Commun. Optim. Theory 20201 (2021), Article ID 2 https://doi.org/10.23952/cot.2021.2

Communications in Optimization Theory
Available online at http://cot.mathres.org

ASYMPTOTIC STABILITY IN HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we prove the asymptotic stability of the zero solution of a nonlinear Hadamard fractional differential equation. We employ the Schauder and Banach fixed point theorems to show the asymptotic stability of the zero solution in a weighted Banach space. Finally, an example is given to illustrate our results.

Keywords. Asymptotic stability; Fixed point theorems; Fractional differential equations; Hadamard fractional derivative.

1. Introduction

Fractional differential equations arise from a variety of applications, such as, applied sciences, physics, chemistry, biology, medicine, etc. In particular, the problems concerning qualitative analysis of linear and nonlinear fractional differential equations have received much attention from many authors; see [1]- [22] and the references therein.

In [13], Ge and Kou investigated the asymptotic stability of the zero solution of the following nonlinear fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}x(t) = kx(t) + f(t, x(t)), \ t \geq 0, \\ x'(0) = 0, \ x(0) = x_0, \end{cases}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order $1 < \alpha < 2$. By employing the Banach contraction mapping principle in a weighted Banach space, they obtained asymptotic stability results.

The nonlinear fractional differential equation

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t)), \ t \geq 0, \\ D^{\alpha-k}x(0) = b_k, \ k = 1, ..., n, \end{cases}$$

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Received June 29, 2020; Accepted December 12, 2020.

was investigated in [18], where D^{α} is the Riemann-Liouville fractional derivative of order $\alpha \in (n-1,n)$, $f(t,0)=0$ and $f:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is a given continuous function. By using the Schauder and Banach fixed point theorems, Li and Kou obtained stability results.

Benchohra and Lazreg, in [9], studied the existence and the Ulam stability of solutions to the following boundary value problem

$$\begin{cases} \mathfrak{D}_1^{\alpha} x(t) = f(t, x(t), \mathfrak{D}_1^{\alpha} x(t)),\ t\in[1,T], \\ x(1) = x_0, \end{cases}$$

where $\mathfrak{D}_1^{\alpha}$ is the Hadamard fractional derivatives of order $0<\alpha\leq 1$. By employing the fixed point theorems, they obtained existence and stability results.

Inspired and motivated by the works mentioned above, we concentrate on the stability of solutions for the nonlinear Hadamard fractional differential equation

$$\begin{cases} \mathfrak{D}_1^{\alpha} x(t) = f(t, x(t)),\ t\geq 1, \\ \mathfrak{D}_1^{\alpha-k} x(1) = b_k,\ k=1,\dots,n, \end{cases} \tag{1.1}$$

where $\alpha \in (n-1,n)$, $f(t,0)=0$ and $f:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is a given continuous function. To show the asymptotic stability of the zero solution, we transform (1.1) into an integral equation and then use the Schauder and Banach fixed point theorems.

This paper is organized as follows. In Section 2, we introduce some notations and lemmas, and state some preliminaries results needed in later section. Also, we present the inversion of (1.1) and the Banach and Schauder fixed point theorems. For details on the Banach and Schauder theorems we refer the reader to [23]. In Section 3, we give and prove our main results on asymptotic stability, and provide an example to illustrate our results.

2. Preliminaries

Let $C_{n-\alpha}([1,\infty)) = \{x:[1,\infty)\to\mathbb{R},\ (\log t)^{n-\alpha} x \in C([1,\infty))\}$. For any $b>1$, we denote $C_{n-\alpha}([1,b]) = \{x:[1,b]\to\mathbb{R},\ (\log t)^{n-\alpha} x\in C([1,b])\}$. $C_{n-\alpha}[1,b]$ is a Banach space equipped with the norm

$$\|x\|_{C_{n-\alpha}([1,b])} = \left\|(\log t)^{n-\alpha} x\right\|_{C([1,b])} = \max_{t\in[1,b]} \left|(\log t)^{n-\alpha} x(t)\right|.$$

We introduce some necessary definitions, lemmas and theorems which will be used in this paper. For more details, we refer to [14, 21].

Definition 2.1. [14] The Hadamard fractional integral of order $\alpha>0$ for a continuous function $x:[1,\infty)\to\mathbb{R}$ is defined as

$$\mathfrak{I}_1^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s},\ \alpha>0.$$

Definition 2.2. [14] *The Hadamard fractional derivative of order $\alpha>0$ for a function $x:[1,\infty)\to\mathbb{R}$ is defined as*

$$\mathfrak{D}_1^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} x(s)\frac{ds}{s},\ n-1<\alpha<n.$$

Lemma 2.3. *[14] Let $n-1<\alpha\leq n$, $n\in\mathbb{N}$. The equality $\left(\mathfrak{J}_1^\alpha\mathfrak{D}_1^\alpha x\right)(t)=0$ is valid if and only if*

$$x(t)=\sum_{k=1}^{n}c_k(\log t)^{\alpha-k}\text{ for each }t\in[1,\infty),$$

where $c_k\in\mathbb{R}$, $k=1,\ldots,n$ are arbitrary constants.

Lemma 2.4. *[14] For all $\mu>0$ and $\nu>-1$,*

$$\frac{1}{\Gamma(\mu)}\int_1^t\left(\log\frac{t}{s}\right)^{\mu-1}(\log s)^{\nu}\frac{ds}{s}=\frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(\log t)^{\mu+\nu}.$$

The following lemma is fundamental to our results.

