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THE EXISTENCE AND UNIQUENESS OF SOLUTIONS TO A NEW FRACTIONAL DIFFERENTIAL SYSTEM WITH p-LAPLACIAN OPERATORS

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Abstract. In this paper, we focus on the following fractional differential system with p-Laplacian operators

$$\begin{cases} D_{0^+}^{\beta_1}(\varphi_p(D_{0^+}^{\alpha_1}u(t)))+f_1(t,u(t),v(t))=0, 0\le t\le 1, \\ D_{0^+}^{\beta_2}(\varphi_p(D_{0^+}^{\alpha_2}v(t)))+f_2(t,u(t),v(t))=0, 0\le t\le 1, \\ u(0)=u(1)=u'(0)=D_{0^+}^{\alpha_1-2}u(0)=D_{0^+}^{\alpha_1}u(0)=0, \\ v(0)=v(1)=v'(0)=D_{0^+}^{\alpha_2-2}v(0)=D_{0^+}^{\alpha_2}v(0)=0, \end{cases}$$

where $3<\alpha_i\le 4$, $0<\beta_i\le 1$, $f_i\in C([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$, $D_{0^+}^{\beta_i}$ and $D_{0^+}^{\alpha_i}$ are the standard Riemann-Liouville fractional derivatives, $i=1,2$, and $\varphi_p(s)$ is the p-Laplacian operator defined by $\varphi_p(s)=|s|^{p-2}s$, and $\varphi_p^{-1}(s)=\varphi_q(s)$ with $\frac{1}{p}+\frac{1}{q}=1$, $p>1$. The existence and uniqueness solutions are obtained via the Leray-Schauder nonlinear alternative and Banach's fixed point theorem. Finally, an example is given to verify the effectiveness and applicability of our main results.

Keywords. Fractional differential system; p-Laplacian; Leray-Schauder nonlinear alternative; Banach's fixed point theorem.

1. INTRODUCTION

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary noninteger order. In recent years, the increasing interest of fractional differential equations is greatly motivated by its extensive applications in various fields of science and engineering, such as porous media, fluid mechanics, heat equations and electromagnetic bioengineering; see, e.g., [1, 2, 3, 4] and the references therein. It is known that the p-Laplacian operator is of importance in both engineering and theoretical research. To discuss the turbulent flow in a porous medium, the differential equation with p-Laplacian operators was first introduced by Leibenson in 1945 [5]. Subsequently, more and more results on differential equations with p-Laplacian operators were investigated; see, e.g., [6, 7, 8, 9, 10] and the references therein. Recently, many authors

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also investigated fractional differential systems, which include Riemann-Liouville fractional derivative, Caputo fractional derivative, Hadamard fractional derivative and so on [9, 10, 11, 12, 13, 14, 15, 16, 17]. Recently, the results on fractional systems with p-Laplacian operators flourished; see, e.g., [18, 19, 20, 21].

In [21], Wang investigated multiple positive solutions for the following mixed fractional differential system

$$\begin{cases} D^{\beta_1}(\varphi_{p_1}({}^cD^{\alpha_1}u(t)))+f_1(t,u(t),v(t))=0, 0<t<1,\\ D^{\beta_2}(\varphi_{p_2}({}^cD^{\alpha_1}v(t)))+f_2(t,u(t),v(t))=0, 0<t<1,\\ u(0)=u'(0)=\cdots=u^{(n-1)}=0,\ {}^cD^{\alpha_1}u(0)=0,\\ v(0)=v'(0)=\cdots=v^{(n-1)}=0,\ {}^cD^{\alpha_2}v(0)=0,\\ {}^cD^{\alpha_1}u(1)=\varepsilon_1\ {}^cD^{\alpha_1}u(\eta_1),\ u(1)=\mu_1\int_0^1 a(s)v(s)dA_1(s),\\ {}^cD^{\alpha_2}v(1)=\varepsilon_2\ {}^cD^{\alpha_2}u(\eta_2),\ v(1)=\mu_2\int_0^1 b(s)u(s)dA_2(s), \end{cases}$$

where $1<\beta_i\le 2,\ 1\le n-1<\alpha_1\le n,\ 1\le m-1<\alpha_2\le m,\ n,m\ge 2,\ \mu_i>0,\ \eta_i\in(0,1)$, $\varepsilon_i>0$, $1-\varepsilon_i^{p_i-1}\eta^{\beta_i-1}>0$, $f_i\in C([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$, D^{β_i} and ${}^cD^{\alpha_i}$ respectively denotes the Riemann-Liouville derivative and Caputo derivative for $i=1,2$, $a,b\in C([0,1],[0,\infty))$, $\int_0^1 a(s)v(s)dA_1(s)$ and $\int_0^1 b(s)u(s)dA_2(s)$ are Riemann-Stieltjes integrals with a signed measure. Guo-Krasnosel'skii fixed point theorem is its main tool to investigate the existence of multiple positive solutions of this system.

