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INFINITELY MANY SOLUTIONS FOR TWO CLASSES OF FRACTIONAL HAMILTONIAN SYSTEMS

MOHSEN TIMOUMI

Department of Mathematics, Faculty of Sciences of Monastir, Monastir, Tunisia

Abstract. In this paper, we study the existence and multiplicity of solutions for a class of fractional Hamiltonian systems

$$\begin{cases} {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_{t}u)(t)+L(t)u(t)=\nabla W(t,u(t)),\ t\in\mathbb{R}, \\ u\in H^{\alpha}(\mathbb{R}), \end{cases}$$

where $L(t)$ satisfies some weaker conditions than the well-known conditions, and $W(t,x)$ is of superquadratic growth as $|x|\longrightarrow\infty$, or satisfies only local conditions near the origin (i.e., it can be subquadratic, superquadratic or asymptotically quadratic as $|x|\longrightarrow\infty$). To the best of our knowledge, there is no result concerning the existence and multiplicity of solutions for the above system with our conditions. The proof is based on variational methods and critical point theory.

Keywords. Fractional Hamiltonian systems; Infinitely many solutions; Critical points; Superquadratic potentials; Local conditions.

1. INTRODUCTION

Fractional differential equations both ordinary and partial ones have been receiving great interest in the few last years. It is mainly due to both the intensive development of the theory of fractional calculus itself and the wide applications of such constructions in physics, mechanics, control theory, biochemistry, bioengineering and economics. Examples include anomalous diffusion [1], chaotic dynamics [2], polymer physics and biophysics [3], control theory [4] and so on. Therefore, the theory of fractional differential equations is an area intensively developed during the last decades. The existence and multiplicity of solutions for fractional differential equations were established by the tools of nonlinear analysis, such as, fixed point theory [2, 5], topological degree [6, 7] and comparison methods [8, 9]. It should be noted that critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions for integer order differential equations. The underlying idea in this approach rests on finding critical points for suitable energy functional defined on an appropriate

E-mail address: m_timoumi@yahoo.com.

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function space. In the three last decades, the critical point theory has become a wonderful tool in studying the existence of solutions to differential equations with variational structures; we refer the reader to [10, 11] and the references listed therein.

Motivated by the previously mentioned classical works, in [12], for the first time, Jiao and Zhou showed that the critical point theory is an effective approach to tackle the existence of solutions for the following fractional boundary value problem

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u)(t) = \nabla W(t,u(t)),\ t \in [0,T] \\ u(0) = u(T), \end{cases}$$

and obtained the existence of at least one nontrivial solution. Inspired by this work, in [13], Torres considered the following fractional Hamiltonian system

$$(\mathscr{FHS}) \qquad \begin{cases} {}_tD_{\infty}^{\alpha}({}_{-\infty}D_t^{\alpha}u)(t) + L(t)u(t) = \nabla W(t,u(t)),\ t \in \mathbb{R} \\ u \in H^{\alpha}(\mathbb{R}), \end{cases}$$

where ${}_{-\infty}D_t^{\alpha}$ and ${}_tD_{\infty}^{\alpha}$ are left and right Liouville-Weyl fractional derivatives of order $\frac{1}{2} < \alpha < 1$ on the whole axis respectively, $L \in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix-valued function, and $W : \mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}$ is a continuous function, differentiable in the second variable with continuous derivative $\nabla W(t,x) = \frac{\partial W}{\partial x}(t,x)$. Assuming that W satisfies the so-called Ambrosetti-Rabinowitz condition and some additional conditions, and L satisfies the following condition, the author showed that $(\mathscr{FHS})$ possesses at least one nontrivial solution via the Mountain Pass Theorem.

(1.1) $L(t)$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$, and there exists an $l \in C(\mathbb{R},\mathbb{R}_+^*)$ such that $l(t) \longrightarrow +\infty$ as $|t| \longrightarrow \infty$, and

$$L(t)x\cdot x \geq l(t)\,|x|^2,\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N,$$

where "$\cdot$" denotes the standard inner product in $\mathbb{R}^N$, and $|.|$ is the induced norm. As it is well known, condition (1.1) is the so-called coercive condition, and is a little bit restrictive. In fact, for a simple choice like $L(t) = \tau I_N$, condition (1.1) is not satisfied, where $\tau > 0$, and I_N is the $N\times N$ identity matrix. Motivated by this point, Yuan and Zhang [14] focused their attentions on the case that $L(t)$ is bounded in the sense that

(1.2) $L(t)$ is a positive definite matrix for all $t \in \mathbb{R}$, and there are $0 < \tau_1 < \tau_2 < \infty$ such that

$$\tau_1\,|x|^2 \leq L(t)x.x \leq \tau_2\,|x|^2,\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N.$$

If the potential $W(t,x)$ is supposed to be subquadratic as $|x| \longrightarrow \infty$, then they showed that $(\mathscr{FHS})$ possessed infinitely many solutions. Since then, conditions (1.1) and (1.2) have been extensively used in the literature; see, e.g., [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. Recently, remarking that $L(t) = (t^2sin^2t+1)I_N$ does not check (1.1) or (1.2), Mèndez and Torres [18] considered the case that $L(t)$ is not coercive in the sens that L satisfies

(1.3) $L(t)$ is positive definite symmetric matrix for all $t \in \mathbb{R}$ and $\inf_{t\in\mathbb{R}} l(t) > 0$, where $l(t) = \inf_{|x|=1} L(t)x\cdot x$ is the first eigenvalue of $L(t)$;

(1.4) There exists $r_0 > 0$ such that for any $b > 0$

$$meas(\{t \in]s-r_0,s+r_0[/l(t) \leq b\}) \longrightarrow 0\ as\ |s| \longrightarrow \infty.$$

Under conditions (1.3), (1.4) and some classical subquadratic conditions on W, they proved the existence of infinitely many nontrivial solutions to $(\mathscr{FHS})$ by using the "genus" properties.

Inspired by the previous results, in this paper, we are interested in the existence of infinitely many solutions for $(\mathscr{F}\mathscr{H}\mathscr{S})$ under some weaker conditions than (1.1), (1.2), (1.3)-(1.4), and we discuss two cases of potentials.

2. Preliminaries

In this Section, for the reader's convenience, we will recall some facts about the fractional calculus on the whole real axis. On the other hand, we will give some preliminary lemmas for using in the sequel.

The Liouville-Weyl fractional integrals of order $0<\alpha<1$ on the whole axis $\mathbb{R}$ are defined as (see [25, 26, 27])

$$
{}_{-\infty}I_t^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-x)^{\alpha-1}u(x)dx, \tag{2.1}
$$

and

$$
{}_tI_{\infty}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_{t}^{\infty}(x-t)^{\alpha-1}u(x)dx. \tag{2.2}
$$

The Liouville-Weyl fractional derivatives of order $0<\alpha<1$ on the whole axis $\mathbb{R}$ are defined as the left-inverse operators of the corresponding Liouville-Weyl fractional integrals (see [25, 26, 27])

$$
{}_{-\infty}D_t^{\alpha}u(t)=\frac{d}{dt}({}_{-\infty}I_t^{1-\alpha}u)(t), \tag{2.3}
$$

and

$$
{}_tD_{\infty}^{\alpha}u(t)=-\frac{d}{dt}({}_tI_{\infty}^{1-\alpha}u)(t). \tag{2.4}
$$

For $0<\alpha<1$, define the semi-norm

$$
|u|_{I_{-\infty}^{\alpha}}=\|{}_{-\infty}D_t^{\alpha}u\|_{L^2},
$$

and the norm

$$
\|u\|_{I_{-\infty}^{\alpha}}=(\|u\|_{L^2}+|u|_{I_{-\infty}^{\alpha}}^2)^{\frac{1}{2}}.
$$

Let

$$
I_{-\infty}^{\alpha}=\overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{I_{-\infty}^{\alpha}}},
$$

where $C_0^{\infty}(\mathbb{R})$ denotes the space of infinitely differentiable functions from $\mathbb{R}$ into $\mathbb{R}^N$ with vanishing property at infinity.