Lemma 2.5. *[14] Let $f:[1,\infty)\times\mathbb{R}\to\mathbb{R}$ be a function such that $f\in C_{n-\alpha}([1,\infty))$ for any $x\in C_{n-\alpha}([1,\infty))$. If $x\in C_{n-\alpha}([1,\infty))$, then x is a solution of (1.1) if and only if it satisfies the Volterra integral equation*

$$x(t)=\sum_{k=1}^{n}\frac{b_k}{\Gamma(\alpha-k+1)}(\log t)^{\alpha-k}+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}f(s,x(s))\frac{ds}{s},\ t\geq 1. \tag{2.1}$$

Lemma 2.6. *Let*

$$E=\left\{x\in C_{n-\alpha}([1,\infty)),\ \lim_{t\to\infty}\frac{(\log t)^{n-\alpha}x(t)}{1+(\log t)^{n+1}}=0\right\},$$

with the norm

$$\|x\|=\sup_{t\geq 1}\frac{(\log t)^{n-\alpha}|x(t)|}{1+(\log t)^{n+1}}.$$

Then, $(E,\|.\|)$ is a Banach space.

Proof. We only prove that the space E is complete. Let $\{x_k\}_{k=1}^{\infty}$ be a Cauchy sequence in E. Then, for any given $\varepsilon>0$ and any $t\geq 1$, there exists a constant $N>0$ such that, for $k,j\geq N$,

$$\frac{(\log t)^{n-\alpha}\left|x_k(t)-x_j(t)\right|}{1+(\log t)^{n+1}}\leq\left\|x_k-x_j\right\|<\frac{\varepsilon}{3}, \tag{2.2}$$

i.e., $\left\{(\log t)^{n-\alpha}x_k(t)/\left(1+(\log t)^{n+1}\right)\right\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Thus, there exists a y such that

$$\lim_{k\to\infty}\frac{(\log t)^{n-\alpha}x_k(t)}{1+(\log t)^{n+1}}=y(t),\ t\geq 1.$$

This means that a function $y:[1,\infty)\to\mathbb{R}$ is well defined. In (2.2), letting $j\to\infty$, we have

$$\left|\frac{(\log t)^{n-\alpha}x_k(t)}{1+(\log t)^{n+1}}-y(t)\right|\leq\frac{\varepsilon}{3},\ t\geq 1,\ k\geq N. \tag{2.3}$$

Next, we show that $y\in C([1,\infty))$. In fact, for any given $t_0\in[1,\infty)$, by (2.3) and the continuity of $\left\{(\log t)^{n-\alpha}x_N(t)/\left(1+(\log t)^{n+1}\right)\right\}$ on $[1,\infty)$, we can see that there exists a $\delta>0$ such that

$|t-t_0|<\delta$ implies

$$\begin{aligned}
&|y(t)-y(t_0)| \\
&\leq \left|y(t)-\frac{(\log t)^{n-\alpha}x_N(t)}{1+(\log t)^{n+1}}\right| + \left|\frac{(\log t)^{n-\alpha}x_N(t)}{1+(\log t)^{n+1}}-\frac{(\log t_0)^{n-\alpha}x_N(t_0)}{1+(\log t_0)^{n+1}}\right| \\
&+\left|\frac{(\log t_0)^{n-\alpha}x_N(t_0)}{1+(\log t_0)^{n+1}}-y(t_0)\right| \\
&\leq \frac{\varepsilon}{3}+2\sup_{t\geq 1}\left|y(t)-\frac{(\log t)^{n-\alpha}x_N(t)}{1+(\log t)^{n+1}}\right| \leq \varepsilon.
\end{aligned}$$

Therefore, $y\in C([1,\infty))$. Letting

$$x(t)=\frac{1+(\log t)^{n+1}}{(\log t)^{n-\alpha}}y(t),$$

we have $(\log t)^{n-\alpha}x\in C([1,\infty))$. With the condition

$$\lim_{t\to\infty}\frac{(\log t)^{n-\alpha}x_N(t)}{1+(\log t)^{n+1}}=0,$$

we have, for $x_N\in E$,

$$\frac{(\log t)^{n-\alpha}|x(t)|}{1+(\log t)^{n+1}}\leq \sup_{t\geq 1}\left|y(t)-\frac{(\log t)^{n-\alpha}x_N(t)}{1+(\log t)^{n+1}}\right|+\frac{(\log t)^{n-\alpha}|x_N(t)|}{1+(\log t)^{n+1}}\to 0,$$

as $t\to\infty$. Thus, we complete the proof. □

Lemma 2.7. *Let Ω be a subset of a Banach space E. Then, Ω is relatively compact in E if the following conditions are satisfied*

(*i*) $\left\{(\log t)^{n-\alpha}x(t)/\left(1+(\log t)^{n+1}\right):x\in\Omega\right\}$ *is uniformly bounded;*

(*ii*) $\left\{(\log t)^{n-\alpha}x(t)/\left(1+(\log t)^{n+1}\right):x\in\Omega\right\}$ *is equicontinuous in* $[1,\infty)$*;*

(*iii*) $\left\{(\log t)^{n-\alpha}x(t)/\left(1+(\log t)^{n+1}\right):x\in\Omega\right\}$ *equiconverges to* 0 *as* $t\to+\infty$*, i.e., for any given* $\varepsilon>0$*, there exists* $T>1$*, such that, for all* $x\in\Omega$ *and* $t>T$*,*

$$\left|(\log t)^{n-\alpha}x_k(t)/\left(1+(\log t)^{n+1}\right)\right|<\varepsilon.$$

Proof. The proof is similar to the proof of Lemma 3.2 in [15] and so the proof is omitted. □

Finally, we state the fixed point theorems, which enable us to prove the stability of the trivial solution of (1.1).

