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GALERKIN METHOD FOR THE BOUSSINESQ EQUATION WITH VISCOELASTIC MEMORY AND INTEGRAL CONDITIONS

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Abstract. This paper deals with the Boussinesq equation with viscoelastic memory and integral conditions. We demonstrate the existence and uniqueness of solutions with the aid of the Faedo-Galerkin's method.

Keywords. Boussinesq equation; Galerkin method; Integral condition; Viscoelastic memory.

1. Introduction

In this paper, we study the existence and uniqueness of solutions of the Boussinesq equation with term viscoelastic memory

$$v_{tt} - \alpha^2 \Delta v - \beta^2 \Delta v_{tt} + \int_0^t h(t-s)\Delta v(s)\,ds = |v|^{p-2} v, \quad (x,t) \in D_T, \tag{1.1}$$

with the initial data

$$v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), \quad x \in \Omega, \tag{1.2}$$

and the integral condition

$$\frac{\partial v}{\partial \eta} = g(x,t) + \int_0^t \int_\Omega v(\xi,\mu)\,d\xi d\mu, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.3}$$

where $D_T := \Omega \times (0,T)$, Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $2 < p \leq \frac{2(N-1)}{N-2}$, $N \geq 3$, η is the unit outward normal on $\partial\Omega$, $T < \infty$, $h(.) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a given function, which will be specified later, and $g(x,t)$, $h(t)$, $v_0(x)$, and $v_1(x)$ are given functions satisfying some conditions.

The convolution term $\int_0^t h(t-s)\Delta v(s)\,ds$ reflects the memory effects of the materials due to the viscoelasticity. Here the convolution kernel h satisfies proper conditions exhibiting "memory character", which will be explained later. Under some assumptions on h, the existence and uniqueness of the generalized solution is established by using the Galerkin Method.

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The aim of this paper is to investigate a non-local problem generated by a Boussinesq equation and an integral condition. The Boussinesq equation is a nonlinear partial differential equation that arises in hydrodynamics and some physical applications. It was subsequently applied to the problems in the percolation of water in porous subsurface strata. Recent developments in numerical schemes for solving Boussinesq-type equations have placed immense interest in nonlinear dispersive wave models. Various Boussinesq type equations can describe varying degrees of accuracy in representing nonlinearity and dispersion. Boussinesq type equations are conventionally associated with relatively shallow water. In many engineering models, such as thermoplasticity, nuclear reactor dynamics, plasma physics, thermal conductivity, radioactive nuclear decay in fluid flows, medical science, chemical diffusion, vibration problems, semiconductor modeling, groundwater flow, population dynamics, control theory, and flows Non-Newtonian, mathematical models of mixed problems with non-local boundaries are created in many models. Sometimes the physical phenomena are modeled by non classical boundary value problems, which involve a boundary condition as an integral condition over the spatial domain of a function of the desired solution. Nonlocal problems are generally encountered thermoelsticity, heat transmission, chemical engineering, heat transmission, plasma physics, and underground water flow; see, e.g., [1, 2, 3] and the references therein. As a special application, Bouziani [4] considered a nonlocal problem, which was proposed in the mathematical modeling of the technologic process of the external elimination of gas, practices in the refining of impurities of Silicon lamina. The nonlocal condition appearing in this mathematical model represents the total mass of impurities in the lamina. For some hyperbolic nonlocal mixed problems, we refer to [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and the references therein. The existence and uniqueness of the generalized solution is constructed by using the Galerkin method for the Boussinesq equation with viscoelastic memory and the integral condition with the following formula $\frac{\partial u}{\partial \eta} = g(x,t) + \int_0^t \int_\Omega u(\xi,\mu)\, d\xi\, d\mu$ with (1.1)-(1.3) is considered a new problem. In [17], Mesloub and Meslou applied the Galerkin method to a higher dimension mixed nonlocal problem for a Boussinesq equation and established the solvability, and the uniqueness of a weak solution. In [18], Boulaaras, Zaraï and Draifia studied the Galerkin method for nonlocal mixed boundary value problem for the Moore-Gibson-Thompson equation with an integral condition and established the solvability of solutions. In [19], Guezane-Lakoud and Boumaza studied the Galerkin method for the Boussinesq equation with an integral condition, and established the solvability and the uniqueness of solutions. In [20], Boumaza and Gheraibia studied the existence of a local solution for an integro-differential equation with an integral boundary condition and established the solvability and the uniqueness of a weak solution. In [21], Mesloub and Mesloub studied the solvability of a mixed nonlocal problem for a nonlinear singular viscoelastic equation, and established the solvability and the uniqueness of a weak solution. However, there are no results on the existence and uniqueness of problem (1.1)-(1.3) for the integral condition with the following formula $\frac{\partial u}{\partial \eta} = g(x,t) + \int_0^t \int_\Omega u(\xi,\mu)\, d\xi\, d\mu$ in the literature. Motivated by the above research, we consider the existence and uniqueness for the integral condition with the following formula $\frac{\partial u}{\partial \eta} = g(x,t) + \int_0^t \int_\Omega u(\xi,\mu)\, d\xi\, d\mu$ of the model (1.1)-(1.3) in this paper.

The outline of the paper is as follows. In Section 2, we define the function spaces, state some inequalities, and supply an appropriate definition of weak solutions of the posed problem. Section 3 is devoted to proving the existence of solutions by using the Faedo–Galerkin's method. Finally, in Section 4, we establish the uniqueness of the generalized solution of the posed problem.

2. PRELIMINARIES

Let

$$W_2^1(D_T) := \left\{ v \in L^2(D_T) : \nabla v(t),\ v_t(t) \in L^2(D_T) \right\}$$

be the usual Sobolev space, whose elements v are in $L^2(D_T)$ along with $\nabla v(t)$ and v_t, and the finite norm

$$\|v(t)\|_{W_2^1(D_T)} := \left(\int_0^T \int_\Omega \left[v^2(t) + |\nabla v(t)|^2 + v_t^2(t) \right] dxdt \right)^{\frac{1}{2}}.$$

The scalar product in $W_2^1(D_T)$ is defined by

$$(v(t), v(t))_{W_2^1(D_T)} := \int_0^T \int_\Omega (v(t)v(t) + \nabla v(t).\nabla v(t) + v_t(t)v_t(t))\, dxdt.$$