Now, Recalling the Fourier transform $\widehat{u}$ of u

$$
\widehat{u}(s)=\int_{-\infty}^{\infty}e^{-ist}u(t)dt,
$$

we can define the fractional Sobolev space $H^{\alpha}(\mathbb{R})$. Define the semi-norm

$$
|u|_{\alpha}=\left\||s|^{\alpha}\widehat{u}\right\|_{L^2},
$$

and the norm

$$
\|u\|_{\alpha}=(\|u\|_{L^2}+|u|_{\alpha}^2)^{\frac{1}{2}}.
$$

Let

$$
H^{\alpha}(\mathbb{R})=\overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{\alpha}}.
$$

Moreover, we note that a function $u\in L^2(\mathbb{R})$ belongs to $I_{-\infty}^{\alpha}$ if and only if

$$
|s|^{\alpha}\widehat{u}\in L^2(\mathbb{R}).
$$

Especially, we have

$$|u|_{I^{\alpha}_{-\infty}} = \left\| |s|^{\alpha}\, \widehat{u} \right\|_{L^2}.$$

Therefore, $I^{\alpha}_{-\infty}$, and $H^{\alpha}(\mathbb{R})$ are isomorphic with equivalent semi-norms and norms. Let $C(\mathbb{R})$ denote the space of continuous functions from $\mathbb{R}$ into $\mathbb{R}^N$. Then we obtain the following Sobolev lemma.

Lemma 2.1. [13, Theorem 2.1] *If* $\alpha > \frac{1}{2}$, *then* $H^{\alpha}(\mathbb{R}) \subset C(\mathbb{R})$, *and there exists a constant* $C = C_{\alpha}$ *such that*

$$\|u\|_{L^{\infty}} = \sup_{t\in\mathbb{R}} |u(t)| \leq C_{\alpha} \|u\|_{\alpha}, \forall u \in H^{\alpha}(\mathbb{R}). \tag{2.5}$$

Remark 2.2. From Lemma 2.1, we know that if $u \in H^{\alpha}(\mathbb{R})$ with $\frac{1}{2} < \alpha < 1$, then $u \in L^p(\mathbb{R})$ for all $p \in [2,\infty]$ due to

$$\int_{\mathbb{R}} |u(t)|^p \, dt \leq \|u\|_{\infty}^{p-2} \|u\|_{L^2}^2.$$

Let χ be the self-adjoint extension of the operator ${}_tD^{\alpha}_{\infty} o {}_{-\infty}D^{\alpha}_t + L(t)$ with the domain $\mathscr{D}(\chi) \subset L^2(\mathbb{R})$. Let $\{E(\lambda)/ -\infty < \lambda < \infty\}$ denote the resolution of χ, and $U = I - E(0) - E(-0)$. It is well known that U commutes with χ, $|\chi|$ and $|\chi|^{\frac{1}{2}}$, and $\chi = |\chi|U$ is the polar decomposition of χ. Set $X^{\alpha} = \mathscr{D}(|\chi|^{\frac{1}{2}})$, and define on X^{α} the inner product

$$< u, v >_{X^{\alpha}} = < |\chi|^{\frac{1}{2}} u, |\chi|^{\frac{1}{2}} v >_{L^2} + < u, v >_{L^2},$$

and the corresponding norm

$$\|u\|_{X^{\alpha}} = < u, u >^{\frac{1}{2}}_{X^{\alpha}}.$$

The main difficulty in dealing with the existence of infinitely many solutions for $(\mathscr{FHS})$ is the lack of compactness of the Sobolev embedding. To overcome this difficulty consider the following assumptions
(L_1) the smallest eigenvalue of $L(t)$ is bounded from below;
(L_{σ}) There exists a constant $\sigma > 1$ such that

$$meas(\{t \in \mathbb{R}/ |t|^{-\sigma} L(t) < bI_N\}) < \infty, \ \forall b > 0,$$

where *meas* denotes the Lebesgue's measure on $\mathbb{R}$. Here, for two $N \times N$ symmetric matrices M_1 and M_2, we say that $M_1 < M_2$ if

$$\min_{x\in\mathbb{R}^N, |x|=1} (M_1 - M_2)x \cdot x < 0$$

and $M_1 \geq M_2$ if $M_1 < M_2$ does not hold.

Remark 2.3. Let $L(t) = (t^2 sin^2 t - 1)I_N$. It is easy to see that L satisfies (L_1) and (L_{σ}) but it is neither positive definite nor coercive.

In the following, we employ the following compact embedding lemma.

Lemma 2.4. *Suppose that* L *satisfies* (L_1) *and* (L_{σ}). *Then* X^{α} *is compactly embedded in* $L^p(\mathbb{R})$ *for* $1 \leq p \leq \infty$.

Proof. First, we consider the case that $l(t) \geq 1$ for all $t \in \mathbb{R}$. Evidently, we have $|\chi| = \chi$ and

$$\|u\|_{X^\alpha}^2 = \int_{\mathbb{R}} [|{}_{-\infty}D_t^\alpha u(t)|^2 + L(t)u(t) \cdot u(t) + |u(t)|^2] dt.$$

By Lemma 2.1, it is clear that

$$\|u\|_{L^\infty} \leq C_\alpha \|u\|_{X^\alpha}, \ \forall u \in X^\alpha$$

and hence for all $p \in [2, \infty[$ and $u \in X^\alpha$, we have

$$\int_{\mathbb{R}} |u(t)|^p \, dt \leq \|u\|_{L^\infty}^{p-2} \int_{\mathbb{R}} |u(t)|^2 \, dt \leq C_\alpha^{p-2} \|u\|_{X^\alpha}^p.$$

Now, for any $\varepsilon > 0$, by condition (L_σ), we choose $r_\varepsilon \geq 1$ such that $meas(B_\varepsilon) \leq \varepsilon$, where

$$B_\varepsilon = \left\{ t \in \mathbb{R} \backslash] - r_\varepsilon, r_\varepsilon[/ \, |t|^{-\sigma} L(t) < \frac{1}{\varepsilon} I_N \right\}.$$

Let

$$D_\varepsilon = \mathbb{R} \setminus (B_\varepsilon \cup] - r_\varepsilon, r_\varepsilon[),$$

and

$$\mu_\varepsilon = \inf_{|\xi|=1, t \in D_\varepsilon} |t|^{-\sigma} L(t) \xi \cdot \xi.$$

Then $\frac{1}{\mu_\varepsilon} \leq \varepsilon$. Let $(u_k) \subset X^\alpha$ be a sequence such that $u_k \rightharpoonup u$ in X^α weakly. The Banach-Steinhaus Theorem implies that

$$M = \sup_{k \in \mathbb{R}} \|u_k - u\|_{X^\alpha} < \infty.$$

Since $X^\alpha \subset H^\alpha(\mathbb{R}) \subset L^p(\mathbb{R})$ for $p \in [2, \infty]$ with continuous embedding, it holds

$$M_p = \sup_{k \in \mathbb{R}} \|u_k - u\|_{L^p} < \infty.$$

Sobolev's embedding Theorem implies that $u_k \longrightarrow u$ uniformly in $\bar{I}_\varepsilon = [-r_\varepsilon, r_\varepsilon]$.