Definition 2.8. Let $(E,\|.\|)$ be a Banach space and $F:E\to E$. The operator F is a contraction operator if there is an $\lambda\in(0,1)$ such that, for any $x,y\in E$,

$$\|Fx-Fy\|\leq\lambda\|x-y\|.$$

Theorem 2.9 (Banach [23]). *Let Ω be a nonempty closed convex subset of a Banach space E and $F:\Omega\to\Omega$ be a contraction operator. Then there is a unique $x\in\Omega$ with $Fx=x$.*

Theorem 2.10 (Schauder [23])**.** *Let Ω be a nonempty bounded closed convex subset of a Banach space E and $F : \Omega \to \Omega$ be a continuous compact operator. Then F has a fixed point in Ω.*

3. MAIN RESULTS

In this section, we prove the asymptotic stability of the zero solution of (1.1).

Definition 3.1. The zero solution $x = 0$ of (1.1) is said to be

(i) stable in the Banach space E if, for any given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, for all $t \geq 1$, $\sum_{k=1}^{n} |b_k| \leq \delta$ implies that the solution $x(t) = x(t, b_1, ..., b_k)$ satisfies $\|x\| \leq \varepsilon$.

(ii) asymptotically stable if it is stable in the Banach space E and there exists a $\sigma > 0$ such that $\sum_{k=1}^{n} |b_k| \leq \sigma$ implies $\lim_{t\to\infty} \frac{(\log t)^{n-\alpha} x(t)}{1+(\log t)^{n+1}} = 0$.

Our first result is based on the Schauder fixed point theorem.

Theorem 3.2. *Let $f : [1,\infty) \times \mathbb{R} \to \mathbb{R}$ be a function such that $f \in C_{n-\alpha}([1,\infty))$ for any $x \in C_{n-\alpha}([1,\infty))$. Suppose that there exist two nonnegative continuous functions p and r defined on $[1,\infty)$ such that*

$$|f(t,x(t))| \leq p(t)\, r\left(\frac{|x(t)|}{1+(\log t)^{n+1}}\right), \tag{3.1}$$

and

$$\sup_{t\geq 1} \int_1^t \frac{(\log t)^{n-\alpha} \left(\log \frac{t}{s}\right)^{\alpha-1} (\log s)^{\alpha-n}}{1+(\log t)^{n+1}} p(s) \frac{ds}{s} < \Gamma(\alpha), \tag{3.2}$$

where p is bounded on $[1,\infty)$ and $r(t) \leq t$. Then, the zero solution $x = 0$ of (1.1) is asymptotically stable in the Banach space E.

Proof. For any given $\varepsilon > 0$, we will prove the existence of $\delta > 0$ such that $\sum_{k=1}^{n} |b_k| \leq \delta$ implies $\|x\| \leq \varepsilon$. We define a mapping F on E as follows

$$(Fx)(t) = \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)} (\log t)^{\alpha-k} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s,x(s)) \frac{ds}{s},\ t \geq 1. \tag{3.3}$$

Here, we divide the proof into three steps.

Step 1. We first prove that F maps E into itself.

We will prove $Fx \in C_{n-\alpha}([1,\infty))$ and $\lim_{t\to\infty} \frac{(\log t)^{n-\alpha}(Fx)(t)}{1+(\log t)^{n+1}} = 0$. Note that

$$\frac{(\log t)^{n-\alpha} (Fx)(t)}{1+(\log t)^{n+1}} = \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)} \frac{(\log t)^{n-k}}{1+(\log t)^{n+1}} + \frac{1}{\Gamma(\alpha)} \int_1^t \frac{(\log t)^{n-\alpha} \left(\log \frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}} f(s,x(s)) \frac{ds}{s}.$$

First, it is obvious that each term of the following formula

$$\sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)} \frac{(\log t)^{n-k}}{1+(\log t)^{n+1}}$$

is continuous on $[1,\infty)$. We only prove

$$\frac{1}{\Gamma(\alpha)}\int_1^t \frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}f(s,x(s))\frac{ds}{s}\in C([1,\infty)).$$

Note that $(\log t)^{n-\alpha}Fx$ is continuous at $t_0=1$. We consider the right-side continuity of

$$\frac{1}{\Gamma(\alpha)}\int_1^t \frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}f(s,x(s))\frac{ds}{s}$$

on the interval $(1,\infty)$. Let $t>t_0$, for the given ε. Then there exists $\delta_1=\min\{\delta_2,2\delta_3,\delta_4\}>0$ such that $0<t-t_0<\delta_1$. Our aim is to prove

$$\left|\int_1^t \frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}f(s,x(s))\frac{ds}{s}-\int_1^{t_0} \frac{(\log t_0)^{n-\alpha}\left(\log\frac{t_0}{s}\right)^{\alpha-1}}{1+(\log t_0)^{n+1}}f(s,x(s))\frac{ds}{s}\right|\leq\varepsilon.$$