Now, let $V(D_T)$ and $W(D_T)$ be the set spaces, respectively, defined by

$$V(D_T) := \left\{ v \in W_2^1(D_T) : v_t \in H^1(D_T) \right\},$$

and

$$W(D_T) := \{ u \in V(D_T) : u(x,T) = 0 \}.$$

Consider the equation

$$\begin{aligned}
&(v_{tt}(t), u(t))_{L^2(D_T)} - \alpha^2 (\Delta v(t), u(t))_{L^2(D_T)} - \beta^2 (\Delta v_{tt}(t), u(t))_{L^2(D_T)} \\
&\quad + \left(\int_0^t h(t-s)\Delta v(s)\, ds, u(t) \right)_{L^2(D_T)} \\
&= \left(|v|^{p-2} v(t), u(t) \right)_{L^2(D_T)},
\end{aligned} \tag{2.1}$$

where $(.,.)_{L^2(D_T)}$ stands for the inner product in $L^2(D_T)$, v is supposed to be a solution of (1.1)-(1.3) and $u \in W(D_T)$. The evaluation of the inner product in (2.1) and the use of boundary condition in (1.1)-(1.3) lead to

$$
\begin{aligned}
&-\left(v_t\left(t\right),u_t\left(t\right)\right)_{L^2(D_T)}+\alpha^2\left(\nabla v,\nabla u\left(t\right)\right)_{L^2(D_T)}-\beta^2\left(\nabla v_t,\nabla u_t\right)_{L^2(Q_T)}\\
&\quad-\left(\int_0^t h\left(t-s\right)\nabla v\left(s\right)ds,\nabla u\left(t\right)\right)_{L^2(D_T)}\\
&=\left(v_t(0),u(0)\right)_{L^2(\Omega)}+\beta^2\left(\nabla v_t(0),\nabla u(0)\right)_{L^2(\Omega)}+\alpha^2\int_0^T\int_{\partial\Omega}g\left(x,t\right)v\left(t\right)dS_xdt\\
&+\alpha^2\int_0^T\int_{\partial\Omega}\left[\left(\int_0^t\int_\Omega v\left(\xi,\mu\right)d\xi d\mu\right)u\left(t\right)\right]dS_xdt+\beta^2\int_0^T\int_{\partial\Omega}g_{tt}\left(x,t\right)v\left(t\right)dS_xdt\\
&+\beta^2\int_0^T\int_{\partial\Omega}\left[\left(\int_\Omega v_t\left(\xi,t\right)d\xi\right)u\left(t\right)\right]dS_xdt-\beta^2\int_0^T\int_{\partial\Omega}\left[\left(\int_\Omega v_t\left(\xi,0\right)d\xi\right)u\left(t\right)\right]dS_xdt\\
&-\int_0^T\int_{\partial\Omega}\left[\left(\int_0^t h\left(t-s\right)g\left(s,x\right)ds\right)u\left(t\right)\right]dS_xdt\\
&-\int_0^T\int_{\partial\Omega}\left[\left(\int_0^t g\left(t-s\right)\left\{\int_0^s\int_\Omega v\left(\xi,\mu\right)d\xi d\mu\right\}ds\right)u\left(t\right)\right]dS_xdt\\
&+\left(\left|v\right|^{p-2}v\left(t\right),u\left(t\right)\right)_{L^2(D_T)}.
\end{aligned}
\tag{2.2}
$$

Definition 2.1. A function $v\in V\left(D_T\right)$ is called a generalized solution of problem (1.1)-(1.3) if it satisfies equation (2.2) for each $u\in W\left(D_T\right)$.

Recall the binary notation

$$
\left(h\circ w\right)\left(t\right):=\int_0^t h\left(t-s\right)\left\|w\left(x,s\right)-w\left(x,t\right)\right\|_{L^2(\Omega)}^2 ds.
$$

We give some useful inequalities next.

Lemma 2.2. [22] *(Trace inequality) Let $\Omega\subset\mathbb{R}^N$ be an open and bounded set with a smooth boundary $\partial\Omega$. Then*

$$
(\mathbf{T.I})\qquad \int_{\partial\Omega}v^2\left(x\right)dS_x\le\gamma_\Omega\left\|v\left(t\right)\right\|_{H^1(\Omega)}^2,\quad \text{for all } v\in W_1^2\left(\Omega\right),
$$

where γ_Ω is a positive constant and depends on the domain Ω only.

Lemma 2.3. [23] *Let $\Omega\subset\mathbb{R}^N$ be an open and bounded set of class C^1. Then the embedding $H^1\hookrightarrow L^p$ is continuous if $1\le p\le\dfrac{2N}{N-2}$, $N\ge3$.*

Lemma 2.4. [22] *Let $\Omega\subset\mathbb{R}^N$ be an open and bounded set with a smooth boundary $\partial\Omega$. Let $2\le p\le\dfrac{2\left(N-1\right)}{N-2}$, $N\ge3$. Then, there exists a constant C_p depending on p, N and Ω such that*

$$
\begin{aligned}
&\left\|\left|v_1\right|^{p-2}v_1-\left|v_2\right|^{p-2}v_2\right\|_{L^2(\Omega)}^2\\
\le\;& C_p\left[1+\left(\left\|v_1\right\|_{H^1(\Omega)}+\left\|v_2\right\|_{H^1(\Omega)}\right)^{\frac{1}{N}}+\left(\left\|v_1\right\|_{H^1(\Omega)}+\left\|v_2\right\|_{H^1(\Omega)}\right)^{p-2}\right]^2\left\|v_1-v_2\right\|_{H^1(\Omega)}^2,
\end{aligned}
$$

for all $v_1,\ v_2\in H^1\left(\Omega\right)$.

3. The Existence of The Generalized Solution

We make the following assumptions:

$(\mathbf{H}_1)$ $2 < p < \dfrac{2(N-1)}{N-2}$, $N \geq 3$.

$(\mathbf{H}_2)$ $h(t) \geq 0$ and $h'(t) \leq 0$ for all $t \geq 0$.

$(\mathbf{H}_3)$ $c^2 - \bar{h} > 0$ where $\bar{h} := \int_0^\infty h(s)\,ds$.

$(\mathbf{H}_4)$ $g \in L^2\left(0,T;L^2(\partial\Omega)\right),\ g',\ g'' \in L^2\left(0,T;L^2(\partial\Omega)\right).$

We now give the main result on the existence of solutions of problem (1.1)-(1.3), and prove it by using the Galerkin method.

Theorem 3.1. *Assume that the hypotheses $(\mathbf{H}_1)-(\mathbf{H}_4)$ hold, and the initial data $v_0(x),\ v_1(x) \in H^1(\Omega)$, then there is at least one generalized solution in $V(D_T)$ to problem (1.1)-(1.3).*

