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DYNAMICS OF A NON-AUTONOMOUS TWO SPECIES LOTKA-VOLTERRA COOPERATIVE SYSTEMS WITH DELAYS

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Abstract. A class of non-autonomous two-species Lotka-Volterra cooperative population systems with time delays is considered. Some new sufficient conditions on the boundedness, permanence, periodic solution, and global attractivity of the systems are established by using the comparison method and the construction of suitable Lyapunov functional.

Keywords. Global attractivity; Lotka-Volterra cooperative system; Lyapunov functional; Permanence; Time delay.

1. INTRODUCTION

The dynamical behavior of population dynamical systems is one of the hotspots of the study of modern mathematical biology. In particular, the dynamical behavior of mathematical population cooperative dynamical systems is one of the important disciplines in modern applied mathematics, and the most popular topics among the scholars. Recently, a lot of results related to the population cooperative dynamical systems were established; see, e.g., [1]-[11] and the references cited therein. Most of these results concerned with the extinction, permanence, global attractivity, and the existence of periodic solutions and so on. In [3], for example, Li and Lu considered the following two species autonomous Lotka-Volterra systems with delays

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[r_1 - a_1x_1(t) + a_{11}x_1(t-\tau_{11}) + a_{12}x_2(t-\tau_{12})], \\ \dot{x}_2(t) &= x_2(t)[r_2 - a_2x_2(t) + a_{21}x_1(t-\tau_{21}) + a_{22}x_2(t-\tau_{22})]. \end{aligned} \tag{1.1}$$

They obtained some sufficient conditions for the permanence of system (1.1) for the competitive case and cooperative case, respectively. In [5], Lu, Lu and Lian considered the following two

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species autonomous Lotka-Volterra cooperative systems with delays

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)[r_1 - a_1x_1(t) - a_{11}x_1(t-\tau_{11}) + a_{12}x_2(t-\tau_{12})],\\ \dot{x}_2(t) &= x_2(t)[r_2 - a_2x_2(t) + a_{21}x_1(t-\tau_{21}) - a_{22}x_2(t-\tau_{22})].\end{aligned} \tag{1.2}$$

They obtained some sufficient conditions for the permanence of system (1.2). In [7], Nakata and Muroya studied the following two species non-autonomous Lotka-Volterra system with delays

$$\begin{aligned}\dot{x}_1(t) = \;& x_1(t)[r_1(t) - a_{11}^1(t)x_1(t-\tau)\\ & -a_{11}^2(t)x_1(t-2\tau) + a_{12}^1(t)x_2(t-\tau)],\\ \dot{x}_2(t) = \;& x_2(t)[r_2(t) + a_{21}^0(t)x_1(t) + a_{21}^1(t)x_1(t-\tau)\\ & -a_{22}^0(t)x_2(t) - a_{22}^1(t)x_2(t-\tau)].\end{aligned} \tag{1.3}$$

They established some sufficient conditions which ensured the system to be permanent. For that reasons and based on the above works, in this paper, we consider the following two species non-autonomous Lotka-Volterra cooperative with discrete time delays

$$\begin{aligned}\dot{x}_1(t) = \;& x_1(t)\left[r_1(t) - a_{11}^0(t)x_1(t) - a_{11}^1(t)x_1(t-\tau) + a_{12}(t)x_2(t-\tau)\right],\\ \dot{x}_2(t) = \;& x_2(t)\left[r_2(t) + a_{21}(t)x_1(t) - a_{22}^0(t)x_2(t) - a_{22}^1(t)x_2(t-\tau)\right],\end{aligned} \tag{1.4}$$

The cooperative relationship between population was, is and will always be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. In addition, an important problem in cooperative theory and related topics in mathematical ecological dynamical systems, concerns the permanence, periodic solution, and global attractivity of considered dynamical system. Hence, in this paper, our main purpose is to establish some sufficient conditions on the boundedness, permanence, periodic solution, and global attractivity for system (1.4) by using the comparison method and the construction of suitable Lyapunov functional.