In [22], Luca discussed the following fractional differential system

$$\begin{cases} D_{0^+}^{\alpha_1}(\varphi_{r_1}(D_{0^+}^{\beta_1}u(t)))+\lambda f(t,u(t),v(t))=0, 0<t\le 1,\\ D_{0^+}^{\alpha_2}(\varphi_{r_2}(D_{0^+}^{\beta_2}u(t)))+\mu g(t,u(t),v(t))=0, 0<t\le 1,\\ u^{(j)}(0)=0,\ j=1,2,\cdots,n-2,\ D_{0^+}^{\beta_1}u(0)=0,\ D_{0^+}^{p_1}u(1)=\sum\limits_{i=1}^{N}a_iD_{0^+}^{q_1}u(\xi_i),\\ v^{(j)}(0)=0,\ j=1,2,\cdots,m-2,\ D_{0^+}^{\beta_2}v(0)=0,\ D_{0^+}^{p_2}v(1)=\sum\limits_{i=1}^{M}b_iD_{0^+}^{q_2}v(\eta_i), \end{cases}$$

where $\alpha_1,\alpha_2\in(0,1]$, $\beta_1\in(n-1,n]$, $\beta_2\in(m-1,m]$, $n,m\in\mathbf{N}$, $n,m\ge 3$, $p_1,p_2,q_1,q_2\in\mathbf{R}$, $p_1\in[1,n-2]$, $p_2\in[1,m-2]$, $q_1\in[0,p_1]$, $q_2\in[0,p_2]$, $\xi_i,a_i\in\mathbf{R}$ for all $i=1,2,\cdots,N(N\in\mathbf{N})$, $0<\xi_1<\xi_2<\cdots\xi_N\le 1$, $\eta_i,b_i\in\mathbf{R}$ for all $i=1,2,\cdots,M(M\in\mathbf{N})$, $0<\eta_1<\eta_2<\cdots\eta_M\le 1$, $\lambda,\mu>0$, $f,g\in C([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$. The author obtained the existence of solutions with the aid of Guo-Kransnosel'skill fixed point theorem, and gave intervals for the parameters λ and μ such that positive solutions exist.

In [20], Yang and Zhu studied the following fractional differential system

$$\begin{cases} D_{0^+}^{\alpha_1}(\varphi_{p_1}(D_{0^+}^{\beta_1}u(t)))+\lambda f(t,u(t),v(t))=0, 0<t\le 1,\\ D_{0^+}^{\alpha_2}(\varphi_{p_2}(D_{0^+}^{\beta_2}u(t)))+\mu g(t,u(t),v(t))=0, 0<t\le 1,\\ u(0)=u(1)=u'(0)=u'(1)=D_{0^+}^{\beta_1}u(0)=0,\ D_{0^+}^{\beta_1}u(1)=b_1D_{0^+}^{\beta_1}u(\eta_1),\\ v(0)=v(1)=v'(0)=v'(1)=D_{0^+}^{\beta_2}v(0)=0,\ D_{0^+}^{\beta_2}v(1)=b_2D_{0^+}^{\beta_2}v(\eta_2), \end{cases}$$

where $1<\alpha_i<2$, $3<\beta_i<4$, $D_{0^+}^{\alpha_i}$ is the standard Riemann-Liouville fractional derivative, $0<\eta_i<1$, $0<b_i<\eta_i^{\frac{1-\alpha_i}{p_i-1}}$, $i=1,2$ and $f,\ g\in C([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$, and λ and μ are two parameters. The primary tool is a fixed point theorem in a normal cone, and the unique positive solution to the system depending on positive constants λ and μ exists in a product set.

They show that the solution can be approximated via an iterative sequence with any initial point in the product set.

Inspired by above results, we discuss the following fractional differential system with the p-Laplacian operators

$$\begin{cases} D_{0^+}^{\beta_1}(\varphi_p(D_{0^+}^{\alpha_1}u(t)))+f_1(t,u(t),v(t))=0, 0<t\leq 1, \\ D_{0^+}^{\beta_2}(\varphi_p(D_{0^+}^{\alpha_2}v(t)))+f_2(t,u(t),v(t))=0, 0<t\leq 1, \\ u(0)=u(1)=u'(0)=D_{0^+}^{\alpha_1-2}u(0)=D_{0^+}^{\alpha_1}u(0)=0, \\ v(0)=v(1)=v'(0)=D_{0^+}^{\alpha_2-2}v(0)=D_{0^+}^{\alpha_2}v(0)=0, \end{cases} \tag{1.1}$$

where $3<\alpha_i\leq 4$, $0<\beta_i\leq 1$, $f_i\in C([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$, $D_{0^+}^{\beta_i}$, $D_{0^+}^{\alpha_i}$ are the standard Riemann-Liouville fractional derivatives, $i=1,2$ and $\varphi_p(s)$ is the p-Laplacian operator defined by $\varphi_p(s)=|s|^{p-2}s$, $\varphi_p^{-1}(s)=\varphi_q(s)$, $\frac{1}{p}+\frac{1}{q}=1$, $p>1$. The methods we will use in this paper are the Leray-Schauder nonlinear alternative and Banach's fixed point theorem, which are widely used in the boundary value problems. The range of solutions can be more accurate if we choose the more appropriate domain of the operator. Up to our knowledge, there are still very few results devoted to the study of positive solutions to the fractional differential system with the p-Laplacian operator, and mixed boundary conditions. So it is desirable investigate system (1.1).

The rest paper is organized as follows. In Section 2, we give some necessary definitions and preliminary facts of fractional calculus, which are needed in the following sections. In Section 3, we prove our main results on the existence and uniqueness of the solutions to our system. In Section 4, the last section, we give an example to illustrate our results.