Proof. Let $\{\psi_l(x)\}_{l\geq 1}$ be a fundamental system in $W_2^1(\Omega)$, and assume it has been orthonormalized in $L^2(\Omega)$, that is, $(\psi_l(x), \psi_k(x))_\Omega = \delta_{l,k}$. We seek an approximate solution $v^m(x)$ in the form $v^m(x,t) = \sum_{l=1}^{l=m} f_l(t)\,\psi_l(x)$, where $f_l(t)$ are defined by

$$f_l(t) = (\psi_l(x), v^m(x,t))_{L^2(\Omega)}, \quad l = 1,\ldots,m,$$

and can be determined from the relations, for all $k = 1,\ldots,m$,

$$\begin{aligned}
&(v_{tt}^m, \psi_k(x))_{L^2(\Omega)} + \alpha^2(\nabla v^m, \nabla\psi_k(x))_{L^2(\Omega)} + \beta^2(\nabla v_{tt}^m, \nabla\psi_k(x))_{L^2(\Omega)} \\
&\quad - \left(\int_0^t h(t-s)\nabla v^m(s)\,ds, \nabla\psi_k(x)\right)_{L^2(\Omega)} \\
&= \alpha^2\int_{\partial\Omega} g(x,t)\,\psi_k(x)\,dS_x + \alpha^2\int_{\partial\Omega}\left(\int_0^t\int_\Omega v^m(\xi,\mu)\,d\xi\,d\mu\right)\psi_k(x)\,dS_x \\
&+ \beta^2\int_{\partial\Omega} g_{tt}(x,t)\,\psi_k(x)\,dS_x + \beta^2\int_{\partial\Omega}\left(\int_0^t\int_\Omega v_{\mu\mu}^m(\xi,\mu)\,d\xi\,d\mu\right)\psi_k(x)\,dS_x \\
&- \int_{\partial\Omega}\left(\int_0^t h(t-s)\,g(x,s)\,ds\right)\psi_k(x)\,dS_x \\
&- \int_{\partial\Omega}\left(\int_0^t h(t-s)\left\{\left(\int_0^s\int_\Omega v^m(\xi,\mu)\,d\xi\,d\mu\right)\right\}ds\right)\psi_k(x)\,dS_x \\
&+ \left(|v^m|^{p-2}v^m(t), \psi_k(x)\right)_{L^2(\Omega)}.
\end{aligned} \tag{3.1}$$

The above system is a system of ordinary differential equations in $f_l(t),\ l = 1,\ldots,m$, and the initial conditions

$$f_l(0) = (\psi_l, v_0(x))_{L^2(\Omega)},\ f_l'(0) = (\psi_l, v_1(x))_{L^2(\Omega)}.$$

From Caratheodory theorem [24], there exists solutions $f_l(t),\ l = 1,\ldots,m,\ t \in [0,t_m)$. We need a priori estimates that permit us to extend the solution to the whole domain $[0,T]$. Thus, for every m, there exists a function $v^m(x)$ satisfying (3.1). We next obtain bounds for v^m, which do not depend on m. In the first key estimate, we put

$$S^m(t) := \|v^m(t)\|_{W_2^1(\Omega)}^2 + \|v_t^m(t)\|_{H^1(\Omega)}^2. \tag{3.2}$$

To do this, we multiply each equation of (3.1) by the appropriate $f_k'(t)$ and add them up from 1 to m, and then integrate with respect o t to τ, with $\tau \leq T$. It follows that

$$\begin{aligned}
&(v_{tt}^m(t), v_t^m(t))_{L^2(D_\tau)} + \alpha^2 (\nabla v^m(t), \nabla v_t^m(t))_{L^2(D_\tau)} + \beta^2 (\nabla v_{tt}^m(t), \nabla v_t^m(t))_{L^2(D_\tau)} \\
&- \left(\int_0^t h(t-s) \nabla v^m(s) ds, \nabla v_t^m(t) \right)_{L^2(D_\tau)} \\
&= \alpha^2 \int_0^\tau \int_{\partial\Omega} g(x,t) v_t^m(t) dS_x dt + \alpha^2 \int_0^\tau \int_{\partial\Omega} \left(\int_0^t \int_\Omega v^m(\xi,\mu) d\xi d\mu \right) v_t^m(t) dS_x dt \\
&+ \beta^2 \int_0^\tau \int_{\partial\Omega} g_{tt}(x,t) v_t^m(t) dS_x dt + \beta^2 \int_0^\tau \int_{\partial\Omega} \left(\int_0^t \int_\Omega v_{\mu\mu}^m(\xi,\mu) d\xi d\mu \right) v_t^m(t) dS_x dt \\
&- \int_0^\tau \int_{\partial\Omega} \left(\int_0^t h(t-s) g(x,s) ds \right) v_t^m(t) dS_x dt \\
&- \int_0^\tau \int_{\partial\Omega} \left(\int_0^t h(t-s) \left\{ \int_0^s \int_\Omega v^m(\xi,\mu) d\xi d\mu \right\} ds \right) v_t^m(t) dS_x dt \\
&+ \left(|v^m|^{p-2} v^m(t), v_t^m(t) \right)_{L^2(D_\tau)}.
\end{aligned} \tag{3.3}$$

Using direct calculation, we obtain

$$(v_{tt}^m(t), v_t^m(t))_{L^2(D_\tau)} = \frac{1}{2} \|v_\tau^m(\tau)\|_{L^2(\Omega)}^2 - \frac{1}{2} \|v_t^m(0)\|_{L^2(\Omega)}^2, \tag{3.4}$$

$$\alpha^2 (\nabla v^m(t), \nabla v_t^m(t))_{L^2(D_\tau)} = \frac{\alpha^2}{2} \|\nabla v^m(\tau)\|_{L^2(\Omega)}^2 - \frac{\alpha^2}{2} \|\nabla v^m(0)\|_{L^2(\Omega)}^2, \tag{3.5}$$

$$\beta^2 (\nabla v_{tt}^m(t), \nabla v_t^m(t))_{L^2(D_\tau)} = \frac{\beta^2}{2} \|\nabla v_\tau^m(\tau)\|_{L^2(\Omega)}^2 - \frac{\beta^2}{2} \|\nabla v_t^m(0)\|_{L^2(\Omega)}^2, \tag{3.6}$$

$$\begin{aligned}
&- \left(\int_0^t h(t-s) \nabla v^m(s) ds, \nabla v_t^m(t) \right)_{L^2(D_\tau)} \\
&= \frac{1}{2} (h \circ \nabla v^m)(\tau) - \frac{1}{2} \left(\int_0^\tau h(s) ds \right) \|\nabla v^m(\tau)\|_{L^2(\Omega)}^2 - \frac{1}{2} \int_0^\tau \left(h' \circ \nabla v^m \right)(t) dt \\
&+ \frac{1}{2} \int_0^\tau h(t) \|\nabla v^m(t)\|_{L^2(\Omega)}^2 dt,
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&\beta^2 \int_0^\tau \int_{\partial\Omega} \left(\int_0^t \int_\Omega v_{\mu\mu}^m(\xi,\mu) d\xi d\mu \right) v_t^m(t) dS_x dt \\
&= \beta^2 \int_0^\tau \int_{\partial\Omega} \left(\int_\Omega v_t^m(\xi,t) d\xi \right) v_t^m(t) dS_x dt - \beta^2 \int_0^\tau \int_{\partial\Omega} \left(\int_\Omega v_t^m(\xi,0) d\xi \right) v_t^m(t) dS_x dt.
\end{aligned} \tag{3.8}$$