2. Preliminaries

In system (1.4), $x_i(t)$, $i = 1,2$, denotes the density of the two cooperative species at time t, respectively; $r_i(t)$, $i = 1,2$, represents the intrinsic growth rate of three species $x_i (i = 1,2)$ at time t. $a_{12}(t)$ and $a_{21}(t)$ represent the cooperative coefficients between species x_1 and x_2 at time t. τ is positive constant. Throughout this paper, we always assume that system (1.4) satisfies the following assumptions

(H_1) $r_i(t), a_{11}^0(t), a_{11}^1(t), a_{12}(t)$, $a_{21}(t)$, $a_{22}^0(t), a_{22}^1(t)$ are continuous, bounded, and strictly positive functions on $[0,\infty)$.

(H_2) $r_i(t), a_{11}^0(t), a_{11}^1(t), a_{12}(t), a_{21}(t), a_{22}^0(t), a_{22}^1(t)$ are all continuously positive ω-periodic functions on $[0,\omega]$.

Throughout this paper, for system (1.4), we consider the solution with the following initial condition

$$x_i(t) = \phi_i(t) \quad \text{for all} \quad t \in [-\tau, 0),\ i = 1,2, \tag{2.1}$$

where $\phi_i(t)$ $(i = 1,2)$ are nonnegative continuous functions defined on $[-2\tau, 0)$ satisfying $\phi_i(0) > 0$ $(i = 1,2)$.

For a continuous and bounded function $f(t)$ defined on $[0,\infty)$, we define $f^L = \inf_{t\in[0,\infty)}\{f(t)\}$ and $f^M = \sup_{t\in[0,\infty)}\{f(t)\}$.

On the global attractivity of system (1.4), we have the following definition.

Definition 2.1. [8] System (1.4) is said to be global attractive if, for any two positive solutions $(x_1(t), x_2(t))$ and $(y_1(t), y_2(t))$ of system (1.4)

$$\lim_{t\to\infty}(x_i(t) - y_i(t)) = 0, \quad i = 1, 2.$$

The following three lemmas will be used in the proofs of the main results on the boundedness.

Lemma 2.2. [8] *Let $y(t) \geq 0$ be a function defined on $[-m\tau, \infty)$ satisfying that*

$$\dot{y}(t) \leq y(t)(\lambda - \sum_{l=0}^{m} \mu^l y(t - l\tau)) + D,$$

where

$$\lambda > 0,\ \mu^l \geq 0 (l = 0, 1, 2, \cdots m),\ \mu = \sum_{l=0}^{m} \mu^l > 0,\ D \geq 0,$$

are constants. Then there exists a positive constant M_y such that

$$\limsup_{t\to\infty} y(t) \leq M_y = -\frac{D}{\lambda} + \big(\frac{D}{\lambda} + y^*\big)\exp(\lambda m\tau), \tag{2.2}$$

where $y = y^*$ is the unique positive solution of equation

$$y(\lambda - \mu y) + D = 0.$$

Lemma 2.3. [11] *Let $\dot{u}(t) = u(t)(d_1 - d_2 u(t))$, where $d_2 > 0$. Then,*
(1) if $d_1 > 0$, then $\lim_{t\to+\infty} u(t) = d_1/d_2$;
(2) if $d_1 < 0$, then $\lim_{t\to+\infty} u(t) = 0$.

Lemma 2.4. [8] *Let $y(t) \geq 0$ be a function defined on $[-m\tau, \infty)$ satisfying that*

$$\dot{y}(t) \geq y(t)(\lambda - \sum_{l=0}^{m} \mu^l y(t - l\tau)) + D,$$

where

$$\lambda > 0,\ \mu^l \geq 0 (l = 0, 1, 2, \cdots m),\ \mu = \sum_{l=0}^{m} \mu^l > 0 \quad \text{and } D \geq 0,$$

are constants. If (2.2) holds, then there exists a positive constant m_y such that

$$\liminf_{t\to\infty} y(t) \geq m_y = \frac{\lambda}{\mu}\exp\{(\lambda - \mu M_y)m\tau\}.$$

Consider the following periodic differential equation with solution $x(t, 0, \Phi)$

$$\frac{dx}{dt} = F(t, x_t), \tag{2.3}$$

where $F(t, x_t)$ is a n-dimensional continuous functional and

$$x(t) \in R^n, x(t, 0, \Phi) = (x_1(t, 0, \Phi), x_2(t, 0, \Phi), \cdots, x_n(t, 0, \Phi))$$

is a solution of the functional differential equation with initial condition $x_0 = \Phi$.