Step 1. We claim that X^α is compactly embedded in $L^2(\mathbb{R})$. In fact, we have

$$\begin{aligned}
\int_{|t| \geq r_\varepsilon} |u_k - u|^2 \, dt &= \int_{B_\varepsilon} |u_k - u|^2 \, dt + \int_{D_\varepsilon} |u_k - u|^2 \, dt \\
&\leq meas(B_\varepsilon) \|u_k - u\|_{L^\infty}^2 + \int_{D_\varepsilon} |t|^\sigma |u_k - u|^2 \, dt \\
&\leq meas(B_\varepsilon) \|u_k - u\|_{L^\infty}^2 + \frac{1}{\mu_\varepsilon} \int_{D_\varepsilon} L(t)(u_k - u) \cdot (u_k - u) dt \\
&\leq \varepsilon M_\infty^2 + \varepsilon \|u_k - u\|_{X^\alpha}^2 \leq \varepsilon (M_\infty^2 + M^2).
\end{aligned}$$

Since $u_k \longrightarrow u$ uniformly in $\bar{I}_\varepsilon$, we get $u_k \longrightarrow u$ in $L^2(\mathbb{R})$ as $k \longrightarrow \infty$.

Step 2: $p \in]2, \infty[$. We claim that X^α is compactly embedded in $L^p(\mathbb{R})$. In fact, we have

$$\|u_k - u\|_{L^p}^p = \int_{\mathbb{R}} |u_k - u|^p \, dt \leq \|u_k - u\|_{L^\infty}^{p-2} \int_{\mathbb{R}} |u_k - u|^2 \, dt \leq M_\infty^{p-2} \|u_k - u\|_{L^2}^2.$$

By Step 1, we deduce that $u_k \longrightarrow u$ in $L^p(\mathbb{R})$.

Step 3: $p \in [1,2[$. We claim that $u_k \longrightarrow u$ in $L^p(\mathbb{R})$. Let $s = \frac{\sigma}{2-p}$. Then $p > \frac{2}{1+\sigma}$ and $sp > 1$. For $v \in L^p(\mathbb{R})$, we have

$$\begin{aligned}
\int_{|t|\geq r_\varepsilon} |v|^p\,dt &= \int_{B_\varepsilon} |v|^p\,dt + \int_{D_\varepsilon} |v|^p\,dt \\
&= \int_{B_\varepsilon} |v|^p\,dt + \int_{\{t\in D_\varepsilon/|t|^s|v(t)|\leq 1\}} |v|^p\,dt + \int_{\{t\in D_\varepsilon/|t|^s|v(t)|\geq 1\}} |v|^p\,dt \\
&\leq (meas(B_\varepsilon))^{\frac{1}{2}} \|v\|^p_{L^{2p}} + \int_{D_\varepsilon} |t|^{-sp}\,dt + \int_{\{t\in D_\varepsilon/|t|^s|v(t)|\geq 1\}} (|t|^s|v|)^p |t|^{-sp}\,dt \\
&\leq (meas(B_\varepsilon))^{\frac{1}{2}} \|v\|^p_{L^{2p}} + \int_{|t|\geq r_\varepsilon} |t|^{-\sigma}\,dt + \int_{D_\varepsilon} (|t|^s|v|)^2 |t|^{-sp}\,dt \\
&\leq \sqrt{\varepsilon}\,\|v\|^p_{L^{2p}} + 2\int_{r_\varepsilon}^{\infty} |t|^{-\sigma}\,dt + \int_{|t|\geq r_\varepsilon} |t|^{(2-p)s} |v|^2\,dt \\
&\leq \sqrt{\varepsilon}\,\|v\|^p_{L^{2p}} + \frac{2r_\varepsilon^{1-\sigma}}{\sigma-1} + \int_{|t|\geq r_\varepsilon} |t|^{\sigma} |v|^2\,dt \\
&\leq \sqrt{\varepsilon}\,\|v\|^p_{L^{2p}} + \frac{2r_\varepsilon^{1-\sigma}}{\sigma-1} + \frac{1}{\mu_\varepsilon}\int_{|t|\geq r_\varepsilon} L(t)v\cdot v dt \\
&\leq \sqrt{\varepsilon}\,\|v\|^p_{L^{2p}} + \frac{2r_\varepsilon^{1-\sigma}}{\sigma-1} + \varepsilon\,\|v\|^2_{X^\alpha}.
\end{aligned}$$

Choosing r_ε large enough such that $r_\varepsilon^{1-\sigma} \leq \sqrt{\varepsilon}$, we obtain

$$\int_{|t|\geq r_\varepsilon} |v|^p \leq \sqrt{\varepsilon}(\|v\|^p_{L^{2p}}\,dt + \frac{2}{\sigma-1} + \sqrt{\varepsilon}\,\|v\|^2_{X^\alpha}).$$

Since $2p \geq 2$, we have $\|u_k - u\|_{L^{2p}} \leq M_{2p}$ for all $k \in \mathbb{N}$, and

$$\int_{|t|\geq r_\varepsilon} |u_k - u|^p\,dt \leq \sqrt{\varepsilon}(M^p_{2p} + \frac{2}{\sigma-1} + \sqrt{\varepsilon}M^2),\ \forall k \in \mathbb{N}.$$

As above, we have $\int_{\bar{I}_\varepsilon} |u_k - u|^p\,dt \longrightarrow 0$ as $k \longrightarrow \infty$. Hence $u_k \longrightarrow u$ in $L^p(\mathbb{R})$. By a standard argument, we prove the general case which does not need the condition $l(t) \geq 1$ for all $t \in \mathbb{R}$. The proof of Lemma 2.4 is completed. □

