{1,\dots,m\}$ and numbers $S_i,\tilde{S}_i\in[0,T]$, $i=1,\dots,q$ such that

$$S_i\le\tilde{S}_i,\ i=1,\dots,q,\ S_{i+1}>\tilde{S}_i,\ i\in\{1,\dots,q\}\setminus\{q\},$$

$$S_i<\tilde{S}_i \text{ if } i\in\{1,\dots,q\} \text{ and } S_i<T$$

and there exist $j_1,\dots,j_q\in\{1,\dots,m\}$ such that

$$S_1\le L_0,\ \tilde{S}_q\ge T-L_0,$$

$$S_p-\tilde{S}_{p-1}\le L_0,\ p\in\{1,\dots,q\}\setminus\{1\},$$

$$\|u(t)-\bar{x}_{j_p}\|\le\varepsilon,\ t\in[S_p,\tilde{S}_p],\ p=1,\dots,q.$$

Theorem 4.1 implies that there exists $L_1>2L_0$ such that the following property holds:

(d) for each $T\ge L_1$ and each $x\in W^{1,1}(0,T)$ such that $x(0)\in B(0,M_0)$ and at least one of the following conditions holds:

$$x(T) \in B(0,M_0),\ I^f(0,T,x) \le U(0,T,x(0),x(T)) + M_1 + 1;$$

$$I^f(0,T,x) \le U(0,T,x(0)) + M_1$$

the inequality

$$\mathrm{mes}(\{t \in [0,T] :\ \|x(t) - \bar{x}_i\| > \delta \text{ for each } i \in \{1,\dots,m\}\}) < L_1.$$

Fix

$$l_0 = 4L_1 + 4,\ l > 4Ql_0. \tag{7.2}$$

Choose an integer

$$Q > 2m(4 + \delta^{-1}M_1). \tag{7.3}$$

Assume that $T > lQ$ and that $x \in W^{1,1}(0,T)$ satisfies at least one of conditions (a) and (b). Together with property (d) this implies that there exists

$$t_0 \in [0,L_1] \tag{7.4}$$

such that

$$\min\{\|x(t_0) - \bar{x}_i\| :\ i = 1,\dots,m\} \le \delta. \tag{7.5}$$

Property (d), conditions (a), (b) and equations (7.1), (7.4) and (7.5) imply that there exists a finite sequence of numbers $t_0 < t_1 \cdots < t_q$ belonging to the interval $[0,T]$ such that for each $k \in \{0,\dots,q\}$,

$$\min\{\|x(t_k) - \bar{x}_i\| :\ i = 1,\dots,m\} \le \delta, \tag{7.6}$$

$$t_{i+1} - t_i \in [L_1, 2L_1],\ i \in \{0,\dots,q\} \setminus \{q\}, \tag{7.7}$$

$$t_q > T - 2L_1. \tag{7.8}$$

Now we construct a strictly increasing sequence of numbers $S_i \in \{t_0,\dots,t_q\}$, $i = 0,\dots,p$. Set

$$S_0 = t_0. \tag{7.9}$$

Assume that $k \ge 0$ is an integer and that we already defined a finite increasing sequence $S_i \in \{t_0,\dots,t_q\}$, $i = 0,\dots,k$. If $S_k = t_q$, then the construction is completed. Assume that

$$S_k < t_q. \tag{7.10}$$

If

$$I(S_k,t_q,x) \le U(S_k,t_q,x(S_k),x(t_q)) + \delta,$$

then we set

$$S_{k+1} = t_q$$

and the construction is completed. Assume that

$$I(S_k,t_q,x) > U(S_k,t_q,x(S_k),x(t_q)) + \delta. \tag{7.11}$$

There exists $j \in \{0,\dots,q\}$ such that

$$S_k = t_j. \tag{7.12}$$

If

$$I(S_k,t_{j+1},x) > U(S_k,t_{j+1},x(S_k),x(t_{j+1})) + \delta,$$

then we set

$$S_{k+1} = t_{j+1}.$$

Assume that

$$I(S_k, t_{j+1}, x) \le U(S_k, t_{j+1}, x(S_k), x(t_{j+1})) + \delta. \tag{7.13}$$

We define

$$S_{k+1} = \min\{t_i : \ i \in \{j+1, \dots, q\} : \\ I(S_k, t_i, x) > U(S_k, t_i, x(S_k), x(t_i)) + \delta\}. \tag{7.14}$$