Lemma 2.5. [11] *If there exist positive constants m and M, for any $\Phi \in C_+^n[-\tau, 0]$, such that*

$$m < \liminf_{t\to\infty} x_i(t, 0, \Phi) \leq \limsup_{t\to\infty} x_i(t, 0, \Phi) < M, \qquad i = 1, 2, \cdots, n.$$

then system (2.3) admits at least one positive ω-periodic solution.

3. Main Results

In this section, we will obtain some sufficient conditions for the boundedness and permanence of system (1.4). First, we denote the following functions

$$b_1(t) = a_{11}^1(t) - a_{21}(t-\tau), \quad b_2(t) = a_{22}^0(t-\tau) - a_{12}(t).$$

Theorem 3.1. *Assume that* (H_1) *holds,* $b_i^L > 0$ $(i = 1,2,\cdots,6)$, $a_{ii}^{1L} > 0$ $(i = 1,2)$ *and* $a_{33}^{2L} > 0$. *Then, for any positive solution* $(x_1(t), x_2(t))$ *of system (1.4),*

$$\limsup_{t\to\infty} x_1(t)x_2(t-\tau) \le A_1 = \frac{(r_1^M + r_2^M)^2}{a_{11}^{1L} a_{22}^{2L}} \exp\{(r_1^M + r_2^M)\tau\}. \tag{3.1}$$

Proof. First, we suppose that $\limsup_{t\to\infty} x_1(t)x_2(t-\tau) = \infty$. Then, we obtain that there exists a time sequence $\{t_k\}_{k=1}^{\infty}$ such that

$$\limsup_{t\to\infty} x_1(t_k)x_2(t_k-\tau) = \infty, \tag{3.2}$$

and

$$\frac{d}{dt}(x_1(t)x_2(t-\tau))|_{t=t_k} \ge 0, \quad k = 1,2,\cdots. \tag{3.3}$$

From system (1.4), we obtain

$$\begin{aligned} & \frac{d}{dt}(x_1(t)x_2(t-\tau)) \\ = \; & x_1(t)x_2(t-\tau)\big(r_1(t) + r_2(t-\tau) - a_{11}^0(t)x_1(t) - a_{11}^1(t)x_1(t-\tau) + a_{12}(t)x_2(t-\tau) \\ & + a_{21}(t-\tau)x_1(t-\tau) - a_{22}^0(t-\tau)x_3(t-\tau) - a_{22}^1(t-\tau)x_2(t-2\tau)\big) \\ \le \; & x_1(t)x_2(t-\tau)\big(r_1^M + r_2^M - b_1^L x_1(t-\tau) - b_2^L x_2(t-\tau) - a_{22}^{1L} x_2(t-2\tau)\big). \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we have

$$\sum_{i=1}^{2} b_i^L x_i(t_k-\tau) + a_{22}^{1L} x_2(t_k - 2\tau) \le r_1^M + r_2^M.$$

Thus,

$$x_1(t_k-\tau) \le \frac{r_1^M + r_2^M}{b_1^L} \quad \text{and} \quad x_2(t_k - 2\tau) \le \frac{r_1^M + r_2^M}{a_{22}^{1L}}.$$

Moreover, integrating both side of 3.4) from $t_k - \tau$ to t_k, we further have

$$x_1(t_k)x_2(t_k-\tau) \le x_1(t_k-\tau)x_2(t_k-2\tau)\exp\{(r_1^M + r_2^M)\tau\}.$$

It follows that

$$x_1(t_k)x_2(t_k-\tau) \le \frac{(r_1^M + r_2^M)^2}{b_1^L a_{22}^{1L}} \exp\{(r_1^M + r_2^M)\tau\}.$$

This leads to a contradiction with (3.2).