By Lemma 2.4, we see that, since the selfadjoint operator χ in $L^2(\mathbb{R})$ is bounded from below, it possesses a compact resolvent. Therefore, the spectrum $\sigma(\chi)$ consists of eigenvalues numbered in $\lambda_1 \leq \lambda_2 \leq ... \longrightarrow +\infty$ (counted in their multiplicities), and a corresponding system of eigenfunctions $(e_j)_{j\in\mathbb{N}}$, $(\chi e_j = \lambda_j e_j)$, forms an orthonormal basis in $L^2(\mathbb{R})$. Let k^- (resp. k^0) be the number of $\lambda_j < 0$ (resp. $\lambda_j = 0$), $\bar{k} = k^- + k^0$, and let $X^- = span\{e_1,,...,e_{k^-}\}$, $X^0 = span\{e_{k^-+1},,...,e_{\bar{k}}\}$, and $X^+ = Cl_{X^\alpha}span\{e_{\bar{k}+1},,...\}$, where $Cl_{X^\alpha}S$ is the closure of the set S in X^α. Then $X^\alpha = X^- \oplus X^0 \oplus X^+$. We introduce on X^α the following inner product

$$< u,v > = < |\chi|^{\frac{1}{2}} u, |\chi|^{\frac{1}{2}} v >_{L^2} + < u^0, v^0 >_{L^2},$$

and the corresponding norm

$$\|u\|^2 = \left\| |\chi|^{\frac{1}{2}} u \right\|^2_{L^2} + \left\| u^0 \right\|^2_{L^2},$$

where $u = u^- + u^0 + u^+, v = v^- + v^0 + v^+ \in X^- \oplus X^0 \oplus X^+$. Clearly, $\|u\|_{L^2}^2 \leq \lambda \|u\|^2$ for all $u \in X^\alpha$, where $\lambda = \max\left\{1, \lambda_{\bar{k}+1}^{-1}, |\lambda_{k^-}|^{-1}\right\}$. Since $\|u\|_{X^\alpha}^2 = \left\|u^- + u^0\right\|_{L^2}^2 + \|u\|^2$ for all $u \in X^\alpha$, one has $\|u\|^2 \leq \|u\|_{X^\alpha}^2 \leq (1+\lambda)\|u\|^2$, i.e., the norms $\|.\|_{X^\alpha}$ and $\|.\|$ are equivalent. From now on, the norm $\|.\|$ on X^α will be used. By Lemma 2.4, for all $p \in [1,\infty]$, there exists a constant $\eta_p > 0$ such that

$$\|u\|_{L^p} \leq \eta_p \|u\|, \ \forall u \in X^\alpha. \tag{2.6}$$

For later use, let

$$a(u,v) = <|\chi|^{\frac{1}{2}} Uu, |\chi|^{\frac{1}{2}} v>_{L^2}, \ \forall u,v \in X^\alpha$$

be the quadratic form associated with χ. For any $u \in \mathscr{D}(\chi)$ and $v \in X^\alpha$, we have

$$a(u,v) = \int_{\mathbb{R}} ({}_{-\infty}D_t^\alpha u(t).{}_{-\infty}D_t^\alpha v(t) + L(t)u(t)\cdot v(t))dt. \tag{2.7}$$

Since $\mathscr{D}(\chi)$ is dense in X^α, (2.7) holds for all $u \in X^\alpha$. Moreover, by definition

$$a(u,u) = \left\|u^+\right\|^2 - \left\|u^-\right\|^2 \tag{2.8}$$

for all $u = u^- + u^0 + u^+ \in X^\alpha = X^- \oplus X^0 \oplus X^+$.

3. Superquadratic Case

In this section, we are interested in the existence of infinitely many solutions of $(\mathscr{FHS})$ when the potential $W(t,x)$ is superquadratic at infinity with respect to x. More precisely, we make the following assumptions:

(W_1) $$\frac{W(t,x)}{|x|^2} \longrightarrow +\infty \ as \ |x| \longrightarrow \infty, \ uniformly \ in \ t \in \mathbb{R};$$

(W_2) $$\nabla W(t,x)\cdot x \geq 2W(t,x) \geq 0, \ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N;$$

(W_3) $$\frac{|\nabla W(t,x)|}{|x|} \longrightarrow 0 \ as \ |x| \longrightarrow 0, \ uniformly \ in \ t \in \mathbb{R};$$

(W_4) there exist constants $\gamma > 0$ and $a > 0$ such that

$$|\nabla W(t,x)| \leq a(|x|^\gamma + 1), \ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N;$$

(W_5) there exist constants $\beta \geq \gamma$, $\beta > 1$, $b > 0$ and $r > 0$ such that

$$\nabla W(t,x)\cdot x - 2W(t,x) \geq b|x|^\beta, \ \forall t \in \mathbb{R}, \ \forall |x| \geq r.$$

Our main results in this Section read as follows.

Theorem 3.1. *Assume that (L_1), (L_σ) and $(W_1)-(W_5)$ hold. Then system $(\mathscr{FHS})$ possesses at least one nontrivial solution.*

Theorem 3.2. *Assume that (L_1), (L_σ) and $(W_1)-(W_5)$ hold and $W(t,x)$ is even in $x \in \mathbb{R}^N$. Then $(\mathscr{FHS})$ has infinitely many distinct solutions.*

Example 3.3. Let

$$W(t,x) = |x|^2 \, ln(1+|x|^2).$$

A straightforward computation shows that W satisfies our Theorem 3.1.

Proof of Theorem 3.1. For system $(\mathscr{F}\mathscr{H}\mathscr{S})$, we associate the following functional defined on the space X^α introduced in Section 2 by

$$f(u)=\frac{1}{2}\int_{\mathbb{R}}[|_{-\infty}D_t^\alpha u(t)|^2+L(t)u(t)\cdot u(t)]dt-\int_{\mathbb{R}}W(t,u)dt,\ u\in X^\alpha.$$

By (2.8), f can be rewritten as

$$f(u)=\frac{1}{2}\left\|u^+\right\|^2-\frac{1}{2}\left\|u^-\right\|^2-g(u),\ \ u\in X^\alpha,$$

where

$$g(u)=\int_{\mathbb{R}}W(t,u(t))dt.$$

It is well known that under assumptions of Theorem 3.1, f is continuously differentiable on X^α, and its critical points on X^α are exactly the solutions of system $(\mathscr{F}\mathscr{H}\mathscr{S})$. Moreover g' is compact, and, for all $u,v\in X^\alpha$,

$$f'(u)v=\int_{\mathbb{R}}({}_{-\infty}D_t^\alpha u(t)\cdot{}_{-\infty}D_t^\alpha v(t)+L(t)u(t)\cdot v(t))dt-\int_{\mathbb{R}}\nabla W(t,u(t))\cdot v(t)dt.$$

For the existence and multiplicity of solutions of $(\mathscr{F}\mathscr{H}\mathscr{S})$, we appeal to the following abstract critical lemmas.

Let X be a Banach space and $f\in C^1(X,\mathbb{R})$. As usual, we say f satisfies the Palais-Smale condition ((PS) for short) if any sequence $(u_k)\subset X$ for which $(f(u_k))$ is bounded, and $f'(u_k)\longrightarrow 0$ as $k\longrightarrow\infty$ possesses a convergent subsequence.

Lemma 3.4. *(Generalized Mountain Pass Theorem [11]) Let X an infinite dimensional Banach space such that $X=V\oplus Z$, where V is finite dimensional. If $f\in C^1(X,\mathbb{R})$ and the following conditions hold*
(f_1) f satisfies the (PS) condition;
(f_2) there are constants $\rho,\delta>0$ such that

$$f_{|\partial B_\rho\cap Z}\geq\delta;$$

where $\partial B_\rho=\{u\in X:\|u\|=\rho\}$;
(f_3) there are constants $r>\rho$, $M>0$ and $e\in Z$ with $\|e\|=1$ such that

$$f_{|\partial\Lambda}\leq 0 \text{ and } f_{|\Lambda}\leq M,$$

where

$$\Lambda=(B_r\cap V)\oplus\{se:0\leq s\leq r\},$$

then f has a critical point u with $f(u)\geq\delta$.