Substituting (3.4)-(3.8) into (3.3), we obtain

$$
\begin{aligned}
&\frac{1}{2}\left(\alpha^2-\int_0^\tau h(s)\,ds\right)\|\nabla v^m(\tau)\|^2_{L^2(\Omega)}+\frac{1}{2}\|v^m_\tau(\tau)\|^2_{L^2(\Omega)}+\frac{\beta^2}{2}\|\nabla v^m_\tau(\tau)\|^2_{L^2(\Omega)}+\frac{1}{2}(h\circ\nabla v^m)(\tau)\\
&=\frac{\alpha^2}{2}\|\nabla v^m(0)\|^2_{L^2(\Omega)}+\frac{1}{2}\|v^m_t(0)\|^2_{L^2(\Omega)}+\frac{\beta^2}{2}\|\nabla v^m_t(0)\|^2_{L^2(\Omega)}\\
&\quad+\frac{1}{2}\int_0^\tau\left(h'\circ\nabla v^m\right)(t)\,dt-\frac{1}{2}\int_0^\tau h(t)\|\nabla v^m(t)\|^2_{L^2(\Omega)}\,dt\\
&\quad-\beta^2\int_0^\tau\int_{\partial\Omega}\left(\int_\Omega v^m_t(\xi,0)\,d\xi\right)v^m_t(t)\,dS_xdt\\
&\quad+\alpha^2\int_0^\tau\int_{\partial\Omega}\left(\int_0^t\int_\Omega v^m(\xi,\mu)\,d\xi d\mu\right)v^m_t(t)\,dS_xdt\\
&\quad+\beta^2\int_0^\tau\int_{\partial\Omega}\left(\int_\Omega v^m_t(\xi,t)\,d\xi\right)v^m_t(t)\,dS_xdt\\
&\quad-\int_0^\tau\int_{\partial\Omega}\left(\int_0^t h(t-s)\left\{\int_0^s\int_\Omega v^m(\xi,\mu)\,d\xi d\mu\right\}ds\right)v^m_t(t)\,dS_xdt\\
&\quad+\alpha^2\int_0^\tau\int_{\partial\Omega}g(x,t)\,v^m_t(t)\,dS_xdt-\int_0^\tau\int_{\partial\Omega}\left(\int_0^t h(t-s)\,g(x,s)\,ds\right)v^m_t(t)\,dS_xdt\\
&\quad+\beta^2\int_0^\tau\int_{\partial\Omega}g_{tt}(x,t)\,v^m_t(t)\,dS_xdt+\left(|v^m|^{p-2}v^m(t),v^m_t(t)\right)_{L^2(D_\tau)}\\
&=\frac{\alpha^2}{2}\|\nabla v^m(0)\|^2_{L^2(\Omega)}+\frac{1}{2}\|v^m_t(0)\|^2_{L^2(\Omega)}+\frac{\beta^2}{2}\|\nabla v^m_t(0)\|^2_{L^2(\Omega)}\\
&\quad+\frac{1}{2}\int_0^\tau\left(h'\circ\nabla v^m\right)(t)\,dt-\frac{1}{2}\int_0^\tau h(t)\|\nabla v^m(t)\|^2_{L^2(\Omega)}\,dt+I_1+\ldots+I_8.
\end{aligned}
\tag{3.9}
$$

Using Young's Inequality, Cauchy-Schwarz Inequality, (**T.I**), and (3.2), we obtain

$$I_1\le\frac{\beta^2|\Omega||\partial\Omega|T}{2}\|v^m_t(0)\|^2_{L^2(\Omega)}+\frac{\beta^2\gamma_\Omega}{2}\int_0^\tau S^m(t)\,dt, \tag{3.10}$$

$$I_2\le\frac{\alpha^2\left\{|\Omega||\partial\Omega|T^2+2\gamma_\Omega\right\}}{4}\int_0^\tau S^m(t)\,dt, \tag{3.11}$$

$$I_3\le\frac{\beta^2\left\{|\Omega||\partial\Omega|+\gamma_\Omega\right\}}{2}\int_0^\tau S^m(t)\,dt, \tag{3.12}$$

$$I_4\le\frac{\left\{(\sup g(t))^2T^4|\Omega||\partial\Omega|+2\gamma_\Omega\right\}}{4}\int_0^\tau S^m(t)\,dt, \tag{3.13}$$

$$I_5\le\frac{\alpha^2}{2}\int_0^\tau\|g(t)\|^2_{L^2(\partial\Omega)}\,dt+\frac{\alpha^2\gamma_\Omega}{2}\int_0^\tau S^m(t)\,dt, \tag{3.14}$$

$$I_6\le\frac{\left(\sup h^2(t)\right)T^2}{4}\int_0^\tau\|g(t)\|^2_{L^2(\partial\Omega)}\,dt+\frac{\gamma_\Omega}{2}\int_0^\tau S^m(t)\,dt, \tag{3.15}$$

and

$$I_7\le\frac{\beta^2}{2}\int_0^\tau\|g_{tt}(t)\|^2_{L^2(\partial\Omega)}\,dt+\frac{\beta^2\gamma_\Omega}{2}\int_0^\tau S^m(t)\,dt. \tag{3.16}$$

Since $1 \leq 2 \leq 2(p-1) \leq \dfrac{2N}{N-2}$, $H^1(\Omega) \hookrightarrow L^{2p-2}$, and (3.2), we have

$$I_8 \leq C_T \|v^m(0)\|_{L^2(\Omega)}^{2p-2} + C_T \int_0^\tau (S^m(t))^{p-1}\, dt + \int_0^\tau S^m(t)\, dt. \tag{3.17}$$

Substituting (3.10)-(3.17) into (3.9), and making use of the following inequality

$$\|v^m(\tau)\|_{L^2(\Omega)}^2 \leq \|v^m(0)\|_{L^2(\Omega)}^2 + \int_0^\tau S^m(t)\, dt,$$

we obtain

$$\begin{aligned}
& \|v^m(\tau)\|_{L^2(\Omega)}^2 + \frac{\bar{h}}{2}\|\nabla v^m(\tau)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|v_\tau^m(\tau)\|_{L^2(\Omega)}^2 + \frac{\beta^2}{2}\|\nabla v_\tau^m(\tau)\|_{L^2(\Omega)}^2 \\
\leq\ & \|v^m(0)\|_{L^2(\Omega)}^2 + \frac{\alpha^2}{2}\|\nabla v^m(0)\|_{L^2(\Omega)}^2 + \frac{\left\{1+\beta^2|\Omega||\partial\Omega|T\right\}}{2}\|v_t^m(0)\|_{L^2(\Omega)}^2 \\
& + \frac{\beta^2}{2}\|\nabla v_t^m(0)\|_{L^2(\Omega)}^2 + C_T\|v^m(0)\|_{L^2(\Omega)}^{2p-2} \\
& + \frac{\left\{2\alpha^2 + \left(\sup h^2(t)\right)T^2\right\}}{2}\int_0^\tau \|g(t)\|_{L^2(\partial\Omega)}^2\, dt + \frac{\beta^2}{2}\int_0^\tau \|g_{tt}(t)\|_{L^2(\partial\Omega)}^2\, dt \\
& + C_1\int_0^\tau S^m(t)\, dt + C_T\int_0^\tau (S^m(t))^{p-1}\, dt,
\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}
C_1\ :\ &= 2 + \frac{\beta^2\gamma_\Omega}{2} + \frac{\alpha^2\left\{|\Omega||\partial\Omega|T^2 + 2\gamma_\Omega\right\}}{4} + \frac{\beta^2\left\{|\Omega||\partial\Omega| + \gamma_\Omega\right\}}{2} \\
& + \frac{\left\{(\sup g(t))^2 T^4|\Omega||\partial\Omega| + 2\gamma_\Omega\right\}}{4} + \frac{\alpha^2\gamma_\Omega}{2} + \frac{\gamma_\Omega}{2} + \frac{\beta^2\gamma_\Omega}{2}.
\end{aligned}$$