Therefor, $\limsup_{t\to\infty}(x_1(t)x_2(t-\tau)) < \infty$. Moreover, we can also obtain that

$$\limsup_{t\to\infty}(x_1(t)x_2(t-\tau)) \le \frac{(r_1^M + r_2^M)^2}{b_1^L a_{22}^{1L}} \exp\{(r_1^M + r_2^M)\tau\},$$

This completes the proof. □

Theorem 3.2. *Assume that* (H_1) *holds, and* $b_i^L > 0$. *Then, for any positive solution* $(x_1(t), x_2(t))$ *of system (1.4),*

$$x_1(t) \le M_1, \quad x_2(t) \le M_2.$$

Proof. First, we show that $x_1(t)$ is bounded. From Theorem 3.1, for any positive constant $\varepsilon_1 > 0$, we see that there exists a positive constant T_1 such that

$$x_1(t)x_2(t-\tau) \leq V_1 + \varepsilon_1 \quad \text{for all} \quad t \geq T_1.$$

Then, from the first equation of system (1.4), we have

$$\dot{x_1}(t) \leq x_1(t)\big(r_1^M - a_{11}^{0L}x_1(t) - a_{11}^{1L}x_1(t-\tau)\big) + a_{12}^M(V_1+\varepsilon_1), \quad t \geq T_1.$$

In view of Lemma 2.2, we have

$$\limsup_{t\to\infty} x_1(t) \leq M_1 \triangleq -\frac{a_{12}^M V_1}{r_1^M} + \big(\frac{a_{12}^M V_1}{r_1^M} + x_1^*\big)\exp\big(r_1^M\tau\big). \tag{3.5}$$

For any positive constant $\varepsilon_2 > 0$, we see from (3.5) that there exists a positive constant T_2 such that

$$x_1(t-\tau) \leq M_1 + \varepsilon_2 \quad t \geq T_2.$$

From the second equation of system (1.4), we have

$$\dot{x_2}(t) \leq x_2(t)\big(r_2^M + a_{21}^M(M_1+\varepsilon_2) - a_{22}^{0L}x_2(t)\big), \quad t \geq T_2.$$

Using Lemma 2.3 and the comparison method, we obtain

$$\lim_{t\to\infty} x_2(t) \leq M_2 \triangleq \frac{r_2^M + a_{21}^M M_1}{a_{22}^{0L}}.$$

Hence, there exists a positive constant T_3 such that

$$x_i(t) \leq M_i (i=1,2),\ t \geq T_3.$$

This completes the proof. □

Theorem 3.3. *Assume that* (H_1) *holds, and* $b_i^L > 0$ $(i=1,2)$. *Then, system (1.4) is permanent.*

Proof. From the first and second equation of system (1.4), we see that

$$\dot{x_1}(t) \geq x_1(t)\big(r_1^L - a_{10}^{1M}x_1(t) - a_{11}^{1M}x_1(t-\tau)\big),$$

and

$$\dot{x_2}(t) \geq x_2(t)\big(r_2^L - a_{22}^{0M}x_2(t) - a_{22}^{1M}x_2(t-\tau)\big).$$

By using Lemma 2.4, we obtain that

$$\liminf_{t\to\infty} x_1(t) \geq m_1, \quad \liminf_{t\to\infty} x_2(t) \geq m_2,$$

where

$$m_1 = \frac{r_1^L}{a_{11}^{0M} + a_{11}^{1M}} \exp\big\{(r_1^L - (a_{11}^{1M} + a_{11}^{2M})M_1)\tau\big\},$$

and

$$m_2 = \frac{r_2^L}{a_{22}^{0M} + a_{22}^{1M}} \exp\big\{(r_2^L - (a_{22}^{0M} + a_{22}^{1M})M_2)\tau\big\}.$$

This completes the proof. □

As a direct result of Lemma 2.5, we have from Theorem 3.1 and Theorem 3.2 the following result.

Corollary 3.4. *Assume that* (H_2) *holds and* $b_i^L > 0$ $(i=1,2)$, *then system (1.4) is permanent and has at least one positive* $\omega-$*periodic solution.*

On the global attractivity of system (1.4), we have the following result.