Lemma 3.5. *(Symmetric Mountain Pass Theorem [11]). Let X be an infinite dimensional Banach space such that $X=V\oplus Z$, where V is finite dimensional. If $f\in C^1(X,\mathbb{R})$ is even and satisfies $f(0)=0$, (f_1), (f_2) and*
$(f_3^{'})$ for each finite dimensional subspace $\tilde{X}\subset X$, there is an $R=R(\tilde{X})>0$ such that $f\leq 0$ on $\tilde{X}\setminus B_R$,
Then f possesses an unbounded sequence of critical values.

In the following, c_n, $n\in\mathbb{N}$, denote some various constants.

Lemma 3.6. *If* (L_1), (L_σ), (W_2), (W_4) *and* (W_5) *hold, then* f *satisfies the* (PS) *condition.*

Proof. Let $(u_k) \subset X^\alpha$ be a (PS) sequence, i.e., there exists a constant $M > 0$ such that

$$|f(u_k)| \leq M, \ \forall k \in \mathbb{N} \ and \ f'(u_k) \longrightarrow 0, \ as \ k \longrightarrow \infty. \tag{3.1}$$

We claim that (u_k) is bounded. If not, passing to a subsequence if necessary, we may assume that $\|u_k\| \longrightarrow \infty$ as $k \longrightarrow \infty$. By (W_2) and (W_5), we have

$$\begin{aligned} 2f(u_k) - f'(u_k)u_k &= \int_{\mathbb{R}} [\nabla W(t,u_k)\cdot u_k - 2W(t,u_k)]dt \\ &\geq b\int_{\{t\in\mathbb{R}:|u_k(t)|\geq r\}} |u_k(t)|^\beta \, dt \end{aligned} \tag{3.2}$$

for all positive integer k, which implies that

$$\frac{1}{\|u_k\|}\int_{\{t\in\mathbb{R}:|u_k(t)|\geq r\}} |u_k(t)|^\beta \, dt \longrightarrow 0 \tag{3.3}$$

as $k \longrightarrow \infty$. Let

$$v_k(t) = \begin{cases} u_k(t), \ if \ |u_k(t)| \leq r, \\ 0, \ if \ |u_k(t)| > r, \end{cases} \tag{3.4}$$

and

$$w_k(t) = u_k(t) - v_k(t) \tag{3.5}$$

for all positive integer k and all $t \in \mathbb{R}$. By (3.2) and (3.5), we get

$$c_1(1+\|u_k\|) \geq b\|w_k\|_{L^\beta}^\beta \tag{3.6}$$

for all positive integer k. It follows from Hölder's inequality, (3.4), (3.5) and the equivalence of the norms on the finite dimensional subspace $X^- \oplus X^0$ that

$$\begin{aligned} \left\|u_k^- + u_k^0\right\|_{L^2}^2 &= < u_k^- + u_k^0, u_k >_{L^2} = < u_k^- + u_k^0, v_k >_{L^2} + < u_k^- + u_k^0, w_k >_{L^2} \\ &\leq \left\|u_k^- + u_k^0\right\|_{L^1} \|v_k\|_{L^\infty} + \left\|u_k^- + u_k^0\right\|_{L^{\beta'}} \|w_k\|_{L^\beta} \\ &\leq c_2 \left\|u_k^- + u_k^0\right\|_{L^2} (1 + \|w_k\|_{L^\beta}) \end{aligned} \tag{3.7}$$

for all positive integer k, where $\beta' = \frac{\beta}{\beta-1}$ $(\beta > 1)$ is the Hölder's conjugate of β. From the equivalence of the norms on the finite dimensional subspace $X^- \oplus X^0$, (3.6), and (3.7), we obtain

$$\left\|u_k^- + u_k^0\right\| \leq c_3 \left\|u_k^- + u_k^0\right\|_{L^2} \leq c_4(1 + \|w_k\|_{L^\beta}) \leq c_5(1 + \|u_k\|^{\frac{1}{\beta}})$$

for all positive integer k, which implies that

$$\frac{\left\|u_k^- + u_k^0\right\|}{\|u_k\|} \longrightarrow 0 \tag{3.8}$$

as $k \longrightarrow \infty$. By (W_4) and (2.6), one sees that

$$\begin{aligned}
f'(u_k)u_k^+ &= \left\|u_k^+\right\|^2 - \int_{\mathbb{R}} \nabla W(t,u_k)\cdot u_k^+ dt \\
&\geq \left\|u_k^+\right\|^2 - \int_{\mathbb{R}} |\nabla W(t,u_k)| \left|u_k^+\right| dt \\
&\geq \left\|u_k^+\right\|^2 - a\int_{\mathbb{R}} |u_k|^\gamma \left|u_k^+\right| dt - a\int_{\mathbb{R}} \left|u_k^+\right| dt \\
&\geq \left\|u_k^+\right\|^2 - a\left\|u_k^+\right\|_{L^\infty} \int_{\{t\in\mathbb{R}:|u_k(t)|\geq r\}} |u_k|^\gamma dt \\
&- ar^\gamma \int_{\{t\in\mathbb{R}:|u_k(t)|<r\}} \left|u_k^+\right| dt - a\int_{\mathbb{R}} \left|u_k^+\right| dt \\
&\geq \left\|u_k^+\right\|^2 - a\left\|u_k^+\right\|_{L^\infty} r^{\gamma-\beta} \int_{\{t\in\mathbb{R}:|u_k(t)|\geq r\}} |u_k|^\beta dt - ar^\gamma \left\|u_k^+\right\|_{L^1} - a\left\|u_k^+\right\|_{L^1} \\
&\geq \left\|u_k^+\right\|^2 - a\eta_\infty \left\|u_k^+\right\| r^{\gamma-\beta} \int_{\{t\in\mathbb{R}:|u_k(t)|\geq r\}} |u_k|^\beta dt - ar^\gamma \eta_1 \left\|u_k^+\right\| - a\eta_1 \left\|u_k^+\right\|,
\end{aligned}$$

which, together with (3.3), implies that

$$\frac{\left\|u_k^+\right\|}{\|u_k\|} \longrightarrow 0 \tag{3.9}$$

as $k \longrightarrow \infty$. Hence by (3.8) and (3.9), we obtain

$$1 = \frac{\|u_k\|}{\|u_k\|} \leq \frac{\left\|u_k^- + u_k^0\right\| + \left\|u_k^+\right\|}{\|u_k\|} \longrightarrow 0$$

as $k \longrightarrow \infty$, which is a contradiction. Hence (u_k) must be bounded. Moreover, we have

$$\left\|u_k^+ - u^+\right\|^2 = (f'(u_k) - f'(u))(u_k^+ - u^+) + (g'(u_k) - g'(u))(u_k^+ - u^+).$$

Going to a subsequence if necessary, we may assume, by using Lemma 2.4, that $u_k \rightharpoonup u$ weakly in X^α, and

$$u_k \longrightarrow u \text{ in } L^2(\mathbb{R}) \text{ and in } L^\infty(\mathbb{R}) \text{ as } k \longrightarrow \infty. \tag{3.10}$$