Set

$$\begin{aligned}
S^m(\tau)\ \leq\ & \omega\Big\{\|v^m(0)\|_{H^1(\Omega)}^2 + \|v_t^m(0)\|_{H^1(\Omega)}^2 + \|v^m(0)\|_{L^2(\Omega)}^{2p-2} \\
& + \int_0^\tau \|g(t)\|_{L^2(\partial\Omega)}^2\, dt + \int_0^\tau \|g_{tt}(t)\|_{L^2(\partial\Omega)}^2\, dt\Big\} \\
& + \omega\int_0^\tau S^m(t)\, dt + \omega\int_0^\tau (S^m(t))^{p-1}\, dt,
\end{aligned} \tag{3.19}$$

where

$$\omega := \frac{\max\left\{\frac{\alpha^2}{2}, \frac{\left\{1+\beta^2|\Omega||\partial\Omega|T\right\}}{2}, \frac{\beta^2}{2}, C_T, \frac{\left\{2\alpha^2+\left(\sup h^2(t)\right)T^2\right\}}{2}, \frac{\beta^2}{2}, C_1\right\}}{\min\left\{\frac{\bar{h}}{2}, \frac{\beta^2}{2}\right\}} > 0.$$

By solving (3.19) (based on the methods in [25]) and integrating from 0 to τ, we obtain

$$\begin{aligned}\int_0^\tau S^m(t)\,dt &\le \omega T\left\{\|v(0)\|_{L^2(\Omega)}^{2p-2}+\|v(0)\|_{H^1(\Omega)}^2+\|v_t(0)\|_{H^1(\Omega)}^2\right.\\ &\quad\left.+\int_0^\tau \|g(t)\|_{L^2(\partial\Omega)}^2\,dt+\int_0^\tau \|g_{tt}(t)\|_{L^2(\partial\Omega)}^2\,dt\right\}. \qquad (3.20)\end{aligned}$$

We deduce from (3.20) that

$$\|v^m(t)\|_{W_2^1(D_T)}^2+\|v_t^m(t)\|_{H^1(D_T)}^2\le A.$$

Therefore, $\{v^m\}_{m\ge1}$ is bounded in $V(D_T)$, and we can extract from it a subsequence for which we use the same notation which converges weakly in $V(D_T)$ to a limit function $v(x,t)$. We show that $v(x,t)$ is a generalized solution of (1.1). Note that $v^m(x,t)\to v(x,t)$ in $L^2(D_T)$ and $v^m(x,0)\to v(x,0)$ in $L^2(\Omega)$. Now, we prove that (2.2) holds. If we let $\phi^m(x,t):=\sum_{k=1}^{k=m} p_k(t)\,\psi_k(x)$, then

$$\begin{aligned}&-(v_t^m,\phi_t^m)_{L^2(Q_T)}+\alpha^2(\nabla v^m,\nabla\phi^m)_{L^2(D_T)}-\beta^2(\nabla v_t^m,\nabla\phi_t^m)_{L^2(Q_T)}\\ &\quad-\left(\int_0^t h(t-s)\nabla v^m(s)\,ds,\nabla\phi^m(t)\right)_{L^2(D_T)}\\ &=(v_t^m(0),\phi^m(0))_{L^2(\Omega)}+\beta^2(\nabla v_t^m(0),\nabla\phi^m(0))_{L^2(\Omega)}+\alpha^2\int_0^T\int_{\partial\Omega} g(x,t)\,\phi^m(t)\,dS_x dt\\ &\quad+\alpha^2\int_0^T\int_{\partial\Omega}\left(\int_0^t\int_\Omega v^m(\xi,\mu)\,d\xi d\mu\right)\phi^m(t)\,dS_x dt+\beta^2\int_0^T\int_{\partial\Omega} g_{tt}(x,t)\,\phi^m(t)\,dS_x dt\\ &\quad+\beta^2\int_0^T\int_{\partial\Omega}\left(\int_\Omega v_t^m(\xi,t)\,d\xi\right)\phi^m(t)\,dS_x dt-\beta^2\int_0^T\int_{\partial\Omega}\left(\int_\Omega v_t^m(\xi,0)\,d\xi\right)\phi^m(t)\,dS_x dt\\ &\quad-\int_0^T\int_{\partial\Omega}\left(\int_0^t h(t-s)\,g(x,s)\,ds\right)\phi^m(t)\,dS_x dt\\ &\quad-\int_0^T\int_{\partial\Omega}\left(\int_0^t h(t-s)\left\{\left(\int_0^s\int_\Omega v^m(\xi,\mu)\,d\xi d\mu\right)\right\}ds\right)\phi^m(t)\,dS_x dt\\ &\quad+\left(|v^m|^{p-2}v^m(t),\phi^m\right)_{L^2(D_T)}.\end{aligned}$$

for all $\phi^m(x,t)$ of the form $\sum_{k=1}^{k=m} p_k(t)\,\psi_k(x)$. Since

$$\begin{aligned}&-\left(\int_0^t h(t-s)\{\nabla v^m(s)-\nabla v(s)\}\,ds,\nabla\phi^m(t)\right)_{L^2(D_T)}\\ &\le\frac{(\sup h(t))\,T}{\sqrt2}\|\nabla v^m(t)-\nabla v(t)\|_{L^2(D_T)}\|\nabla\phi^m(t)\|_{L^2(D_T)},\end{aligned}$$

$$\int_0^t\int_\Omega (v^m(\xi,\mu)-v(\xi,\mu))\,d\xi d\mu\le\sqrt{T\,|\Omega|}\,\|v^m(t)-v(t)\|_{L^2(D_T)},$$

$$\int_0^T\left(\int_\Omega (v_t^m(\xi,t)-v_t(\xi,t))\,d\xi\right)\phi^m(t)\,dt\le\sqrt{|\Omega|}\,\|v_t^m(t)-v_t(t)\|_{L^2(D_T)}\left(\int_0^T(\phi^m(t))^2\,dt\right)^{1/2},$$

$$\int_0^T \left(\int_\Omega (v_t^m(\xi,0)-v_t(\xi,0))\,d\xi\right)\phi^m(t)\,dt \leq \sqrt{|\Omega|}T\left\|v_t^m(0)-v_t(0)\right\|_{L^2(\Omega)}\left(\int_0^T(\phi^m(t))^2\,dt\right)^{1/2},$$