2. Preliminaries

In this section, we recall some basic definitions and preliminary facts of fractional calculus that will be needed in the sequel.

Definition 2.1. [23, 24] Let $u:(0,\infty)\to\mathbf{R}$ be a function and $\alpha>0$. The Riemann-Liouville fractional integral of order α of u is defined by

$$I_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)ds,$$

provided that the right-hand side is point-wise defined on $(0,+\infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. [23, 24] Let $u:(0,\infty)\to\mathbf{R}$ be a function and $\alpha>0$. The Riemann-Liouville fractional derivative of order α of u is defined by

$$D_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}u^{(n)}(s)ds,$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$, where $n=[\alpha]+1$. If $\alpha=n$, then $D_{0^+}^{\alpha}u(t)=u^{(n)}(t)$.

Lemma 2.3. [23, 24] *If $u\in C(0,1)\bigcap L^1(0,1)$ such that $D_{0^+}^{\alpha}u\in C(0,1)\bigcap L^1(0,1)$, then*

$$I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t)=u(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_{n-1}t^{\alpha-n},$$

for some $c_i\in\boldsymbol{R}$, $i=1,2,\cdots$, where $n=[\alpha]+1$.

Lemma 2.4. [23, 24] *(i) If $q > -1$, then $D_{0^+}^{\alpha} t^q = \frac{\Gamma(q+1)}{\Gamma(q+1-\alpha)} t^{q-\alpha}$, and $D_{0^+}^{\alpha} t^{\alpha-q} = 0$, $q = 1, 2 \cdots n$, where $n = [\alpha] + 1$.*
(ii) $I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t)$ for all $u \in C(0,1) \bigcap L^1(0,1)$.
(iii) If $u \in L^1(0,1)$, $\alpha > 0$, $\beta > 0$, then $D_{0^+}^{\beta} I_{0^+}^{\alpha} u(t) = I_{0^+}^{\alpha-\beta} u(t)$.

Lemma 2.5. [25] *(Leray-Schauder nonlinear alternative theorem) Let E be a Banach space, and let Ω be a bounded open subset of E with $\theta \in \Omega$. Then every completely continuous map $T : \overline{\Omega} \to E$ has one of the following two properties:*
(i) T has a fixed point in $\overline{\Omega}$.
(ii) There is an $x \in$ and $\lambda \in (0,1)$ with $x = \lambda T x$.

Lemma 2.6. [26] *(Banach's fixed point theorem) Let E be a Banach space, D a closed subset of E, and $T : D \to D$ a strict contraction, i.e., there exists a constant $\gamma \in (0,1)$ such that $\|T(u) - T(v)\| \leq \gamma \|u - v\|$, $\forall u, v \in D$. Then T has a unique fixed point.*

Lemma 2.7. *If $\alpha_1 \in (3,4]$ and $h \in C([0,1])$, then the following fractional boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha_1} u(t) + h(t) = 0, \\ u(0) = u(1) = u'(0) = D_{0^+}^{\alpha_1 - 2} u(0) = 0, \end{cases}$$

has a unique solution given by

$$u(t) = \int_0^1 G_1(t,s) h(s) ds,$$

where

$$G_1(t,s) = \frac{1}{\Gamma(\alpha_1)} \begin{cases} (1-s)^{\alpha_1 - 1} t^{\alpha_1 - 1} - (t-s)^{\alpha_1 - 1}, 0 \leqslant s \leqslant t \leqslant 1, \\ (1-s)^{\alpha_1 - 1} t^{\alpha_1 - 1}, \quad 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

Proof. According to Lemma 2.3, we have

$$u(t) = -I_{0^+}^{\alpha_1} h(t) + c_1 t^{\alpha_1 - 1} + c_2 t^{\alpha_1 - 2} + c_3 t^{\alpha_1 - 3} - c_4 t^{\alpha_1 - 4},$$

for some $c_i \in \mathbf{R}$, $i = 1, 2, 3, 4$. By the boundary condition $u(0) = 0$, we get $c_4 = 0$. Thus,

$$u'(t) = -I_{0^+}^{\alpha_1 - 1} h(t) + (\alpha_1 - 1) c_1 t^{\alpha_1 - 2} + (\alpha_1 - 2) c_2 t^{\alpha_1 - 3} + (\alpha - 3) c_3 t^{\alpha_1 - 4}.$$

Substituting 0 into above equation, one has $c_3 = 0$, and

$$u(t) = -I_{0^+}^{\alpha_1} h(t) + c_1 t^{\alpha_1 - 1} + c_2 t^{\alpha_1 - 2}.$$

From Lemma 2.4 (i), we know

$$D_{0^+}^{\alpha_1 - 2} u(t) = \int_0^t (t-s) h(s) ds + c_1 \Gamma(\alpha_1) t + c_2 \Gamma(\alpha_1 - 1),$$

which together with $D_{0^+}^{\alpha_1 - 2} u(0) = 0$ implies $c_2 = 0$, that is,

$$u(t) = -I_{0^+}^{\alpha_1} h(t) + c_1 t^{\alpha_1 - 1}.$$

Finally, using $u(1) = 0$, one has

$$c_1 = \frac{1}{\Gamma(\alpha_1)} \int_0^1 (1-s)^{\alpha_1 - 1} h(s) ds.$$

Thus,

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}h(s)ds + \frac{t^{\alpha_1-1}}{\Gamma(\alpha_1)}\int_0^1 (1-s)^{\alpha_1-1}h(s)ds \\ &= \int_0^1 G_1(t,s)h(s)ds. \end{aligned}$$