Since g' is compact, we deduce that $g'(u_k) \longrightarrow g'(u)$ and therefore $u_k^+ \longrightarrow u^+$ in X^α. From (3.10) and the equivalence of the norms on the finite dimensional subspace $X^- \oplus X^0$, we obtain that $u_k^0 \longrightarrow u^0$ and $u_k^- \longrightarrow u^-$ in X^α as $k \longrightarrow \infty$. Hence (u_k) has a convergent subsequence, which shows that the (PS) condition holds. The proof of Lemma 3.6 is achieved. □

Lemma 3.7. *Assume that (L_1), (L_σ), (W_2) and (W_3) are satisfied. Then there are constants $\rho > 0$ and $\delta > 0$ such that*

$$f_{|S} \geq \delta$$

where

$$S = \left\{u \in X^+ : \|u\| = \rho\right\}.$$

Proof. By (W_3), for all $\varepsilon > 0$, there exists $\nu > 0$ such that

$$|\nabla W(t,x)| \leq \varepsilon |x|, \; \forall t \in \mathbb{R}, \; \forall |x| \leq \nu,$$

which together with (W_2) and the Mean Value Theorem gives

$$W(t,x)=\int_0^1 \nabla W(t,sx)\cdot xds \leq \frac{\varepsilon}{2}|x|^2,\ \forall t\in\mathbb{R},\ \forall |x|\leq \nu.$$

Choose $\varepsilon=(2\eta_2^2)^{-1}$ and take $\rho=\frac{\nu}{\eta_\infty}$, $\delta=\frac{\rho^2}{4}$. By Lemma 2.4, we get

$$\begin{aligned} f(u) &= \frac{1}{2}\|u\|^2-\int_{\mathbb{R}} W(t,u(t))dt \geq \frac{1}{2}\|u\|^2-\frac{\varepsilon}{2}\int_{\mathbb{R}}|u(t)|^2dt \\ &\geq \frac{1}{2}\|u\|^2-\frac{\varepsilon}{2}\eta_2^2\|u\|^2=\frac{1}{4}\|u\|^2=\frac{\rho^2}{4}=\delta \end{aligned}$$

for all $u\in S$. The proof of Lemma 3.7 is completed. □

Proof of Theorem 3.1.

Lemma 3.8. *Assume that* (L_1), (L_σ), (W_1), (W_2) *and* (W_5) *are satisfied./ Let* $e\in X^+$ *with* $\|e\|=1$. *There exist* $r_1,r_2>0$ *such that*

$$f(u)\leq 0,\ \forall u\in\partial\Lambda,$$

where

$$\Lambda=\{se: 0\leq s\leq r_1\}\oplus\left\{u\in X^-\oplus X^0:\|u\|\leq r_2\right\}.$$

Proof. Let $e\in X^+$ with $\|e\|=1$, and $F=span\{e\}\oplus X^-\oplus X^0$. By a standard argument, there exists a constant $\varepsilon_0>0$ such that

$$meas(\{t\in\mathbb{R}:|u(t)|\geq\varepsilon_0\|u\|\})\geq\varepsilon_0,\ \forall u\in F\setminus\{0\}. \tag{3.11}$$

For $u=u^-+u^0+u^+\in F$, let

$$\Omega_u=\{t\in\mathbb{R}/\,|u(t)|\geq\varepsilon_0\|u\|\}.$$

By (W_1), for $d=\frac{1}{2\varepsilon_0^3}>0$, there exists $R_1>0$ such that

$$W(t,x)\geq d|x|^2,\ \forall |x|\geq R_1,\ \forall t\in\mathbb{R}.$$

Hence one has

$$W(t,u(t))\geq d|u(t)|^2\geq d\varepsilon_0^2\|u\|^2 \tag{3.12}$$

for all $u\in F$ with $\|u\|\geq\frac{R_1}{\varepsilon_0}$ and $t\in\Omega_u$. It follows from (W_2), (W_5), (3.11) and (3.12) that

$$\begin{aligned} f(u) &= \frac{1}{2}\left\|u^+\right\|^2-\frac{1}{2}\left\|u^-\right\|^2-\int_{\mathbb{R}}W(t,u(t))dt \\ &\leq \frac{1}{2}\left\|u^+\right\|^2-\int_{\Omega_u}W(t,u(t))dt \\ &\leq \frac{1}{2}\left\|u^+\right\|^2-d\varepsilon_0^2\|u\|^2 meas(\Omega_u) \\ &\leq \frac{1}{2}\left\|u^+\right\|^2-d\varepsilon_0^3\|u\|^2=\frac{1}{2}\left\|u^+\right\|^2-\frac{1}{2}\|u\|^2\leq 0, \end{aligned} \tag{3.13}$$

for all $u\in F$ with $\|u\|\geq\frac{R_1}{\varepsilon_0}$. Let $r_1>0$ and denote

$$\Lambda=\{se/0\leq s\leq r_1\}\oplus\left\{u\in X^-\oplus X^0:\|u\|\leq r_1\right\}.$$

It follows that

$$\partial\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3,$$

where

$$\Lambda_1 = \left\{u \in X^- \oplus X^0 : \|u\| \le r_1\right\},$$
$$\Lambda_2 = r_1 e + \left\{u \in X^- \oplus X^0 / \|u\| \le r_1\right\},$$

and

$$\Lambda_3 = \{se/0 \le s \le r_1\} \oplus \left\{u \in X^- \oplus X^0 / \|u\| = r_1\right\}.$$

By (3.13), one has

$$f(u) \le 0,\ \forall u \in \Lambda_2 \cup \Lambda_3$$

for all $r_1 \ge \frac{R}{\varepsilon_0}$. From (W_2), we have

$$f(u) \le 0,\ \forall u \in X^- \oplus X^0,$$

which implies that

$$f(u) \le 0,\ \forall u \in \Lambda_1.$$

Hence, $f(u) \le 0$, $\forall u \in \partial\Lambda$, for all $r_1 > max\left\{\rho, \frac{R_1}{\varepsilon_0}\right\}$, where ρ is defined in Lemma 3.7. This complete the proof of Lemma 3.8. □

By Lemma 3.4, f has a critical point u satisfying $f(u) \ge \delta > 0$, where δ is given by Lemma 3.7. Since $f(0) = 0$, then u is nontrivial and $(\mathscr{F}\mathscr{H}\mathscr{S})$ possesses a nontrivial solution. The proof of Theorem 3.1 is achieved.

Proof of Theorem 3.2. Note that $f(0) = 0$. Since $W(t,x)$ is even with respect to the second variable, then f is even. The assumptions (f_1) and (f_2) are proved above. Let us prove $(f_3^{'})$. Let $\tilde{X} \subset X^\alpha$ be a finite dimensional subspace of X^α, there exists $m \ge 1$ such that $\tilde{X} \subset X^- \oplus X^0 \oplus span\{w_1, \dots, w_m\} = X^m$, where $w_k = e_{n^- + k^0 + k}$, $k \ge 1$. Replacing the subspace $F = span\{e\} \oplus X^- \oplus X^0$, introduced in the proof of Lemma 3.8, by the subspace X^m and following the same steps, we obtain $R_m > 0$ such that

$$f(u) \le \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\|u\|^2 \le 0,\ \forall u \in X^m,\ \|u\| \ge R_m.$$

Hence $(f_3^{'})$ is verified. Therefore, by Lemma 3.5, f possesses an unbounded sequence of critical points. Hence $(\mathscr{F}\mathscr{H}\mathscr{S})$ possesses infinitely may solutions. The proof of Theorem 3.2 is completed.