$$\begin{aligned}&- \int_0^T\left[\left(\int_0^t h(t-s)\left\{\int_0^s\int_\Omega\{v^m(\xi,\mu)-v(\xi,\mu)\}\,d\xi\,d\mu\right\}ds\right)\phi^m(t)\right]dt\\ &\leq \frac{(\sup h(t))\,T^2\sqrt{|\Omega|}}{2\sqrt{2}}\left\|v^m(t)-v(t)\right\|_{L^2(D_T)}\left(\int_0^T(\phi^m(t))^2\,dt\right)^{1/2},\end{aligned}$$

$$\left\|v^m(t)-v(t)\right\|_{W_2^1(D_T)}\longrightarrow 0,\quad \text{as } m\longrightarrow\infty,$$

and

$$\left\|v_t^m(t)-v_t(t)\right\|_{H^1(D_T)}\longrightarrow 0,\quad \text{as } m\longrightarrow\infty.$$

Then

$$\begin{aligned}&-\left(\int_0^t h(t-s)\nabla v^m(s)\,ds,\nabla\phi^m(t)\right)_{L^2(D_T)}\\ \longrightarrow\ &-\left(\int_0^t h(t-s)\nabla v(s)\,ds,\nabla\phi(t)\right)_{L^2(D_T)},\quad \text{as } m\longrightarrow\infty,\end{aligned}$$

$$\begin{aligned}&\alpha^2\int_0^T\int_{\partial\Omega}\left(\int_0^t\int_\Omega v^m(\xi,\mu)\,d\xi\,d\mu\right)\phi^m(t)\,dS_x dt\\ \longrightarrow\ &\alpha^2\int_0^T\int_{\partial\Omega}\left(\int_0^t\int_\Omega v(\xi,\mu)\,d\xi\,d\mu\right)\phi(t)\,dS_x dt,\quad \text{as } m\longrightarrow\infty,\end{aligned}$$

$$\begin{aligned}&\beta^2\int_0^T\int_{\partial\Omega}\left(\int_\Omega v_t^m(\xi,t)\,d\xi\right)\phi^m(t)\,dS_x dt\\ \longrightarrow\ &\beta^2\int_0^T\int_{\partial\Omega}\left(\int_\Omega v_t(\xi,t)\,d\xi\right)\phi(t)\,dS_x dt,\quad \text{as } m\longrightarrow\infty,\end{aligned}$$

$$\begin{aligned}&-\beta^2\int_0^T\int_{\partial\Omega}\left(\int_\Omega v_t^m(\xi,0)\,d\xi\right)\phi^m(t)\,dS_x dt\\ \longrightarrow\ &-\beta^2\int_0^T\int_{\partial\Omega}\left(\int_\Omega v_t(\xi,0)\,d\xi\right)\phi(t)\,dS_x dt,\quad \text{as } m\longrightarrow\infty,\end{aligned}$$

and

$$\begin{aligned}&-\int_0^T\int_{\partial\Omega}\left(\int_0^t h(t-s)\left\{\left(\int_0^s\int_\Omega v^m(\xi,\mu)\,d\xi\,d\mu\right)\right\}ds\right)\phi^m(t)\,dS_x dt\\ \longrightarrow\ &-\int_0^T\int_{\partial\Omega}\left(\int_0^t h(t-s)\left\{\left(\int_0^s\int_\Omega v(\xi,\mu)\,d\xi\,d\mu\right)\right\}ds\right)\phi(t)\,dS_x dt,\quad \text{as } m\longrightarrow\infty.\end{aligned}$$

By means of the continuity of the function $t\longmapsto|t|^{t-2}t$, we have $|v^m|^{p-2}v^m(t)\longrightarrow|v|^{p-2}v(t)$, a.e. in D_T. On the other hand, one has

$$\left\||v^m|^{p-2}v^m(t)\right\|_{L^2(D_T)}^2\leq C_T''.$$

Using [26, Lemm 1.3], we conclude that

$$\left(|v^m|^{p-2}v^m(t),\phi^m(t)\right)_{L^2(D_T)} \longrightarrow \left(|v|^{p-2}v(t),\phi(t)\right)_{L^2(D_T)}, \quad \text{as } m \longrightarrow \infty.$$

Thus, v satisfies (3.1) for every $\phi^m(x,t) := \sum_{k=1}^{k=m} p_k(t)\psi_k(x)$. We denote by $\mathbb{Q}_m$ the totality of all functions of the form $\phi^m(x,t) := \sum_{k=1}^{k=m} p_k(t) Z_k(x)$, where $p_k(t) \in W_2^1(0,T)$, and $p_k(T) = 0$. But $\cup_{k=1}^{k=m}\mathbb{Q}_k$ is dense in $W(D_T)$. Hence, relation (3.1) holds for all $v \in W(D_T)$. Thus we have shown that the limit function $v(x,t)$ is a generalized solution of problem (1.1)-(1.3) in $V(D_T)$. □

4. The Unicity of The Generalized Solution

Theorem 4.1. *The problem (1.1)-(1.3) cannot have more than one generalized solution in* $V(Q_T)$.

Proof. Let $v_1 \in V(D_T)$, and $v_2 \in V(D_T)$ be two solutions of problem (1.1)-(1.3) such that v_1 is different from v_2. Then $v := v_1 - v_2$ solves

$$\begin{cases} v_{tt} - \alpha^2\Delta v - \beta^2\Delta v_{tt} + \int_0^t h(t-s)\Delta v(s)\,ds \\ \qquad = |v_1|^{p-2}v_1 - |v_2|^{p-2}v_2, \\ v(x,0) = v_t(x,0) = 0, \\ \dfrac{\partial v}{\partial \eta} = \int_0^t\int_\Omega v(\xi,\tau)\,d\xi d\tau, \quad (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{4.1}$$

and (2.2) gives

$$\begin{aligned} &- \quad (v_t(t),u_t(t))_{L^2(D_T)} + \alpha^2(\nabla v,\nabla u(t))_{L^2(D_T)} - \beta^2(\nabla v_t,\nabla u_t)_{L^2(Q_T)} \\ &- \quad \left(\int_0^t h(t-s)\nabla v(s)\,ds, \nabla u(t)\right)_{L^2(D_T)} \\ &= \quad \alpha^2\int_0^T\int_{\partial\Omega}\left[\left(\int_0^t\int_\Omega v(\xi,\mu)\,d\xi d\mu\right)u(t)\right]dS_x dt \\ &+ \quad \beta^2\int_0^T\int_{\partial\Omega}\left[\left(\int_\Omega v_t(\xi,t)\,d\xi\right)u(t)\right]dS_x dt \\ &\quad -\int_0^T\int_{\partial\Omega}\left[\left(\int_0^t g(t-s)\left\{\int_0^s\int_\Omega v(\xi,\mu)\,d\xi d\mu\right\}ds\right)u(t)\right]dS_x dt \\ &+ \quad \left(|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2, u(t)\right)_{L^2(D_T)}. \end{aligned} \tag{4.2}$$