Therefore by induction we defined a strictly increasing sequence of numbers

$$S_i \in \{t_0, \dots, t_q\}, \ i = 0, \dots, p$$

such that

$$S_0 = t_0, \ S_p = t_q,$$

for each $j \in \{0, \dots, p-1\} \setminus \{p-1\}$,

$$I(S_j, S_{j+1}, x) > U(S_j, S_{j+1}, x(S_j), x(S_{j+1})) + \delta, \tag{7.15}$$

for each $j \in \{0, \dots, p-1\}$, if $i \in \{0, \dots, q\}$ and

$$S_j < t_i < S_{j+1},$$

then

$$I(S_j, t_i, x) \le U(S_j, t_i, x(S_j), x(t_i)) + \delta, \tag{7.16}$$

Conditions (a), (b) and (7.15) imply that

$$M_1 \ge I(0, T, x) - U(0, T, x(0), x(T)) \\ \ge \sum \{I(S_j, S_{j+1}, x) - U(S_j, S_{j+1}, x(S_j), x(S_{j+1})) : \ j \in \{0, \dots, p-1\} \setminus \{p-1\}\} \\ \ge (p-1)\delta$$

and

$$p \le 1 + \delta^{-1} M_1. \tag{7.17}$$

Assume that $j \in \{0, \dots, p-1\}$ satisfies

$$S_{j+1} - S_j \ge l_0. \tag{7.18}$$

Property (d), conditions (a) and (b) and equations (7.2), (7.6), (7.14) and (7.18) imply that there exists

$$\tilde{S}_j \in [S_{j+1} - 3L_1, S_{j+1} - 2L_1] \tag{7.19}$$

such that

$$\min\{\|x(\tilde{S}_j) - \bar{x}_i\| : \ i = 1, \dots, m\} \le \delta. \tag{7.20}$$

It follows from (7.7), (7.16) and (7.19) that

$$I(S_j, \tilde{S}_j, x) \le U(S_j, \tilde{S}_j, x(S_j), x(\tilde{S}_j)) + \delta. \tag{7.21}$$

Property (c) and equations (7.2), (7.18)-(7.21) imply that there exist an integer $q_j \ge 1$ and numbers $S_{j,i}, \tilde{S}_{j,i} \in [S_j, \tilde{S}_j]$, $i = 1, \dots, q_j$ such that

$$S_{j,i} \le \tilde{S}_{j,i}, \ i = 1, \dots, q_j, \ S_{j,i+1} > \tilde{S}_{j,i}, \ i \in \{1, \dots, q_j\} \setminus \{q_j\},$$

$$S_{j,i} < \tilde{S}_{j,i} \text{ if } i \in \{1, \dots, q_j\} \text{ and } S_{j,i} < T$$

and there exist $k_{j,1}, \dots, k_{j,q_j} \in \{1, \dots, m\}$ such that

$$S_{j,1} \le L_0 + S_j, \ \tilde{S}_{j,q_j} \ge \tilde{S}_j - L_0,$$

$$S_{j,p} - \tilde{S}_{j,p-1} \le L_0, \ j \in \{1, \dots, q_j\} \setminus \{1\},$$

$$\|u(t) - \bar{x}_{j,k_\tau}\| \leq \varepsilon, \ t \in [S_{j,\tau}, \tilde{S}_{j,\tau}], \ \tau = 1, \dots, q_j.$$

Consider the collection of intervals $[S_{j,\tau}, \tilde{S}_{j,\tau}]$, $\tau = 1, \dots, q_j$, $j \in \{0, \dots, p-1\}$ such that $S_{j+1} - S_j \geq l_0$. The number of these intervals does not exceed $(p+2)m < Q$ (see (7.2) and (7.17)). The completion of their union in $[0,T]$ is also a union of a finite collection of subintervals of $[0,T]$ and in view of (7.3), (7.17) their number does not exceed

$$2(p+2)m \leq 2m(4 + \delta^{-1}M_1) < Q.$$

By (7.2), (7.4) and (7.8) the measure of this complement does not exceed

$$Ql_0 + 3QL_1 + 3L_1 < Ql_0 + 3(Q+1)L_1 < l.$$

Theorem 7.1 is proved. □

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