4. Local Conditions

In this section, we are interested in a general case where the potential $W(t,x)$ satisfies only locally conditions near the origin with respect to x, and do not satisfy any additional hypotheses at infinity. More precisely, we present the following assumptions:

(W_6) There exist constants $r, c > 0$ and $\nu \in]0,1[$ such that

$$|\nabla W(t,x)| \le c\,|x|^\nu,\ \forall t \in \mathbb{R},\ |x| \le r;$$

(W_7)
$$\lim_{|x|\to 0} \frac{|W(t,x)|}{|x|^2} = +\infty,\ uniformly\ for\ all\ t \in \mathbb{R}.$$

(W_8) There exists $\rho \in]0,r]$ such that

$$W(t,-x) = W(t,x),\ and\ W(t,x) \geq 0,\ \forall t \in \mathbb{R}\ |x| \leq \rho.$$

Our main result in this section reads as follows.

Theorem 4.1. *Assume that (L_1), (L_σ) and $(W_6)-(W_8)$ are satisfied. Then $(\mathscr{FHS})$ possesses infinitely many nontrivial solutions (u_k) such that $u_k \neq 0$ and $\|u_k\|_{L^\infty} \longrightarrow$ as $k \longrightarrow \infty$.*

In the following, we give some examples which satisfy our assumptions.

Example 4.2. (The subquadratic case at infinity). Let

$$W(t,x) = h(t)\,|x|^\theta\, ln(1+|x|^2),$$

where $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, $\theta \in]1,2[$ and $h \in C(\mathbb{R},\mathbb{R})$ with $0 < \inf_{t\in\mathbb{R}} h(t) \leq \sup_{t\in\mathbb{R}} h(t) < \infty$. It is easy to see that $W(t,x)$ satisfies the conditions $(W_6)-(W_8)$ and the subquadratic condition at infinity, i.e.,

$$\lim_{|x|\longrightarrow\infty} \frac{W(t,x)}{|x|^2} = 0.$$

Example 4.3. (The superquadratic case at infinity). Let

$$W(t,x) = h(t)(|x|^\nu + |x|^\theta\, ln(1+|x|^2)),$$

where $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, $\theta \in]1,2[$, $\nu \in]2,\infty[$, and $h \in C(\mathbb{R},\mathbb{R})$ with $0 < \inf_{t\in\mathbb{R}} h(t) \leq \sup_{t\in\mathbb{R}} h(t) < \infty$. It is easy to see that $W(t,x)$ satisfies the conditions $(W_6)-(W_8)$ and the superquadratic condition at infinity, i.e.,

$$\lim_{|x|\longrightarrow\infty} \frac{W(t,x)}{|x|^2} = +\infty.$$

Example 4.4. (The asymptotically quadratic case at infinity). Let

$$W(t,x) = \frac{1}{2}S(t)x\cdot x + |x|^\theta\, ln(1+|x|^2),$$

where $S:\mathbb{R}\longrightarrow\mathbb{R}^{N^2}$ is a bounded symmetric $N\times N$ matrix-valued function and $\theta \in]1,2[$. It is clear that $W(t,x)$ is asymptotically quadratic at infinity with respect to x and satisfies the conditions $(W_6)-(W_8)$.

Proof of Theorem 4.1. Consider the functional $f : X^\alpha \longrightarrow \mathbb{R}$ introduced in Section 3. Under assumptions of Theorem 4.1, the functional f is continuously differentiable on X^α, and the critical points of f on X^α are the solutions of the system $(\mathscr{FHS})$.

We shall use the following Variant Symmetric Mountain Pass Lemma due to Kajikiya [28] to prove our result. We will first recall the notion of genus.

Let X be a Banach space, and let A be a subset of X. A is said to be symmetric if $u \in A$ implies $-u \in A$. For a closed symmetric set A, which does not contain the origin, we define the genus $\gamma(A)$ of A by the smallest integer k for which there exists an odd continuous mapping from A to $\mathbb{R}^k\setminus\{0\}$. If such a k does not exist, we define $\gamma(A) = +\infty$. Moreover, we set $\gamma(\phi) = 0$. Let

$$\Gamma_k = \{A \subset X / A\ is\ a\ closed\ symmetric\ subset,\ 0 \notin A,\ \gamma(A) \geq k\}.$$

The properties of genus used in the proof of our main result are summarized as follows.

Lemma 4.5. [28] *Let A and B be closed symmetric subsets of X that do not contain the origin. Then the following assertions hold.*
a) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.
b) The $n-$dimensional sphere S^n has a genus of $n+1$ by the Borsuk-Ulam theorem.

Lemma 4.6 (10)**.** *Let X be an infinite-dimensional Banach space, and let $f \in C^1(X,\mathbb{R})$ satisfy the following*
(f_1) $f(0)=0$, f is even and bounded from below and f satisfies the $(PS)-$condition;
(f_2) for each $k \in \mathbb{N}$, there exists $A_k \subset \Gamma_k$ such that

$$\sup_{u \in A_k} f(u) < 0.$$

Then f possesses a sequence of critical points (u_k) such that

$$f(u_k) \leq 0,\ u_k \neq 0,\ \forall k \in \mathbb{N} \text{ and } \lim_{k \longrightarrow \infty} u_k = 0.$$

Now, let $\theta \in C^1(\mathbb{R}^+,\mathbb{R}^+)$ satisfy

$$\begin{cases} \theta(s) = 1 \ for\ s \in [0, \frac{\rho}{2\eta_\infty}],\ \theta(s) = 0 \ for\ s \geq \frac{\rho}{\eta_\infty}, \\ \theta'(s) < 0 \ for\ \frac{\rho}{2\eta_\infty} < s < \frac{\rho}{\eta_\infty}, \end{cases} \tag{4.1}$$

where ρ is defined in (W_8). Consider the new functional h defined on X^α by

$$h(u) = \frac{1}{2}\|u\|^2 - \theta(\|u\|)\Big(\left\|u^-\right\|^2 + \frac{1}{2}\left\|u^0\right\|^2 + \int_{\mathbb{R}} W(t,u)dt\Big).$$

Remark 4.7. It is clear that $h \in C^1(X^\alpha,\mathbb{R})$, $h(u) = f(u)$ for all $\|u\| \leq \frac{\rho}{2\eta_\infty}$. Thus, critical points of h satisfying $\|u\| \leq \frac{\rho}{2\eta_\infty}$ are exactly critical points of f. Consequently, to prove our result, we will apply Lemma 4.6 to the functional h instead of f.