Observe that

$$\begin{aligned}
&\left|\int_1^t \frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}f(s,x(s))\frac{ds}{s}-\int_1^{t_0} \frac{(\log t_0)^{n-\alpha}\left(\log\frac{t_0}{s}\right)^{\alpha-1}}{1+(\log t_0)^{n+1}}f(s,x(s))\frac{ds}{s}\right|\\
&\leq\left|\int_1^{t_0}\left(\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}-\frac{(\log t_0)^{n-\alpha}\left(\log\frac{t_0}{s}\right)^{\alpha-1}}{1+(\log t_0)^{n+1}}\right)f(s,x(s))\frac{ds}{s}\right|\\
&+\left|\int_{t_0}^t \frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}f(s,x(s))\frac{ds}{s}\right|\\
&\leq\|f(s,x(s))\|_{C_{n-\alpha}([1,t_0+\delta_1])}\\
&\times\left[\int_1^{t_0}\left(\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}-\frac{(\log t_0)^{n-\alpha}\left(\log\frac{t_0}{s}\right)^{\alpha-1}}{1+(\log t_0)^{n+1}}\right)(\log s)^{\alpha-n}\frac{ds}{s}\right.\\
&\left.+\int_{t_0}^t \frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}(\log s)^{\alpha-n}\frac{ds}{s}\right].
\end{aligned}$$

We analyze the two terms in brackets, respectively. Clearly, we have

$$\int_{t_0}^t \frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}(\log s)^{\alpha-n}\frac{ds}{s}\leq(\log t_0)^{\alpha-n}\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{t_0}\right)^{\alpha}}{\alpha\left(1+(\log t)^{n+1}\right)}$$

by integral. Thus, for the given ε, there exists $\delta_2>0$, such that, for $0<t-t_0<\delta_2$,

$$\int_{t_0}^t \frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}(\log s)^{\alpha-n}\frac{ds}{s}\leq\frac{\varepsilon}{2}. \tag{3.4}$$

Now we analyze the rest. It is easy to show

$$\int_1^t \frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}(\log s)^{\alpha-n}\frac{ds}{s}=\frac{(\log t)^{\alpha}}{1+(\log t)^{n+1}}B(\alpha-n+1,\alpha). \tag{3.5}$$

Thus, for the given ε, there exists a small enough $\delta_3 > 1$ such that, for $1 < t \leq \delta_3$,

$$\left| \int_1^t \frac{(\log t)^{n-\alpha} \left(\log \frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}} (\log s)^{\alpha-n} \frac{ds}{s} \right| \leq \frac{\varepsilon}{8}. \tag{3.6}$$

Letting

$$g(t) = \frac{(\log t)^{n-\alpha} \left(\log \frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}},$$

$t \geq s$, we have

$$\begin{aligned}
g'(t) = {} & \frac{1}{\left(1+(\log t)^{n+1}\right)^2} \left\{ \left[(n-\alpha) \frac{(\log t)^{n-\alpha-1}}{t} \left(\log \frac{t}{s}\right)^{\alpha-1} \right.\right. \\
& \left. + (\alpha-1) \frac{\left(\log \frac{t}{s}\right)^{\alpha-2}}{t} (\log t)^{n-\alpha} \right] \left(1+(\log t)^{n+1}\right) \\
& \left. - \frac{(n+1)(\log t)^n}{t} \left[(\log t)^{n-\alpha} \left(\log \frac{t}{s}\right)^{\alpha-1} \right] \right\} \\
= {} & \frac{(\log t)^{n-\alpha-1} \left(\log \frac{t}{s}\right)^{\alpha-2}}{t\left(1+(\log t)^{n+1}\right)^2} \left[(n-1)(\log t) - (n-\alpha)(\log s) \right. \\
& + (n-1)(\log t)^{n+2} - (n-\alpha)(\log s)(\log t)^{n+1} \\
& \left. - (n+1)(\log t)^{n+2} + (n+1)(\log s)(\log t)^{n+1} \right] \\
= {} & \frac{(\log t)^{n-\alpha-1} \left(\log \frac{t}{s}\right)^{\alpha-2}}{t\left(1+(\log t)^{n+1}\right)^2} \left\{ \left[(\alpha+1)(\log t)^{n+1} - (n-\alpha) \right] (\log s) \right. \\
& \left. + \left[(n-1) - 2(\log t)^{n+1} \right] (\log t) \right\} \\
\geq {} & \frac{(\log t)^{n-\alpha-1} \left(\log \frac{t}{s}\right)^{\alpha-2}}{t\left(1+(\log t)^{n+1}\right)^2} \left[(\alpha-1)(\log t)^{n+1} + (\alpha-1) \right] (\log s) \\
\geq {} & 0,
\end{aligned}$$

through calculation. Thus, g is a monotonous increasing function on $[1, \infty)$. We divide the interval $[1, t_0]$ of s into the small enough interval $[1, \delta_3]$ and the other interval $[\delta_3, t_0]$. From

(3.6), we get that

$$\begin{aligned}
&\left|\int_1^{\delta_3}\left(\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}-\frac{(\log t_0)^{n-\alpha}\left(\log\frac{t_0}{s}\right)^{\alpha-1}}{1+(\log t_0)^{n+1}}\right)(\log s)^{\alpha-n}\frac{ds}{s}\right| \\
&\leq\left|\int_1^{\delta_3}\frac{(\log t_0)^{n-\alpha}\left(\log\frac{t_0}{s}\right)^{\alpha-1}}{1+(\log t_0)^{n+1}}(\log s)^{\alpha-n}\frac{ds}{s}\right| \\
&+\left|\int_1^{\delta_3}\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}(\log s)^{\alpha-n}\frac{ds}{s}\right| \\
&\leq 2\left|\int_1^{\delta_3}\frac{(\log\delta_3)^{n-\alpha}\left(\log\frac{\delta_3}{s}\right)^{\alpha-1}}{1+(\log\delta_3)^{n+1}}(\log s)^{\alpha-n}\frac{ds}{s}\right| \\
&\leq\frac{\varepsilon}{4}. \qquad (3.7)
\end{aligned}$$