Define the function $u(x,t)$ by

$$u(x,t) := \begin{cases} \int_t^\tau v(x,s)\,ds, & 0 \le t \le \tau, \\ 0, & \tau \le t \le T. \end{cases} \tag{4.3}$$

It is obvious that $v \in W(D_T)$ and $u_t(x,t) = -v(x,t)$ for all $t \in [0,\tau]$. Integration by parts in the **LHS** of (4.2) gives

$$-(v_t,u_t)_{L^2(D_T)} = \frac{1}{2}\|v(\tau)\|^2_{L^2(\Omega)}, \tag{4.4}$$

$$\alpha^2 (\nabla v, \nabla u)_{L^2(D_T)} = \frac{\alpha^2}{2} \|\nabla u(0)\|^2_{L^2(\Omega)}, \tag{4.5}$$

and

$$-\beta^2 (\nabla v_t, \nabla u_t)_{L^2(D_T)} = \frac{\beta^2}{2} \|\nabla v(\tau)\|^2_{L^2(\Omega)}. \tag{4.6}$$

Substituting (4.4)-(4.6), we arrive at

$$\begin{aligned}
&\frac{1}{2}\|v(\tau)\|^2_{L^2(\Omega)} + \frac{\beta^2}{2}\|\nabla v(\tau)\|^2_{L^2(\Omega)} + \frac{\alpha^2}{2}\|\nabla u(0)\|^2_{L^2(\Omega)} \\
&= -\left(\int_0^t h(t-s)\nabla v(s)\,ds, \nabla u(t)\right)_{L^2(D_T)} \\
&\quad + \alpha^2 \int_0^T \int_{\partial\Omega} \left[\left(\int_0^t \int_\Omega v(\xi,\mu)\,d\xi d\mu\right) u(t)\right] dS_x dt \\
&\quad + \beta^2 \int_0^T \int_{\partial\Omega} \left[\left(\int_\Omega v_t(\xi,t)\,d\xi\right) u(t)\right] dS_x dt \\
&\quad - \int_0^T \int_{\partial\Omega} \left[\left(\int_0^t g(t-s)\left\{\int_0^s \int_\Omega v(\xi,\mu)\,d\xi d\mu\right\} ds\right) u(t)\right] dS_x dt \\
&\quad + \left(|v_1|^{p-2} v_1 - |v_2|^{p-2} v_2, u(t)\right)_{L^2(D_T)} \\
&= j_1 + \ldots + j_5.
\end{aligned} \tag{4.7}$$

Put

$$S(t) := \|v(t)\|^2_{W_2^1(\Omega)} + \|v_t(t)\|^2_{H^1(\Omega)} + \|\nabla\theta(t)\|^2_{L^2(\Omega)}, \tag{4.8}$$

where

$$\theta(x,t) := \int_0^t v(x,s)\,ds. \tag{4.9}$$

Using (4.4), and (4.9), we have

$$u(x,t) = \theta(x,\tau) - \theta(x,t), \; \nabla u(x,0) = \nabla\theta(x,\tau),$$

and

$$\int_0^\tau \|\nabla u(t)\|^2_{L^2(\Omega)} dt \le 2\tau \|\nabla\theta(\tau)\|^2_{L^2(\Omega)} + 2\int_0^\tau \|\nabla\theta(t)\|^2_{L^2(\Omega)} dt. \tag{4.10}$$

Using (4.4), and (4.8), we have

$$\int_0^\tau \|u(t)\|^2_{L^2(\Omega)} dt \le T^2 \int_0^\tau S(t)\,dt. \tag{4.11}$$

Using Young's inequality, Cauchy-Schwarz inequality, $(\mathbf{T.I})$, (4.8), (4.10), and (4.11), we have

$$j_1 \le \frac{\left\{(\sup h(t))^2 T^2 + 4\right\}}{4} \int_0^\tau S(t)\,dt + \tau\|\nabla\theta(\tau)\|^2_{L^2(\Omega)}, \tag{4.12}$$

$$j_2 \le \frac{\alpha^2\left\{T^2\left[2\gamma_\Omega + |\Omega|\,|\partial\Omega|\right] + 4\gamma_\Omega\right\}}{4} \int_0^\tau S(t)\,dt + \tau\alpha^2\gamma_\Omega \|\nabla\theta(\tau)\|^2_{L^2(\Omega)}, \tag{4.13}$$

$$j_3 \le \frac{\beta^2\left\{|\Omega|\,|\partial\Omega| + 3\gamma_\Omega\right\}}{2} \int_0^\tau S(t)\,dt + \tau\beta^2\gamma_\Omega \|\nabla\theta(\tau)\|^2_{L^2(\Omega)}, \tag{4.14}$$

and

$$j_4 \le \frac{\left\{T^2\left[(\sup h(t))^2 T^2 |\Omega| |\partial\Omega| + 8\gamma_\Omega\right] + 16\gamma_\Omega\right\}}{16}\int_0^\tau S(t)\,dt + \gamma_\Omega \tau \|\nabla\theta(\tau)\|^2_{L^2(\Omega)}. \tag{4.15}$$

Since $1 \le 2 \le 2(p-1) \le \frac{2N}{N-2}$, $H^1(\Omega) \hookrightarrow L^{2p-2}$, we conclude from (4.8) and (4.11) that

$$I_5 \le C'_T \int_0^\tau (S(t))^{p-1}\,dt + \frac{\{2+T^2\}}{2}\int_0^\tau S(t)\,dt. \tag{4.16}$$

Substituting (4.12)-(4.16) into (4.7), we have

$$\begin{aligned}&\frac{1}{2}\|v(\tau)\|^2_{L^2(\Omega)} + \frac{\beta^2}{2}\|\nabla v(\tau)\|^2_{L^2(\Omega)} + \frac{\alpha^2}{2}\|\nabla\theta(\tau)\|^2_{L^2(\Omega)}\\ &\le C'_T\int_0^\tau (S(t))^{p-1}\,dt + C_2\int_0^\tau S(t)\,dt + \tau\left\{1+\left[\alpha^2+\beta^2+1\right]\gamma_\Omega\right\}\|\nabla\theta(\tau)\|^2_{L^2(\Omega)},\end{aligned} \tag{4.17}$$

where

$$\begin{aligned}C_2 \;:&= 2 + \frac{T^2}{2} + \frac{(\sup h(t))^2 T^2 + 4}{4} + \frac{\alpha^2 T^2\left[2\gamma_\Omega + |\Omega| |\partial\Omega|\right] + 4\gamma_\Omega}{4}\\ &+ \frac{\beta^2\{|\Omega| |\partial\Omega| + 3\gamma_\Omega\}}{2} + \frac{T^2\left[(\sup h(t))^2 T^2 |\Omega| |\partial\Omega| + 8\gamma_\Omega\right] + 16\gamma_\Omega}{16}.\end{aligned}$$