This completes the proof. □

Lemma 2.8. *The function $G_1(t,s)$ in Lemma 2.7 satisfies the following properties:*
(1) *$G_1(t,s)$ is continuous on $[0,1]\times[0,1]$ and*

$$0 \le G_1(t,s) \le \frac{1}{\Gamma(\alpha_1-1)}(1-s)^{\alpha_1-2},\ (t,s)\in[0,1]\times[0,1];$$

(2) *$\frac{\partial}{\partial t}G_1(t,s)$ is continuous and $|\frac{\partial}{\partial t}G_1(t,s)| \le \frac{2}{\Gamma(\alpha_1-1)}(1-s)^{\alpha_1-2}$.*

Proof. First, we consider (1).

(i) If $0\le t\le s\le 1$, then

$$0 \le G_1(t,s) = \frac{1}{\Gamma(\alpha_1)}(1-s)^{\alpha_1-1}t^{\alpha_1-1} \le \frac{1}{\Gamma(\alpha_1-1)}(1-s)^{\alpha_1-2}.$$

(ii) If $0\le s\le t\le 1$, then

$$\begin{aligned} G_1(t,s) &= \frac{1}{\Gamma(\alpha_1)}[(1-s)^{\alpha_1-1}t^{\alpha_1-1}-(t-s)^{\alpha_1-1}] \\ &= \frac{1}{\Gamma(\alpha_1)}t^{\alpha_1-1}[(1-s)^{\alpha_1-1}-(1-\frac{s}{t})^{\alpha_1-1}] \ge 0, \end{aligned}$$

and

$$\begin{aligned} G_1(t,s) &= \frac{1}{\Gamma(\alpha_1)}[(1-s)^{\alpha_1-1}t^{\alpha_1-1}-(t-s)^{\alpha_1-1}] \\ &\le \frac{1}{\Gamma(\alpha_1)}t^{\alpha_1-1}(1-s)^{\alpha_1-1} \le \frac{1}{\Gamma(\alpha_1-1)}(1-s)^{\alpha_1-2}. \end{aligned}$$

It follows that

$$0 \le G_1(t,s) \le \frac{1}{\Gamma(\alpha_1-1)}(1-s)^{\alpha_1-2},\ (t,s)\in[0,1]\times[0,1].$$

Next, we take (2) into consideration. According to the formula of $G_1(t,s)$, we get the form of $\frac{\partial}{\partial t}G_1(t,s)$ as follows:

$$\frac{\partial}{\partial t}G_1(t,s) = \frac{1}{\Gamma(\alpha_1-1)}\begin{cases} (1-s)^{\alpha_1-1}t^{\alpha_1-2}-(t-s)^{\alpha_1-2}, & 0\leqslant s\leqslant t\leqslant 1, \\ (1-s)^{\alpha_1-1}t^{\alpha_1-2}, & 0\leqslant t\leqslant s\leqslant 1. \end{cases}$$

So

$$|\frac{\partial}{\partial t}G_1(t,s)| \le \frac{1}{\Gamma(\alpha_1-1)}\max\{|(1-s)^{\alpha_1-1}t^{\alpha_1-2}-(t-s)^{\alpha_1-2}|,(1-s)^{\alpha_1-1}t^{\alpha_1-2}\}.$$

In view of

$$\begin{aligned} |(1-s)^{\alpha_1-1}t^{\alpha_1-2}-(t-s)^{\alpha_1-2}| &\le |(1-s)^{\alpha_1-1}t^{\alpha_1-2}| + |(t-s)^{\alpha_1-2}| \\ &\le (1-s)^{\alpha_1-1}+(1-s)^{\alpha_1-2} \\ &\le 2(1-s)^{\alpha_1-2}, \end{aligned}$$

and $(1-s)^{\alpha_1-1}t^{\alpha_1-2} \leq (1-s)^{\alpha_1-2}$, we have

$$|\frac{\partial}{\partial t}G_1(t,s)| \leq \frac{2}{\Gamma(\alpha_1-1)}(1-s)^{\alpha_1-2}.$$

Similarly, $G_2(t,s)$ has the same properties. □

Lemma 2.9. *(u,v) is a solution of (1.1) if and only if it satisfies the integral equations*

$$u(t) = \int_0^1 G_1(t,s)\varphi_q\Big(\int_0^s \frac{1}{\Gamma(\beta_1)}(s-r)^{\beta_1-1}f_1(r,u,v)dr\Big)ds,$$

and

$$v(t) = \int_0^1 G_2(t,s)\varphi_q\Big(\int_0^s \frac{1}{\Gamma(\beta_2)}(s-r)^{\beta_2-1}f_2(r,u,v)dr\Big)ds,$$

where

$$G_1(t,s) = \frac{1}{\Gamma(\alpha_1)}\begin{cases} (1-s)^{\alpha_1-1}t^{\alpha_1-1}-(t-s)^{\alpha_1-1}, 0\leqslant s\leqslant t\leqslant 1, \\ (1-s)^{\alpha_1-1}t^{\alpha_1-1}, \qquad 0\leqslant t\leqslant s\leqslant 1, \end{cases}$$

and

$$G_2(t,s) = \frac{1}{\Gamma(\alpha_2)}\begin{cases} (1-s)^{\alpha_2-1}t^{\alpha_2-1}-(t-s)^{\alpha_2-1}, 0\leqslant s\leqslant t\leqslant 1, \\ (1-s)^{\alpha_2-1}t^{\alpha_2-1}, \qquad 0\leqslant t\leqslant s\leqslant 1. \end{cases}$$