Meanwhile, we have

$$\begin{aligned}
&\left|\int_{\delta_3}^{t_0}\left(\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}-\frac{(\log t_0)^{n-\alpha}\left(\log\frac{t_0}{s}\right)^{\alpha-1}}{1+(\log t_0)^{n+1}}\right)(\log s)^{\alpha-n}\frac{ds}{s}\right| \\
&\leq(\log\delta_3)^{\alpha-n}\left|\int_{\delta_3}^{t_0}\left(\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}-\frac{(\log t_0)^{n-\alpha}\left(\log\frac{t_0}{s}\right)^{\alpha-1}}{1+(\log t_0)^{n+1}}\right)\frac{ds}{s}\right| \\
&\leq\frac{(\log\delta_3)^{\alpha-n}}{\alpha}\left(\left|\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{\delta_3}\right)^{\alpha}}{1+(\log t)^{n+1}}-\frac{(\log t_0)^{n-\alpha}\left(\log\frac{t_0}{\delta_3}\right)^{\alpha}}{1+(\log t_0)^{n+1}}\right|\right. \\
&\left.+\left|\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{t_0}\right)^{\alpha}}{1+(\log t)^{n+1}}\right|\right),
\end{aligned}$$

which implies that there exists a δ_4 such that, for $0<t-t_0<\delta_4$,

$$\int_{\delta_3}^{t_0}\left(\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}-\frac{(\log t_0)^{n-\alpha}\left(\log\frac{t_0}{s}\right)^{\alpha-1}}{1+(\log t_0)^{n+1}}\right)(\log s)^{\alpha-n}\frac{ds}{s}\leq\frac{\varepsilon}{4}. \qquad (3.8)$$

Then, (3.7) together with (3.8) leads to

$$\int_1^{t_0}\left(\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}-\frac{(\log t_0)^{n-\alpha}\left(\log\frac{t_0}{s}\right)^{\alpha-1}}{1+(\log t_0)^{n+1}}\right)(\log s)^{\alpha-n}\frac{ds}{s}\leq\frac{\varepsilon}{2}. \qquad (3.9)$$

Therefore, by (3.9) and (3.4), we complete the proof of the right-side continuity of

$$\frac{1}{\Gamma(\alpha)}\int_1^t\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}f(s,x(s))\frac{ds}{s},$$

i.e., $(\log t)^{n-\alpha} Fx$ is right-continuous on $(1,+\infty)$. Similar procedure can be applied on left-continuous discussion. Next, we only prove

$$\lim_{t\to\infty}\frac{(\log t)^{n-\alpha}(Fx)(t)}{1+(\log t)^{n+1}}=0.$$

By (3.1), we have

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\alpha)}\int_1^t \frac{(\log t)^{n-\alpha}(t-s)^{\alpha-1}}{1+(\log t)^{n+1}} f(s,x(s))\frac{ds}{s}\right| \\
&\le \left|\frac{1}{\Gamma(\alpha)}\int_1^t \frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}\left(p(s)\,r\left(\frac{|x(s)|}{1+(\log s)^{n+1}}\right)\right)\frac{ds}{s}\right| \\
&\le \frac{\|p\|_\infty}{\Gamma(\alpha)}\int_1^t \frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}(\log s)^{\alpha-n}\frac{(\log s)^{n-\alpha}x(s)}{1+(\log s)^{n+1}}\frac{ds}{s} \\
&\le \frac{\|x\|\,\|p\|_\infty}{\Gamma(\alpha)}\int_1^t \frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}(\log s)^{\alpha-n}\frac{ds}{s},
\end{aligned}$$

where $\|p\|_\infty=\sup_{t\ge 1}|p(t)|$. By (3.5), we obtain the result

$$\begin{aligned}
\left|\frac{(\log t)^{n-\alpha}(Fx)(t)}{1+(\log t)^{n+1}}\right| &\le \left|\sum_{k=1}^{n}\frac{b_k}{\Gamma(\alpha-k+1)}\frac{(\log t)^{\alpha-k}}{1+(\log t)^{n+1}}\right| \\
&\quad+\frac{\|x\|\,\|p\|_\infty B(\alpha-n+1,\alpha)}{\Gamma(\alpha)}\left|\frac{(\log t)^{\alpha}}{1+(\log t)^{n+1}}\right| \\
&\to 0
\end{aligned}\tag{3.10}$$

as $t\to\infty$.

Step 2. We prove that F is continuous.

According to the definition of the continuity for a mapping, we let $\{x_k\}\subset E$. For the given ε, there exists a $N>0$ such that, for any $k>N$, $\|x_k-x\|\le\varepsilon$. Our aim is to prove $\|Fx_k-Fx\|\le\varepsilon$. In view of $\lim_{k\to\infty}\|x_k-x\|=0$, there exists a $K>0$ such that $\|x_k\|\le K$ $(k=1,2,\ldots)$ and $\|x\|\le K$. By virtue of (3.5) and (3.1), for any k, we have

$$\begin{aligned}
&\|Fx_k-Fx\| \\
&=\sup_{t\ge 1}\left|\frac{(\log t)^{n-\alpha}Fx_k(t)}{1+(\log t)^{n+1}}-\frac{(\log t)^{n-\alpha}(Fx)(t)}{1+(\log t)^{n+1}}\right| \\
&=\sup_{t\ge 1}\left|\frac{1}{\Gamma(\alpha)}\int_1^t\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}\left[f(s,x_k(s))-f(s,x(s))\right]\frac{ds}{s}\right| \\
&\le \frac{2K\|p\|_\infty}{\Gamma(\alpha)}B(\alpha-n+1,\alpha)\left|\frac{(\log t)^{\alpha}}{1+(\log t)^{n+1}}\right|.
\end{aligned}\tag{3.11}$$

