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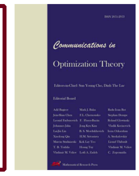

EXISTENCE RESULTS ON GENERALIZED STRONG OPERATOR EQUILIBRIUM PROBLEMS IN HAUSDORFF TVS

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Abstract. In this paper, a generalized strong operator equilibrium problem is considered in Hausdorff topological vector spaces. An existence result for solutions of the generalized strong operator equilibrium problem is obtained in the compact and noncomapct setting by employing the Minty type lemma, the KKM theorem, and the coercive condition.

Keywords. Equilibrium problem; Hemicontinuous; KKM mapping; Pseudomonotone.

1. Introduction

The minimax inequalities of Fan are fundamental in nonlinear analysis and optimization. Their equivalence with equilibrium problems was introduced by Takashashi [1], Blum and Oettli [2], and Noor and Oettli [3]. The terminology of "Equilibrium Problems" was first coined by Blum and Oettli [2], and Noor and Oettli [3]. It is a generalization of many nonlinear problems, such as, variational inequalities, complementary problems, fixed point problems, saddle point problems, and the Nash equilibria; see, e.g., [1, 2, 3, 4] and the references therein.

It is known that equilibrium problems have many important real applications in industry, finance, transportation, and machine learning, and the solutions (existence, solution methods, and stability) of the equilibrium problems have been studied many authors; see, e.g., [5, 6, 7, 8, 9] and the references therein. It also deserves mentioning that equilibrium problems have been generalized to vector valued cases recently; see, e.g., [4, 5, 6, 7, 8] and the references therein. In the literatures, only the weak version of equilibrium problem was investigated. For the results on the strong version of equilibrium problems, we refer to [10, 11, 12, 13, 14, 15, 16, 17, 18] and the references therein.

The study of vector variational inequalities was initiated by Giannessi [19] in 1980 with applications in finite dimensional Euclidean spaces. Since then, it has been extended and generalized in various directions due to its wide applications in pure and applied mathematics. In 2002, Domokos and Kolumban [20] gave a fascinating analysis of variational inequalities and

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vector variational inequalities in terms of variational inequalities with operator solutions in the setting of Banach spaces. They considered the variational inequalities with operator solutions to provide unified approaches to various kinds of variational inequalities and vector variational inequalities in Banach spaces, and effectively depicted these problems in the framework of the variational inequalities with operator solutions. The effectiveness of the variational inequalities with operator solutions can serve as a source of inspiration for various vector optimization problems in abstract spaces.

In 2005, Kazmi and Raouf [21] introduced the operator equilibrium problems and established existence results for the problem with single-valued mappings by employing KKM mappings. In the same year, Kum and Kim [22] generalized the operator equilibrium problem, considered by Kazmi and Raouf [21], with set-valued quasi-equilibrium problems, and obtained an existence result by employing a fixed-point theorem.

In this paper, motivated and inspired by the results presented in Kazmi and Khan[13], Kazmi and Raouf [21], Kum and Kim [22], Ansari et al. [23], we consider a generalized strong operator equilibrium problem, which is an extension of the problems considered in [13, 15, 24, 25]. Using the KKM-Fan lemma [26] and the coercive condition, we prove an existence result for the generalized strong operator equilibrium problem in compact/noncompact settings. The results presented in this paper mainly improve and extend the results in [11, 12, 13, 16, 22, 27, 28].

Next, we list some facts on the framework. Let X, Y be two Hausdorff topological vector spaces, and let $K \subseteq L(X,Y)$. We denote by $L(X,Y)$ the space of all continuous operators from X to Y, by $2^{L(X,Y)}$ the family of all subsets of $L(X,Y)$, by $int_X K$ the interior of K in X, and by $co(K)$ the convex hull of K. Let Y be an ordered Hausdorff topological vector space, and let $C : K \to 2^Y$ be a multifunction such that, for each $f \in K$, $C(f)$ is a closed convex cone with $int_Y C(f) \neq \phi$. It is clear that, for each $f \in K$, the cone $C(f)$ can define on Y a partial order $\leq_{C(f)}$ by $g \leq_{C(f)} h$ if and only if $h - g \in C(f)$. We can also write $g <_{C(f)} h \Leftrightarrow h - g \in int_Y C(f)$ in the case that $int_Y C(f) \neq \phi$. Let $P := \bigcap_{f \in K} C(f)$.