Proof. From Lemma 2.3, we have

$$\varphi_p\big(D_{0^+}^{\alpha_1}u(t)\big) = -I_{0^+}^{\beta_1}f\big(t,u(t),v(t)\big)+ct^{\beta_1-1},$$

where c is a constant. With $D_{0^+}^{\alpha_1}u(0)=0$, we get $c=0$, that is,

$$\varphi_p\big(D_{0^+}^{\alpha_1}u(t)\big) = -\frac{1}{\Gamma(\beta_1)}\int_0^t (t-s)^{\beta_1-1}f_1\big(t,u(t),v(t)\big)ds.$$

Similarly, we can obtain

$$\varphi_p\big(D_{0^+}^{\alpha_2}v(t)\big) = -\frac{1}{\Gamma(\beta_2)}\int_0^t (t-s)^{\beta_2-1}f_2\big(t,u(t),v(t)\big)ds.$$

Therefore, problem (1.1) can be written as follows:

$$\begin{cases} D_{0^+}^{\alpha_1}u(t)+\varphi_q(\int_0^t \frac{1}{\Gamma(\beta_1)}(t-s)^{\beta_1-1}f_1(t,u(t),v(t)))=0, 0\leq t\leq 1, \\ D_{0^+}^{\alpha_2}v(t)+\varphi_q(\int_0^t \frac{1}{\Gamma(\beta_2)}(t-s)^{\beta_2-1}f_2(t,u(t),v(t)))=0, 0\leq t\leq 1, \\ u(0)=u(1)=u'(0)=D_{0^+}^{\alpha_1-2}u(0)=D_{0^+}^{\alpha_1}u(0)=0, \\ v(0)=v(1)=v'(0)=D_{0^+}^{\alpha_2-2}v(0)=D_{0^+}^{\alpha_2}v(0)=0. \end{cases}$$

From Lemma 2.7, we get the expression of u and v. This completes the proof. □

3. Main Results

We now turn to the existence and uniqueness of the solutions to problem (1.1). In this paper, we use the space $E=X\times Y$ with the norm

$$\|(u,v)\| = \|u\|_X+\|v\|_Y,\ (u,v)\in E,$$

where $X=Y=C^1[0,1]$, and their norms are defined as follows:

$$\|u\|_X = \max_{0\leq t\leq 1}\{\|u\|_0, \|u'\|_0\},\ u\in X,$$

where

$$\|u\|_0 = \max_{0\le t\le 1} |u(t)|,\ \|u'\|_0 = \max_{0\le t\le 1} |u'(t)|,\ u \in X,$$

and

$$\|v\|_Y = \max_{0\le t\le 1} \{\|v\|_0, \|v'\|_0\},\ \ v \in Y,$$

where

$$\|v\|_0 = \max_{0\le t\le 1} |v(t)|,\ \|v'\|_0 = \max_{0\le t\le 1} |v'(t)|,\ v \in Y.$$

Evidently, $(E, \|\cdot\|)$ is a Banach space.

Define an operator $T : E \to E$ by $T(u,v)(t) = \big(T_1(u,v)(t), T_2(u,v)(t)\big)$, where

$$T_1(u,v)(t) = \int_0^1 G_1(t,s)\varphi_q\Big(\int_0^s \frac{1}{\Gamma(\beta_1)}(s-r)^{\beta_1-1} f_1(r,u,v)dr\Big)ds, \tag{3.1}$$

and

$$T_2(u,v)(t) = \int_0^1 G_2(t,s)\varphi_q\Big(\int_0^s \frac{1}{\Gamma(\beta_2)}(s-r)^{\beta_2-1} f_2(r,u,v)dr\Big)ds,$$

where $G_1(t,s),\ G_2(t,s)$ are given by Lemma 2.9. Clearly, the solution to problem (1.1) is equal to the fixed point of operator T.