From the above formula, because the fact (3.11) tends to zero at infinity and continuous dependence at $t=1$, for the given ε, we find that there exists $T>1$ and $\delta_5>1$ such that

$$\left|\frac{1}{\Gamma(\alpha)}\int_1^t \frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}\left[f(s,x_k(s))-f(s,x(s))\right]\frac{ds}{s}\right|<\varepsilon,\ t\geq T, \tag{3.12}$$

and

$$\left|\frac{1}{\Gamma(\alpha)}\int_1^t \frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}\left[f(s,x_k(s))-f(s,x(s))\right]\frac{ds}{s}\right|<\varepsilon,\ 1\leq t\leq \delta_5. \tag{3.13}$$

Now we analyze the case when $t\in[\delta_5,T]$. For the above $k\geq N$,

$$\begin{aligned}&\left|\frac{1}{\Gamma(\alpha)}\int_1^t \frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}\left[f(s,x_k(s))-f(s,x(s))\right]\frac{ds}{s}\right|\\&\leq \sup_{t\in[1,T]}\left|f(s,x_k(s))-f(s,x(s))\right|\frac{1}{\Gamma(\alpha)}\sup_{t\in[\delta_5,T]}\int_1^t\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}\frac{ds}{s}\\&\leq \frac{1}{\alpha\Gamma(\alpha)}\sup_{t\in[1,T]}\left|f(s,x_k(s))-f(s,x(s))\right|.\end{aligned}$$

In view of $\lim_{k\to\infty}\|x_k-x\|=0$, we know

$$\lim_{k\to\infty}\sup_{s\in[1,T]}|x_k(s)-x(s)|=0$$

and f is uniformly continuous on any compact subsets. Thus, we have

$$\sup_{s\in[1,T]}\left|f(s,x_k(s))-f(s,x(s))\right|\to 0,\ k\to\infty.$$

Therefore, we have

$$\left|\frac{1}{\Gamma(\alpha)}\int_1^t \frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}\left[f(s,x_k(s))-f(s,x(s))\right]\frac{ds}{s}\right|<\varepsilon,\ \delta_5\leq t\leq T,$$

which together with (3.12) and (3.13) obtains $\|Fx_k-Fx\|\leq\varepsilon$, i.e., mapping F is continuous.

Step 3. We prove mapping F is compact.

Let $\Omega\subset E$ be a bounded set. For any $x\in\Omega$, there exists $M>0$ such that $\|x\|\leq M$. Now we only need to utilize Lemma 2.7 to prove that $F(\Omega)$ is a relatively compact set in E. In fact, it is easy to get

$$\sup_{t\geq 1}\left|\frac{(\log t)^{\beta}}{1+(\log t)^{n+1}}\right|\leq M_1$$

when $0\leq\beta\leq n$, which together with (3.10), yields that

$$\left\{(\log t)^{n-\alpha}(Fx)(t)/\left(1+(\log t)^{n+1}\right)\right\}$$

is uniformly bounded in E and equiconvergent at infinity. Finally, we verify

$$\left\{(\log t)^{n-\alpha}(Fx)(t)/\left(1+(\log t)^{n+1}\right)\right\}$$

is equicontinuous on $[1,\infty)$. For any $x \in \Omega$ and $t_1, t_2 \in [1,\infty)$, $t_1 < t_2$,

$$\begin{aligned}
&\left|\frac{(\log t_2)^{n-\alpha}(Fx)(t_2)}{1+(\log t_2)^{n+1}} - \frac{(\log t_1)^{n-\alpha}(Fx)(t_1)}{1+(\log t_1)^{n+1}}\right| \\
&\le \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)} \left|\frac{(\log t_2)^{n-k}}{1+(\log t_2)^{n+1}} - \frac{(\log t_1)^{n-k}}{1+(\log t_1)^{n+1}}\right| \\
&+ \frac{1}{\Gamma(\alpha)} \left| \int_1^{t_2} \frac{(\log t_2)^{n-\alpha}\left(\log \frac{t_2}{s}\right)^{\alpha-1}}{1+(\log t_2)^{n+1}} f(s,x(s)) \frac{ds}{s} \right| \\
&+ \frac{1}{\Gamma(\alpha)} \left| \int_1^{t_1} \frac{(\log t_1)^{n-\alpha}\left(\log \frac{t_1}{s}\right)^{\alpha-1}}{1+(\log t_1)^{n+1}} f(s,x(s)) \frac{ds}{s} \right| \\
&\le \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)} \left|\frac{(\log t_2)^{n-k}}{1+(\log t_2)^{n+1}} - \frac{(\log t_1)^{n-k}}{1+(\log t_1)^{n+1}}\right| \\
&+ \|f(s,x(s))\|_{C_{n-\alpha}([1,t_2])} \frac{1}{\Gamma(\alpha)} \left| \int_1^{t_2} \frac{(\log t_2)^{n-\alpha}\left(\log \frac{t_2}{s}\right)^{\alpha-1}}{1+(\log t_2)^{n+1}} (\log s)^{\alpha-n} \frac{ds}{s} \right| \\
&+ \|f(s,x(s))\|_{C_{n-\alpha}([1,t_1])} \frac{1}{\Gamma(\alpha)} \left| \int_1^{t_1} \frac{(\log t_1)^{n-\alpha}\left(\log \frac{t_1}{s}\right)^{\alpha-1}}{1+(\log t_1)^{n+1}} (\log s)^{\alpha-n} \frac{ds}{s} \right| \\
&\le \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)} \left|\frac{(\log t_2)^{n-k}}{1+(\log t_2)^{n+1}} - \frac{(\log t_1)^{n-k}}{1+(\log t_1)^{n+1}}\right| \\
&+ \|f(s,x(s))\|_{C_{n-\alpha}([1,t_2])} \frac{B(\alpha-n+1,\alpha)}{\Gamma(\alpha)} \left|\frac{(\log t_2)^{\alpha}}{1+(\log t_2)^{n+1}} - \frac{(\log t_1)^{\alpha}}{1+(\log t_1)^{n+1}}\right| \\
&\to 0,
\end{aligned}$$