Multiplying the differential equation in (4.1) by v_t and integrating over $D_\tau := \Omega\times(0,\tau)$, we obtain

$$\begin{aligned}&(v_{tt}, v_t)_{L^2(D_\tau)} - \alpha^2(\Delta v, v_t)_{L^2(D_\tau)} - \beta^2(\Delta v_{tt}, v_t)_{L^2(D_\tau)} + \left(\int_0^t h(t-s)\Delta v(s)\,ds, v_t\right)_{L^2(D_\tau)}\\ &= \left(|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2, v_t\right)_{L^2(D_\tau)}.\end{aligned} \tag{4.18}$$

This implies that

$$\begin{aligned}&\frac{1}{2}\left(\alpha^2 - \int_0^\tau h(s)\,ds\right)\|\nabla v(\tau)\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_\tau(\tau)\|^2_{L^2(\Omega)} + \frac{\beta^2}{2}\|\nabla v_\tau(\tau)\|^2_{L^2(\Omega)} + \frac{1}{2}(h\circ\nabla v)(\tau)\\ &\quad - \frac{1}{2}\int_0^\tau (h'\circ\nabla v)(t)\,dt + \frac{1}{2}\int_0^\tau h(t)\|\nabla v(t)\|^2_{L^2(\Omega)}\,dt\\ &= \alpha^2\int_0^\tau\int_{\partial\Omega}\left(\int_0^t\int_\Omega v(\xi,\mu)\,d\xi d\mu\right)v_t(t)\,dS_x dt + \beta^2\int_0^\tau\int_{\partial\Omega}\left(\int_\Omega v_t(\xi,t)\,d\xi\right)v_t(t)\,dS_x dt\\ &- \int_0^\tau\int_{\partial\Omega}\left(\int_0^t h(t-s)\left\{\int_0^s\int_\Omega v(\xi,\mu)\,d\xi d\mu\right\}ds\right)v_t(t)\,dS_x dt\\ &\quad + \left(|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2, v_t\right)_{L^2(D_\tau)}\\ &= j_6 + \ldots + j_9.\end{aligned} \tag{4.19}$$

Using Young's inequality, Cauchy-Schwarz inequality, $(\mathbf{T}.\mathbf{I})$, and (4.8), we have

$$j_6 \leq \frac{\alpha^2 \left\{T^2 |\Omega| |\partial\Omega| + 2\gamma_\Omega\right\}}{4} \int_0^\tau S(t)\,dt, \tag{4.20}$$

$$j_7 \leq \frac{\beta^2 \left\{|\Omega| |\partial\Omega| + \gamma_\Omega\right\}}{2} \int_0^\tau S(t)\,dt, \tag{4.21}$$

and

$$j_8 \leq \frac{\left\{(\sup h(t))^2 T^4 |\Omega| |\partial\Omega| + 8\gamma_\Omega\right\}}{16} \int_0^\tau S(t)\,dt. \tag{4.22}$$

Since $1 \leq 2 \leq 2(p-1) \leq \frac{2N}{N-2}$, $H^1(\Omega) \hookrightarrow L^{2p-2}$, we obtain from (4.8) that

$$I_9 \leq C_T' \int_0^\tau (S(t))^{p-1}\,dt + \frac{3}{2}\int_0^\tau S(t)\,dt. \tag{4.23}$$

Substituting (4.20)- (4.23) into (4.19), we see that

$$\begin{aligned} &\frac{\bar{h}}{2}\|\nabla v(\tau)\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_\tau(\tau)\|^2_{L^2(\Omega)} + \frac{\beta^2}{2}\|\nabla v_\tau(\tau)\|^2_{L^2(\Omega)} \\ \leq\ & C_T' \int_0^\tau (S(t))^{p-1}\,dt + C_3\int_0^\tau S(t)\,dt, \end{aligned} \tag{4.24}$$

where

$$\begin{aligned} C_3 \ :\ =\ & \frac{\alpha^2 \left\{T^2 |\Omega| |\partial\Omega| + 2\gamma_\Omega\right\}}{4} + \frac{\beta^2 \left\{|\Omega| |\partial\Omega| + \gamma_\Omega\right\}}{2} \\ & + \frac{\left\{(\sup h(t))^2 T^4 |\Omega| |\partial\Omega| + 8\gamma_\Omega\right\}}{16} + \frac{3}{2}. \end{aligned}$$

From (4.17) and (4.24), we obtain

$$\begin{aligned} &\frac{1}{2}\|v(\tau)\|^2_{L^2(\Omega)} + \frac{\left\{\beta^2 + \bar{h}\right\}}{2}\|\nabla v(\tau)\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_\tau(\tau)\|^2_{L^2(\Omega)} + \frac{\beta^2}{2}\|\nabla v_\tau(\tau)\|^2_{L^2(\Omega)} \\ &\quad + \left\{\frac{\alpha^2}{2} - \tau\left(1 + \left[\alpha^2 + \beta^2 + 1\right]\gamma_\Omega\right)\right\}\|\nabla\theta(\tau)\|^2_{L^2(\Omega)} \\ &\leq 2C_T' \int_0^\tau (S(t))^{p-1}\,dt + \{C_2 + C_3\}\int_0^\tau S(t)\,dt. \end{aligned}$$

Since τ is arbitrary, we assume that $\left(\frac{\alpha^2}{2} - \tau\left(1 + \left[\alpha^2 + \beta^2 + 1\right]\gamma_\Omega\right)\right) > 0$. Hence,

$$S(t) \leq \omega' \int_0^\tau (S(t))^{p-1}\,dt + \omega' \int_0^\tau S(t)\,dt, \tag{4.25}$$

where

$$\omega' := \frac{\max\left\{2C_T', C_2 + C_3\right\}}{\min\left\{\frac{1}{2}, \frac{\beta^2}{2}, \frac{\alpha^2}{2} - \tau\left(1 + \left[\alpha^2 + \beta^2 + 1\right]\gamma_\Omega\right)\right\}} > 0.$$

Solving (4.25) (based on the methods in [25]), we obtain

$$\|v(t)\|^2_{W_2^1(\Omega)} + \|v_t(t)\|^2_{H^1(\Omega)} + \|\nabla\theta(t)\|^2_{L^2(\Omega)} \leq 0, \quad \forall \tau \in \left[0, \frac{\alpha^2}{2\left(1 + \left[\alpha^2 + \beta^2 + 1\right]\gamma_\Omega\right)}\right].$$

Proceeding in the same way for

$$\tau \in \left[\frac{(m-1)\,\alpha^2}{2\left(1+\left[\alpha^2+\beta^2+1\right]\gamma_\Omega\right)}, \frac{m\alpha^2}{2\left(1+\left[\alpha^2+\beta^2+1\right]\gamma_\Omega\right)}\right]$$

to cover the whole interval $[0,T]$, we have $v(x,\tau)=0$, for all τ in $[0,T]$. □

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