In addition, we assume the following conditions:

(H_1) there exists $a_i, b_i \in C[0,1],\ i = 1,2$ such that

$$|f_i(t,u_1,v_1) - f_i(t,u_2,v_2)| \le a_i(t)|u_1 - u_2| + b_i(t)|v_1 - v_2|,\ t \in [0,1],\ u_i, v_i \in \mathbf{R}, i = 1,2;$$

(H_2) there exists constants c and d satisfying the following inequality:

$$c \le f_i(t,u,v) \le d,\ t \in [0,1],\ (u,v) \in \mathbf{R},\ i = 1,2.$$

For the sake of convenience, we give the following notations:

(1) $\Omega := \{(u,v) \in E : \|(u,v)\| \le r,\ r > 0\}$;
(2) $V := \{T(u,v) : (u,v) \in \Omega\}$;
(3) $L_i := \max\limits_{t\in[0,1],(u,v)\in\Omega} |f_i(t,u,v)|,\ i = 1,2$;
(4) $A_i := \int_0^s \frac{1}{\Gamma(\beta_i)}(s-r)^{\beta_i-1} f_i(r,u,v)dr$, obviously, $|A_i| \le \frac{L_i s^{\beta_i}}{\Gamma(\beta_i+1)},\ (u,v) \in \Omega,\ i = 1,2.$

Theorem 3.1. *(Existence). Problem (1.1) has at least a solution $(u,v) \in \Omega$ if the following inequality holds*

$$\sum_{i=1}^{2} \frac{2}{\Gamma(\alpha_i - 1)}\Big(\frac{L_i}{\Gamma(\beta_i + 1)}\Big)^{q-1} \mathscr{B}\big(\alpha_i - 1, \beta_i(q-1) + 1\big) < r, \tag{3.2}$$

where $\mathscr{B}(\cdot,\cdot)$ denotes the Beta function.

Proof. T is well defined thanks to the continuity of $G_i(t,s)$ and $f_i,\ i = 1,2$. Now we are going to divide the progress into two steps.

Step 1. Show that T is compact.

First, we prove that $T(\Omega)$ is uniformly bounded. Let $(u,v)\in\Omega$. It follows from Lemma 2.8 that

$$
\begin{aligned}
|T_1(u,v)(t)| &\le \int_0^1 |G_1(t,s)||\varphi_q(A_1)|ds\\
&\le \frac{1}{\Gamma(\alpha_1-1)}\int_0^1 (1-s)^{\alpha_1-2}\Big(\frac{L_1}{\Gamma(\beta_1+1)}\Big)^{q-1} s^{\beta_1(q-1)}ds\\
&= \frac{1}{\Gamma(\alpha_1-1)}\Big(\frac{L_1}{\Gamma(\beta_1+1)}\Big)^{q-1}\mathscr{B}\big(\alpha_1-1,\beta_1(q-1)+1\big),
\end{aligned}
$$

and

$$
\begin{aligned}
|T_1'(u,v)(t)| &\le \int_0^1 |\frac{\partial}{\partial t}G_1(t,s)||\varphi_q(A_1)|ds\\
&\le \frac{2}{\Gamma(\alpha_1-1)}\int_0^1 (1-s)^{\alpha_1-2}\Big(\frac{L_1}{\Gamma(\beta_1+1)}\Big)^{q-1} s^{\beta_1(q-1)}ds\\
&= \frac{2}{\Gamma(\alpha_1-1)}\Big(\frac{L_1}{\Gamma(\beta_1+1)}\Big)^{q-1}\mathscr{B}\big(\alpha_1-1,\beta_1(q-1)+1\big),
\end{aligned}
$$

that is

$$
\|T_1(u,v)\|_0 \le \frac{1}{\Gamma(\alpha_1-1)}\Big(\frac{L_1}{\Gamma(\beta_1+1)}\Big)^{q-1}\mathscr{B}\big(\alpha_1-1,\beta_1(q-1)+1\big),
$$

and

$$
\|T_1'(u,v)\|_0 \le \frac{2}{\Gamma(\alpha_1-1)}\Big(\frac{L_1}{\Gamma(\beta_1+1)}\Big)^{q-1}\mathscr{B}\big(\alpha_1-1,\beta_1(q-1)+1\big).
$$

Consequently,

$$
\|T_1(u,v)\|_X \le \frac{2}{\Gamma(\alpha_1-1)}\Big(\frac{L_1}{\Gamma(\beta_1+1)}\Big)^{q-1}\mathscr{B}\big(\alpha_1-1,\beta_1(q-1)+1\big). \tag{3.3}
$$

Similarly, one has

$$
\|T_2(u,v)\|_Y \le \frac{2}{\Gamma(\alpha_2-1)}\Big(\frac{L_2}{\Gamma(\beta_2+1)}\Big)^{q-1}\mathscr{B}\big(\alpha_2-1,\beta_2(q-1)+1\big). \tag{3.4}
$$

Combining (3.3) with (3.4) we have

$$
\|T(u,v)\| \le \sum_{i=1}^{2}\frac{2}{\Gamma(\alpha_i-1)}\Big(\frac{L_i}{\Gamma(\beta_i+1)}\Big)^{q-1}\mathscr{B}\big(\alpha_i-1,\beta_i(q-1)+1\big), \tag{3.5}
$$

which shows that $T(\Omega)$ is uniformly bounded.