as $t_1 \to t_2$, we could know

$$\left\{ (\log t)^{n-\alpha} (Fx)(t) / \left(1+(\log t)^{n+1}\right) \right\}$$

is equicontinuous on $[1,\infty)$. Thus, mapping F is compact.

Clearly, all the hypotheses of the Schauder fixed point theorem are satisfied. Hence, F has a fixed point $x \in E$.

Now, we consider the stability of (1.1). For any given $\varepsilon > 0$, there exists

$$\delta = \frac{1}{\sum\limits_{k=1}^{n} \frac{1}{\Gamma(\alpha-k+1)}} \left[1 - \frac{M_1 \|P\|_\infty B(\alpha-n+1,\alpha)}{\Gamma(\alpha)}\right] \varepsilon > 0,$$

such that $\sum_{k=1}^{n}|b_k|<\delta$, which implies

$$\begin{aligned}
\|x\| &= \sup_{t\geq 1}\left|\frac{(\log t)^{n-\alpha}(Fx)(t)}{1+(\log t)^{n+1}}\right| \\
&\leq \sup_{t\geq 1}\left|\sum_{k=1}^{n}\frac{b_k}{\Gamma(\alpha-k+1)}\frac{(\log t)^{n-k}}{1+(\log t)^{n+1}}\right| \\
&+\|x\|\sup_{t\geq 1}\left|\frac{1}{\Gamma(\alpha)}\int_1^t\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{\alpha-n}}{1+(\log t)^{n+1}}p(s)\frac{ds}{s}\right| \\
&\leq \sup_{t\geq 1}\left|\sum_{k=1}^{n}\frac{b_k}{\Gamma(\alpha-k+1)}\frac{(\log t)^{n-k}}{1+(\log t)^{n+1}}\right| \\
&+\sup_{t\geq 1}\frac{\|x\|.\|P\|_{\infty}B(\alpha-n+1,\alpha)}{\Gamma(\alpha)}\left|\frac{(\log t)^{\alpha}}{1+(\log t)^{n+1}}\right| \\
&\leq \sum_{k=1}^{n}\frac{1}{\Gamma(\alpha-k+1)}M_1\delta+\frac{M_1\|P\|_{\infty}B(\alpha-n+1,\alpha)}{\Gamma(\alpha)} \\
&\leq \varepsilon.
\end{aligned}$$

Then, $x=0$ is stable in the Banach space E. Also, the expression

$$\lim_{t\to\infty}\frac{(\log t)^{n-\alpha}x(t)}{1+(\log t)^{n+1}}=0,$$

in the definition of E means that $x=0$ is asymptotically stable in E. □

Our second result is based on the Banach fixed point theorem.

Theorem 3.3. *Let $f:[1,\infty)\times\mathbb{R}\to\mathbb{R}$ be a function such that $f\in C_{n-\alpha}([1,\infty))$ for any $x\in C_{n-\alpha}([1,\infty))$, $f(t,0)=0$. Suppose that the following requirements are satisfied*

(i) $|f(t,x)-f(t,y)|\leq\frac{p(s)}{1+(\log t)^{n+1}}|x-y|$, $t\geq 1$, $x,y\in\mathbb{R}$,

(ii) $\sup_{t\geq 1}\int_1^t\frac{(\log t)^{n-\alpha}(t-s)^{\alpha-1}(\log s)^{\alpha-n}}{1+(\log t)^{n+1}}p(s)\frac{ds}{s}<\Gamma(\alpha)$ *and*

$$\lim_{t\to\infty}\sup_{t\geq 1}\int_1^t\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{\alpha-n}}{1+(\log t)^{n+1}}p(s)\frac{ds}{s}=0.$$

Then the zero solution $x=0$ of (1.1) is asymptotically stable in the Banach space E.

Proof. By a similar argument to the proof of Theorem 3.2, we can get $Fx \in C_{n-\alpha}([1,\infty))$. Also, we have

$$\begin{aligned}
&\left| \frac{1}{\Gamma(\alpha)} \int_1^t \frac{(\log t)^{n-\alpha} \left(\log \frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}} f(s,x(s)) \frac{ds}{s} \right| \\
&\le \frac{1}{\Gamma(\alpha)} \sup_{t\ge 1} \int_1^t \frac{(\log t)^{n-\alpha} \left(\log \frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}} \left| f(s,x(s)) - f(s,0) \right| \frac{ds}{s} \\
&+ \sup_{t\ge 1} \frac{1}{\Gamma(\alpha)} \int_1^t \frac{(\log t)^{n-\alpha} \left(\log \frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}} \left| f(s,0) \right| \frac{ds}{s} \\
&\le \frac{\|x\|}{\Gamma(\alpha)} \sup_{t\ge 1} \int_1^t \frac{(\log t)^{n-\alpha} \left(\log \frac{t}{s}\right)^{\alpha-1} (\log s)^{\alpha-n}}{1+(\log t)^{n+1}} p(s) \frac{ds}{s} \\
&\to 0, \ (t\to\infty).
\end{aligned}$$