Theorem 3.5. *Suppose that* (H_1) *and* $c_i^L > 0$ $(i = 1,2)$ *hold. Then, system* (1.4) *is globally attractive, where* $c_1(t) = a_{11}^0(t) - a_{11}^1(t+\tau) - a_{21}(t), andc_2(t) = a_{22}^0(t) - a_{22}^1(t+\tau) - a_{12}(t+\tau)$.

Proof. Let $(x_1(t), x_2(t))$ and $(y_1(t), y_2(t))$ be any two positive solutions of system (1.4). Let

$$\begin{aligned} V(t) = & \sum_{i=1}^{2} |\ln x_i(t) - \ln y_i(t)| + \int_{t-\tau}^{t} a_{11}^1(s+\tau)|x_1(s) - y_1(s)|ds \\ & + \int_{t-\tau}^{t} a_{12}(s+\tau)|x_2(s) - y_2(s)|ds + \int_{t-\tau}^{t} a_{22}^1(s+\tau)|x_2(s) - y_2(s)|ds. \end{aligned}$$

Calculating the right-upper derivative of $V(t)$ along system (1.4), we have

$$\begin{aligned} D^+W_1(t) = \ & \mu_1 \text{sign}(x_1(t) - y_1(t))\big[-a_{11}^0(t)(x_1(t) - y_1(t)) \\ & -a_{11}^1(t)(x_1(t-\tau) - y_1(t-\tau)) + a_{12}(t)(x_2(t-\tau) - y_2(t-\tau))\big] \\ & +\mu_2 \text{sign}(x_2(t) - y_2(t))\big[a_{21}(t)(x_1(t) - y_1(t)) \\ & -a_{22}^0(t)(x_2(t) - y_2(t)) - a_{22}^1(t)(x_2(t-\tau) - y_2(t-\tau))\big] \\ & +a_{11}^1(t+\tau)|x_1(t) - y_1(t)| + (a_{12}(t+\tau) + a_{22}^1(t+\tau))|x_2(t) - y_2(t)| \\ & -a_{11}^1(t)|x_1(t-\tau) - y_1(t-\tau)| - (a_{12}(t) + a_{22}^1(t))|x_2(t-\tau) - y_2(t-\tau)| \\ \le \ & -(a_{11}^0(t) - a_{11}^1(t+\tau) - a_{21}(t))|x_1(t) - y_1(t)| \\ & -(a_{22}^0(t) - a_{22}^1(t+\tau) - a_{12}(t+\tau))|x_2(t) - y_2(t)| \\ \le \ & -c_1^L|x_1(t) - y_1(t)| - c_2^L|x_2(t) - y_2(t)| \\ \le \ & -c(|x_1(t) - y_1(t)| + |x_2(t) - y_2(t)|), \end{aligned} \tag{3.6}$$

where $c = \min\{c_i^L (i = 1,2)\}$. Integrating from 0 to t on both sides of (3.6) yields

$$V(t) + c\int_0^t \big(\sum_{i=1}^{2} |x_i(s) - y_i(s)|\big)ds \le V(0).$$

Hence, $V(t)$ is bounded on $[0, \infty)$, and

$$\int_0^t \big(\sum_{i=1}^{2} c|x_i(s) - y_i(s)|\big)ds < \infty.$$

From Theorem 3.2, we can obtain that $(x_i(t) - y_i(t))(i = 1,2)$ and their derivatives remain bounded on $[0, \infty)$. As a consequence, $|x_i(t) - y_i(t)|$ $(i = 1,2)$ is uniformly continuous on $[0, \infty)$. By Barbalat's lemma, it follows that

$$\lim_{t\to\infty} \sum_{i=1}^{2} |x_i(t) - y_i(t)| = 0.$$

Hence,

$$\lim_{t\to\infty} (x_i(t) - y_i(t)) = 0, \quad i = 1,22.$$

This completes the proof. □

From Corollary 3.4 and Theorem 3.3, we have the following result.

Corollary 3.6. *If the conditions of Corollary 3.4 and Theorem 3.5 hold, system (1.4) has a globally attractive positive-ω periodic solution.*

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