Lemma 4.8. *Assume that (L_1), (L_σ), (W_6) and (W_8) are satisfied. Then h satisfies the (PS)-condition.*

Proof. Let (u_n) be a Palais-Smale sequence, that is, $(h(u_n))$ is bounded, and $h'(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$. If $u \in X^\alpha$ with $\|u\| \geq \frac{\rho}{\eta_\infty}$, then by the definition of θ and h, we have

$$h(u) = \frac{1}{2}\|u\|^2,$$

which implies that

$$h(u) \longrightarrow +\infty \ as\ \|u\| \longrightarrow \infty. \tag{4.2}$$

Since $(h(u_n))$ is bounded, then (4.2) implies that (u_n) is bounded. Thus, passing to a subsequence if necessary, we can assume by Lemma 2.4 that $u_n \rightharpoonup u = u^- + u^0 + u^+$, $u_n^+ \rightharpoonup u^+$ and $u_n^+ \longrightarrow u^+$ in $L^1(\mathbb{R})$. On the other hand, if $\|u_n\| \geq \frac{\rho}{\eta_\infty}$, we have $h'(u_n)u_n = \|u_n\|^2 \geq \frac{\rho^2}{\eta_\infty^2}$, which contradicts the fact that $h'(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, we can assume that $\|u_n\| \leq \frac{\rho}{\eta_\infty}$ for all $n \in \mathbb{N}$, which together with (2.6) implies that

$$|u_n(t)| \leq \|u_n\|_{L^\infty} \leq \eta_\infty \|u_n\| \leq \rho.$$

It follows from (W_6) that

$$\begin{aligned}\left|\int_{\mathbb{R}} \nabla W(t,u_n)\cdot(u_n^+ - u^+)dt\right| &\leq \int_{\mathbb{R}} \left|\nabla W(t,u_n)\right|\left|u_n^+ - u^+\right| dt \\ &\leq c\left\|u_n\right\|_{L^\infty}^{\nu} \int_{\mathbb{R}} \left|u_n^+ - u^+\right| dt \\ &\leq c\rho^{\nu} \int_{\mathbb{R}} \left|u_n^+ - u^+\right| dt \longrightarrow 0\end{aligned} \tag{4.3}$$

as $n \longrightarrow \infty$.

Now, we have

$$\begin{aligned} h'(u_n)(u_n^+ - u^+) =& < u_n, u_n^+ - u^+ > -\theta'(\|u_n\|) < \frac{u_n}{\|u_n\|}, u_n^+ - u^+ > \\ & \Big(\frac{1}{2}\left\|u_n^0\right\|^2 + \left\|u_n^-\right\|^2 + \int_{\mathbb{R}} W(t,u_n)dt\Big) \\ & -\theta(\|u_n\|)\int_{\mathbb{R}} \nabla W(t,u_n)\cdot(u_n^+ - u^+)dt, \end{aligned}$$

which together with (4.3) and the fact that $h'(u_n) \longrightarrow 0$ implies

$$\Big[1 - \frac{\theta'(\|u_n\|)}{\|u_n\|}\Big(\frac{1}{2}\left\|u_n^0\right\|^2 + \left\|u_n^-\right\|^2 + \int_{\mathbb{R}} W(t,u_n)dt\Big)\Big] < u_n, u_n^+ - u^+ > \longrightarrow 0 \tag{4.3}$$

as $n \longrightarrow \infty$. Since $\|u_n\| \leq \frac{\rho}{\eta_\infty}$, then $|u_n(t)| \leq \rho$ and $W(t,u_n(t)) \geq 0$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$ due to (W_8). Hence, the definition of θ implies

$$1 - \frac{\theta'(\|u_n\|)}{\|u_n\|}\Big(\frac{1}{2}\left\|u_n^0\right\|^2 + \left\|u_n^-\right\|^2 + \int_{\mathbb{R}} W(t,u_n)dt\Big) \geq 1,\ \forall n \in \mathbb{N}.$$

It follows from (4.3) that

$$< u_n, u_n^+ - u^+ > \longrightarrow 0 \ as\ n \longrightarrow \infty.$$

By virtue of $u_n^+ \rightharpoonup u^+$, we have $\|u_n^+\| \longrightarrow \|u^+\|$ and then $u_n^+ \longrightarrow u^+$. Since X^- and X^0 are finite dimensional subspaces, we have $u_n^- \longrightarrow u^-$, and $u_n^0 \longrightarrow u^0$. Therefore $u_n \longrightarrow u$ in X^α, and h satisfies the $(PS)-$condition. The proof of Lemma 4.8 is completed. □

Now, the definitions of h and θ imply that $h(u) = \frac{1}{2}\|u\|^2 = h(-u)$ for all $\|u\| \geq \frac{\rho}{\eta_\infty}$. If $\|u\| \leq \frac{\rho}{\eta_\infty}$, we have as above $|u(t)| \leq \rho$ for all $t \in \mathbb{R}$, which together with (W_8) implies $W(t,-u(t)) = W(t,u(t))$ for all $t \in \mathbb{R}$ and $h(-u) = h(u)$. Thus h is even in X^α.

We claim that h is bounded from below. If not, there exists a sequence (u_n) such that

$$h(u_n) \longrightarrow -\infty \ as\ n \longrightarrow \infty. \tag{4.4}$$

By (W_6), (W_8) and the definitions of h and θ, it is easy to verify that h maps bounded sets into bounded sets. It follows from (4.4) that $\|u_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Thus (4.2) implies that $h(u_n) \longrightarrow +\infty$ as $n \longrightarrow \infty$, which contradicts (4.4). Hence the condition (f_1) of Lemma 4.6 is verified.

Finally, we show that h satisfies the condition (f_2) of Lemma 4.6. For any positive integer k, let

$$X_k = \oplus_{m=1}^k E_m,\ E_m = \mathbb{R}e_m,$$

where the sequence (e_m) is defined in Section 2. Since X_k is finite dimensional, there exists a positive constant β_k such that

$$\|u\| \leq \beta_k \|u\|_{L^2}, \ \forall u \in X_k. \tag{4.5}$$

By (W_7), there exists a constant $R > 0$ such that

$$W(t,x) \geq \beta_k^2 |x|^2, \ \forall t \in \mathbb{R}, \ |x| \leq R. \tag{4.6}$$

Let $u \in X^\alpha$ such that $\|u\| \leq \frac{R}{\eta_\infty}$. We have $|u(t)| \leq R$ for all $t \in \mathbb{R}$. It follows from (4.6) that

$$W(t,u(t)) \geq \beta_k^2 |u(t)|^2, \ \forall t \in \mathbb{R}. \tag{4.7}$$

Therefore, by (4.7), for all $u \in X_k$ with $0 < \|u\| = r_k \leq \min\left\{\frac{\rho}{2\eta_\infty}, \frac{R}{\eta_\infty}\right\}$, we have

$$\begin{aligned} h(u) &= \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}} W(t,u)dt \\ &\leq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \beta_k^2 |u(t)|^2 dt \\ &\leq \frac{1}{2}\|u\|^2 - \|u\|^2 = -\frac{1}{2}\|u\|^2 = -\frac{1}{2}r_k^2, \end{aligned}$$

which implies

$$\{u \in X_k \setminus \{0\} \,/\, \|u\| = r_k\} \subset A_k \tag{4.8}$$

where

$$A_k = \left\{u \in X^\alpha / h(u) \leq -\frac{1}{2}\eta_k^2\right\}.$$

Thus Lemma 4.5 and (4.8) imply

$$\gamma(A_k) \geq \gamma\Big(\{u \in X_k \setminus \{0\} \,/\, \|u\| = r_k\}\Big) \geq k.$$

Hence, by the definition of Γ_k, we have $A_k \subset \Gamma_k$. Moreover, the definition of A_k implies

$$\sup_{u \in A_k} h(u) \leq -\frac{1}{2}\eta_k^2 < 0,$$

which is the condition (f_2) of Lemma 4.6. Consequently, all the conditions of Lemma 4.6 hold and the proof of Theorem 4.1 is finished by this Lemma.

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