Let $F : K \times K \to 2^Y$ be multifunction with $\{0\} \subseteq F(f,f), \forall f \in K$. We consider the following *generalized strong operator equilibrium problem* (GSOEP):

Find $f_0 \in K$ such that

$$F(f_0, g) \not\subseteq -C(f_0) \setminus \{0\}, \text{ for all } g \in K. \tag{1.1}$$

Special cases:

(i) If F is a single-valued mapping, then problem (1.1) reduces to a strong operator equilibrium problem (SOEP), which is the problem of finding $f_0 \in K$ such that

$$F(f_0, g) \notin -C(f_0) \setminus \{0\}, \text{ for all } g \in K. \tag{1.2}$$

(ii) If $K \subset X$ and $C(x) := C$, for all $x \in K$, then problem (1.1) reduces to the problem of finding $x_0 \in K$ such that

$$F(x_0, y) \not\subseteq -C \setminus \{0\}, \text{ for all } y \in K. \tag{1.3}$$

Problem (1.3) is called the multivalued generalized system (MGS), studied by Kum and Wong [15]. It is also known as the generalized strong vector equilibrium problem (GSVEP).

(iii) If F is a single-valued mapping, $K \subset X$, and $C(f) := C$, then problem (1.1) reduces to the problem of finding $x_0 \in K$ such that

$$F(x_0, y) \notin -C \setminus \{0\}, \text{for all } y \in K, \tag{1.4}$$

Problem (1.4) is called the strong vector equilibrium problem (SVEP), studied by Ansari et al. [23] and Kazmi and Khan[13].

(iv) If $F(f,g) = \langle T(f), f-g \rangle$, where $K \subseteq X$, $T : K \to L(X,Y)$ is a nonlinear mapping, and $C(f) := C$, then problem (1.1) reduces to the following strong vector variational inequality (SVVI):

$$\text{Find } f \in K : \langle T(f), g-f \rangle \notin -C \setminus \{0\}, \text{for all } g \in K,$$

which was studied by Fang and Huang [11].

The main objective of this paper is to present existence results of solutions for generalized strong operator equilibrium problems in Hausdorff topological vector spaces under different conditions. The rest of the paper is organized as follows. In Section 2, we first give some important definitions and KKM Fan lemma. These tools are essential for our main results in the sequel. Section 3 is devoted to the main results of this paper, namely, the existence results for the generalized strong operator equilibrium problem under compact/noncompact settings. Some sub-results are also presented as corollaries to justify the existence results for the generalized strong operator equilibrium problems.

2. Preliminaries

In this section, we recall some definitions and results, which are needed in the sequel to obtain the main results of this paper.

Definition 2.1. Let X be a Hausdorff topological vector space, and let K be a nonempty convex subset of X. Let Y be another Hausdorff topological vector space. A nonempty subset P of Y is said to be a *convex cone* if

$$\lambda P \subseteq P, \text{ for all } \lambda > 0 \text{ and } P + P = P.$$

Definition 2.2. Let X, Y be nonempty topological spaces, and let $T : X \to 2^Y$ be a multifunction. Then a multifunction $T : X \to 2^Y$ is said to be *upper semi-continuous* on X if, for each $x \in X$ and each open set V in Y containing $T(x)$, there exists an open neighborhood U of x in X such that $T(y) \subseteq V, \forall y \in U$. A multifunction $T : X \to 2^Y$ is said to be *lower semi-continuous* on X if, for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \phi$, there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \phi$, for each $y \in U$. T is said to be *continuous* if it is both *upper semi-continuous* and *lower semi-continuous*. It is also known that T is *lower semi-continuous* if and only if, for each open set V in Y, the set $\{x \in X : T(x) \subset V\}$ is closed in X.