Next, we prove V is equicontinuous. For $(u,v)\in E,\ t_1,\ t_2\in[0,1]$, we find from (3.1) that

$$
\begin{aligned}
|T_1(u,v)(t_1)-T_1(u,v)(t_2)| &\le \int_0^1 |G_1(t_1,s)-G_1(t_2,s)||\varphi_q(A_1)|ds\\
&\le \Big(\frac{L_1}{\Gamma(\beta_1+1)}\Big)^{q-1}\int_0^1 |G_1(t_1,s)-G_1(t_2,s)|s^{\beta_1(q-1)}ds\\
&\le \Big(\frac{L_1}{\Gamma(\beta_1+1)}\Big)^{q-1}\int_0^1 |G_1(t_1,s)-G_1(t_2,s)|ds,
\end{aligned}
$$

and

$$\begin{aligned}|T_1'(u,v)(t_1)-T_1'(u,v)(t_2)| &\le \int_0^1 |\frac{\partial}{\partial t}G_1(t_1,s)-\frac{\partial}{\partial t}G_1(t_2,s)||\varphi_q(A_1)|ds\\ &\le \Big(\frac{L_1}{\Gamma(\beta_1+1)}\Big)^{q-1}\int_0^1 |\frac{\partial}{\partial t}G_1(t_1,s)-\frac{\partial}{\partial t}G_1(t_2,s)|s^{\beta_1(q-1)}ds\\ &\le \Big(\frac{L_1}{\Gamma(\beta_1+1)}\Big)^{q-1}\int_0^1 |\frac{\partial}{\partial t}G_1(t_1,s)-\frac{\partial}{\partial t}G_1(t_2,s)|ds.\end{aligned}$$

According to the continuity of $G_1(t,s)$, and $\frac{\partial}{\partial t}G_1(t,s)$, we have

$$\|T_1(u,v)(t_1)-T_1(u,v)(t_2)\|_0 \to 0(t_1\to t_2),$$

and

$$\|T_1'(u,v)(t_1)-T_1'(u,v)(t_2)\|_0 \to 0(t_1\to t_2).$$

Therefore,

$$\|T_1(u,v)(t_1)-T_1(u,v)(t_2)\|_X \to 0(t_1\to t_2). \tag{3.6}$$

Similarly, we can obtain that

$$\|T_2(u,v)(t_1)-T_2(u,v)(t_2)\|_Y \to 0(t_1\to t_2). \tag{3.7}$$

Combining (3.6) with (3.7), we have

$$\|T(u,v)(t_1)-T(u,v)(t_2)\| \to 0(t_1\to t_2).$$

It follows that V is equicontinuous. Hence, $T:E\to E$ is completely continuous according to Ascoli-Arzela theorem.

Step 2. Show $x\neq \lambda Tx$, $\forall x\in\partial\Omega$, $\forall\lambda\in(0,1)$.

Assume that $(u_0,v_0)\in\partial\Omega$ and $\lambda_0\in(0,1)$ such that $(u_0,v_0)=\lambda_0 T(u_0,v_0)$. From (3.5), we obtain

$$\begin{aligned}\|(u_0,v_0)\| &= \|u_0\|_X+\|v_0\|_Y\\ &= \lambda_0\|T_1(u_0,v_0)\|_X+\lambda_0\|T_2(u_0,v_0)\|_Y\\ &\le \sum_{i=1}^{2}\frac{2}{\Gamma(\alpha_i-1)}\Big(\frac{L_i}{\Gamma(\beta_i+1)}\Big)^{q-1}\mathscr{B}\big(\alpha_i-1,\beta_i(q-1)+1\big),\end{aligned}$$

thus,

$$r\le \sum_{i=1}^{2}\frac{2}{\Gamma(\alpha_i-1)}\Big(\frac{L_i}{\Gamma(\beta_i+1)}\Big)^{q-1}\mathscr{B}\big(\alpha_i-1,\beta_i(q-1)+1\big),$$

which contradicts to (3.2). From Lemma 2.7, we know that problem (1.1) has at least a solution $(u,v)\in\Omega$. This completes the proof. □

Theorem 3.2. *(Uniqueness). Problem (1.1) has a unique solution if (H_1), (H_1) holds, $p<2$, and*

$$\gamma := (q-1)d^{q-2}\sum_{i=1}^{2}\frac{2}{\Gamma(\alpha_i-1)}\Big(\frac{1}{\Gamma(\beta_i)}\Big)^{q-1}\int_0^1 a_i(r)+b_i(r)dr<1.$$

Proof. From the continuity of T, we only need to show it is a contracting operator. Let (u_1,v), $(u_2,v_2)\in E$. Note that

$$
\begin{aligned}
&|T_1(u_1,v_1)(t)-T_1(u_2,v_2)(t)|\\
&\le \int_0^1 |G_1(t,s)||\varphi_q\Big(\int_0^s \frac{1}{\Gamma(\beta_1)}(s-r)^{\beta_1-1}f_1(r,u_1,v_1)dr\Big)\\
&\quad -\varphi_q\Big(\int_0^s \frac{1}{\Gamma(\beta_1)}(s-r)^{\beta_1-1}f_1(r,u_2,v_2)dr\Big)|ds\\
&\le \frac{1}{\Gamma(\alpha_1-1)}\Big(\frac{1}{\Gamma(\beta_1)}\Big)^{q-1}\int_0^1(1-s)^{\alpha_1-2}|\varphi_q\Big(\int_0^s(s-r)^{\beta_1-1}f_1(r,u_1,v_1)dr\Big)\\
&\quad -\varphi_q\Big(\int_0^s(s-r)^{\beta_1-1}f_1(r,u_2,v_2)dr\Big)|ds\\
&\le \frac{1}{\Gamma(\alpha_1-1)}\Big(\frac{1}{\Gamma(\beta_1)}\Big)^{q-1}|\varphi_q\Big(\int_0^1 f_1(r,u_1,v_1)dr\Big)-\varphi_q\Big(\int_0^1 f_1(r,u_2,v_2)dr\Big)|.
\end{aligned}
$$