Then, F maps E into itself. We will use the Banach fixed point to prove the stability of (1.1). In fact,

$$\begin{aligned}
&\left| \frac{(\log t)^{n-\alpha} (Fx)(t)}{1+(\log t)^{n+1}} - \frac{(\log t)^{n-\alpha} (Fy)(t)}{1+(\log t)^{n+1}} \right| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(\log t)^{n-\alpha} \left(\log \frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}} \left| f(s,x(s)) - f(s,y(s)) \right| \frac{ds}{s} \\
&\le \|x-y\| \frac{1}{\Gamma(\alpha)} \sup_{t\ge 1} \int_1^t \frac{(\log t)^{n-\alpha} \left(\log \frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}} (\log s)^{\alpha-n} p(s) \frac{ds}{s} \\
&\le \lambda \|x-y\|,
\end{aligned}$$

where

$$\begin{aligned}
\lambda &= \frac{1}{\Gamma(\alpha)} \sup_{t\ge 1} \int_1^t \frac{(\log t)^{n-\alpha} \left(\log \frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}} (\log s)^{\alpha-n} p(s) \frac{ds}{s} \\
&< 1.
\end{aligned}$$

Thus, F is a contraction mapping and F has a fixed point $x \in E$. Note that, for any given $\varepsilon > 0$, there exists a

$$\delta = \frac{1-\lambda}{M_1 \sum\limits_{k=1}^{n} \frac{1}{\Gamma(\alpha-k+1)}} \varepsilon > 0$$

such that $\sum\limits_{k=1}^{n}|b_k|<\delta$ implies

$$\begin{aligned}\|x\| &= \sup_{t\geq 1}\left|\frac{(\log t)^{n-\alpha}(Fx)(t)}{1+(\log t)^{n+1}}\right| \\ &\leq \sup_{t\geq 1}\left|\sum_{k=1}^{n}\frac{b_k}{\Gamma(\alpha-k+1)}\frac{(\log t)^{n-k}}{1+(\log t)^{n+1}}\right| \\ &+\sup_{t\geq 1}\frac{1}{\Gamma(\alpha)}\int_1^t\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}\left|f(s,x(s))-f(s,0)\right|\frac{ds}{s} \\ &+\sup_{t\geq 1}\frac{1}{\Gamma(\alpha)}\int_1^t\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}}{1+(\log t)^{n+1}}\left|f(s,0)\right|\frac{ds}{s} \\ &\leq \sup_{t\geq 1}\left|\sum_{k=1}^{n}\frac{b_k}{\Gamma(\alpha-k+1)}\frac{(\log t)^{n-k}}{1+(\log t)^{n+1}}\right| \\ &+\|x\|\sup_{t\geq 1}\frac{1}{\Gamma(\alpha)}\int_1^t\frac{(\log t)^{n-\alpha}\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{\alpha-n}}{1+(\log t)^{n+1}}p(s)\frac{ds}{s} \\ &\leq \varepsilon.\end{aligned}$$

Therefore, the zero solution $x=0$ of (1.1) is stable in the Banach space E. Moreover, the expression

$$\lim_{t\to\infty}\frac{(\log t)^{n-\alpha}x(t)}{1+(\log t)^{n+1}}=0,$$

in the definition of E implies that $x=0$ is asymptotically stable in the Banach space E. □

Example 3.4. Consider the following nonlinear Hadamard fractional differential equation

$$\begin{cases}\mathfrak{D}_1^{\frac{7}{3}}x(t)=\frac{3}{4}\left(\frac{1}{1+(\log t)^6}\right)^{\frac{2}{3}}\sin\left(\frac{x(t)}{1+(\log t)^4}\right)^{\frac{3}{5}},\ t\geq 1, \\ \mathfrak{D}_1^{\frac{7}{3}-1}x(1)=\mathfrak{D}_1^{\frac{7}{3}-2}x(1)=\mathfrak{D}_1^{\frac{7}{3}-3}x(1)=1,\end{cases} \tag{3.14}$$

where $\alpha=\frac{7}{3}$. Assume that $p(t)=\frac{3}{4}\left(\frac{1}{1+(\log t)^6}\right)^{\frac{2}{3}}$, $r(t)=\sin t^{\frac{3}{5}}$. Then

$$|f(t,x(t))|\leq p(t)\,r\left(\frac{|x(t)|}{1+(\log t)^4}\right), \tag{3.15}$$

and

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \sup_{t \geq 1} \int_1^t \frac{(\log t)^{3-\alpha} \left(\log \frac{t}{s}\right)^{\alpha-1} (\log s)^{\alpha-3}}{1+(\log t)^4} \frac{3}{4} \left(\frac{1}{1+(\log s)^6}\right)^{\frac{2}{3}} \frac{ds}{s} \\
&\leq \frac{3}{4\Gamma(\alpha)} \sup_{t \geq 1} \int_1^t \frac{(\log t)^{3-\alpha} \left(\log \frac{t}{s}\right)^{\alpha-1} (\log s)^{\alpha-3}}{1+(\log t)^4} \frac{ds}{s} \\
&\leq \frac{3}{4\Gamma(\alpha)} \sup_{t \geq 1} \frac{(\log t)^{\alpha}}{1+(\log t)^4} B(\alpha-2, \alpha) \\
&= \frac{3\Gamma(\alpha-2)}{8\Gamma(2\alpha-2)} < 1. \qquad (3.16)
\end{aligned}
$$

Note that (3.15) and (3.16) satisfy Theorem 3.2. So, the zero solution of (3.14) is asymptotically stable in the Banach space E.

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