From the Lagrange mean value theorem, $p>1$ and $(H_1),(H_2)$, there exists $c\le\xi\le d$ such that

$$
\begin{aligned}
&|\varphi_q\Big(\int_0^1 f_1(r,u_1,v_1)dr\Big)-\varphi_q\Big(\int_0^1 f_1(r,u_2,v_2)dr\Big)|\\
&\le (q-1)\xi^{q-2}\int_0^1 |f_1(r,u_1,v_1)-f_1(r,u_2,v_2)|dr\\
&\le (q-1)d^{q-2}\int_0^1 a_1(r)|u_1-u_2|+b_1(r)|v_1-v_2|dr\\
&\le (q-1)d^{q-2}\Big(\|u_1-u_2\|_X\int_0^1 a_1(r)dr+\|v_1-v_2\|_Y\int_0^1 b_1(r)dr\Big)\\
&\le (q-1)d^{q-2}\int_0^1 a_1(r)+b_1(r)dr\|(u_1,v_1)-(u_2,v_2)\|.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|T_1(u_1,v_1)-T_1(u_2,v_2)\|_0\\
&\le \frac{1}{\Gamma(\alpha_1-1)}\Big(\frac{1}{\Gamma(\beta_1)}\Big)^{q-1}(q-1)d^{q-2}\int_0^1 a_1(r)+b_1(r)dr\|(u_1,v_1)-(u_2,v_2)\|.
\end{aligned}
$$

Similarly, we can get the following inequality

$$
\begin{aligned}
&\|T_1'(u_1,v_1)-T_1'(u_2,v_2)\|_0\\
&\le \frac{2}{\Gamma(\alpha_1-1)}\Big(\frac{1}{\Gamma(\beta_1)}\Big)^{q-1}(q-1)d^{q-2}\int_0^1 a_1(r)+b_1(r)dr\|(u_1,v_1)-(u_2,v_2)\|,
\end{aligned}
$$

thus,

$$
\begin{aligned}
&\|T_1(u_1,v_1)-T_1(u_2,v_2)\|_X\\
&\le \frac{2}{\Gamma(\alpha_1-1)}\Big(\frac{1}{\Gamma(\beta_1)}\Big)^{q-1}(q-1)d^{q-2}\int_0^1 a_1(r)+b_1(r)dr\|(u_1,v_1)-(u_2,v_2)\|.
\end{aligned}
$$

Similarly, we have

$$\begin{aligned}&\|T_2(u_1,v_1)-T_2(u_2,v_2)\|_Y\\&\leq \frac{2}{\Gamma(\alpha_2-1)}\Big(\frac{1}{\Gamma(\beta_2)}\Big)^{q-1}(q-1)d^{q-2}\int_0^1 a_2(r)+b_2(r)dr\|(u_1,v_1)-(u_2,v_2)\|.\end{aligned}$$

Hence, $\|T(u_1,v_1)-T(u_2,v_2)\|\leq\gamma\|(u_1,v_1)-(u_2,v_2)\|$. From Lemma 2.8, we conclude that operator T has a unique solution. This completes the proof. □

4. The Example

In this section, we give an example to illustrate our main results.

Example 4.1. Considering the following system:

$$\begin{cases} D^{\frac{1}{2}}_{0^+}(\varphi_{\frac{3}{2}}(D^{\frac{7}{2}}_{0^+}u(t)))+2\sin(u+t)+\cos v=0,\ 0\leq t\leq 1,\\ D^{\frac{1}{2}}_{0^+}(\varphi_{\frac{3}{2}}(D^{\frac{7}{2}}_{0^+}u(t)))+\arctan(u+v)-t=0,\ 0\leq t\leq 1,\\ u(0)=u(1)=u'(0)=D^{\frac{3}{2}}_{0^+}u(0)=D^{\frac{7}{2}}_{0^+}u(0)=0,\\ v(0)=v(1)=v'(0)=D^{\frac{3}{2}}_{0^+}v(0)=D^{\frac{7}{2}}_{0^+}v(0)=0,\end{cases}$$

where $\beta_1=\beta_2=\frac{1}{2}$, $\alpha_1=\alpha_2=\frac{7}{2}$, $p=\frac{3}{2}$, and

$$f_1(t,u,v)=|\sin(u+t)|+|\cos v|,\ f_2(t,u,v)=\arctan(u+v)+t.$$

Obviously, $f_1,f_2\in C([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$. Set $r=3$, $\Omega=\{(u,v)\in E:\|(u,v)\|\leq 3\}$. On other hand, we obtain

$$L_1=\max_{t\in[0,1],(u,v)\in\Omega}|\sin(u+r)|+|\cos v|=2,$$

$$L_2=\max_{t\in[0,1],(u,v)\in\Omega}|\arctan(u+v)+t|=2.284,$$

and

$$\frac{2}{\Gamma(\frac{5}{2})}\Big(\frac{1}{\Gamma(\frac{3}{2})}\Big)^2\mathscr{B}(\frac{5}{2},2)(2^2+2.284^2)=2.018<3,$$

which means that (3.2) holds. Hence, the above system has at least a solutions in $\overline{\Omega}$ by Theorem 3.1.

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