Definition 2.3. Let $T : X \to 2^Y$ be a multifunction. The graph of T is denoted by $G(T)$ and is defined by

$$G(T) = \{(x,y) \in X \times Y : x \in X, y \in T(x)\}.$$

Then inverse of T, denoted by T^{-1} is a multifunction $T^{-1} : R(T) \to X$ and is defined by

$$x \in T^{-1}(y) \text{ if and only if } y \in T(x).$$

Definition 2.4. Let X and Y be topological vector spaces. Let $K \subset L(X,Y)$ be a convex subset, and let $P \subset Y$ be a convex cone. Let $F : K \times K \to 2^Y$, and let $T : K \to 2^Y$ be two multifunctions. Then

(i) T is said to be *P-convex* if, for all $f, g \in K$ and $\lambda \in [0,1]$,

$$T(\lambda f + (1-\lambda)g)) \subset \lambda T(f) + (1-\lambda)T(g) - P.$$

(ii) F is said to be P-monotone if, for all $f, g \in K$,

$$F(f,g) + F(g,f) \subseteq -P.$$

Definition 2.5. Let K be a nonempty convex subset of a vector space X. A multifunction $F : K \to 2^X$ is called a *KKM-mapping* if, for each finite subset $\{x_1, x_2 \cdots, x_n\} \subset K$,

$$co\{x_1, x_2 \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i),$$

where $co\{x_1, x_2 \cdots, x_n\}$ denotes convex hull of a set $\{x_1, x_2 \cdots, x_n\}$.

Lemma 2.1. [29] *(Fan-KKM Lemma) Let X be a Hausdorff topological vector space, and let D be a non empty convex subset of X. Let* $F : D \to 2^X$ *be a KKM -mapping. If each* $F(x)$ *is closed and at least one* $F(x)$ *is compact, then* $\bigcap_{x \in D} F(x) \neq \phi$.

Definition 2.6. Let $K \subset X$ and $T : K \to L(X,Y)$ be nonlinear mappings. Then T is said to be:

(i) *hemicontinuous* if, for any given $x, y, z \in K$ and for $\lambda \in [0,1]$, the mapping

$$\lambda \to \langle T(x + \lambda(y-x), z)\rangle$$

is continuous at 0^+.

(ii) *$C-$strongly pseudomonotone* if, for any $x, y \in K$,

$$\langle T(x), y-x)\rangle \notin -C \setminus \{0\} \text{ implies } \langle T(y), x-y)\rangle \in -C.$$

Definition 2.7. Let X and Y be Hausdorff topological vector spaces. Let $K \subseteq L(X,Y)$ be a nonempty convex, and let $C(f)$ be a pointed closed convex cone in Y with $int_Y C(f) \neq \phi$, for all $f \in K$.

(i) A multifunction $F : K \times K \to 2^Y$ is said to be *C(f)-strongly pseudomonotone* if it satisfies

$$F(f,g) \nsubseteq -C(f) \setminus \{0\} \Rightarrow F(g,f) \subseteq -C(f), \text{ for all} f, g \in K.$$

(ii) A multifunction $G : K \to Y$ is said to be *C(f)-convex* if, for all $f, g \in K$ and for all $\lambda \in [0,1]$,

$$G(\lambda g + (1-\lambda)h) \subseteq \lambda G(g) + (1-\lambda)G(h) - C(f).$$

(iii) the mapping G is said to be *generalized hemicontinuous* (for short, g.h.c.) if, for all $g, h \in K$ and for all $\lambda \in [0,1]$, $\lambda \to G(g + \lambda(h-g))$ is upper semicontinuous at 0^+.

Remark 2.1. If $K \subset X$ and $C(f) := C$, then the definition of $C(f)$-strongly pseudomonotonicity reduces to that obtained in [15]. For $K \subset X$ and $F : K \times K \to Y$, $C(x) := C$ for all $x \in K$, $C(f)$-strongly pseudomonotonicity reduces to the definition 2.1 in [13]. Moreover, if $F(x,y) = \langle T(x), y-x\rangle$, where $T : K \to 2^{L(X,Y)}$ is a nonlinear mapping, and $C(x) = C$, for all $x \in K$, then the above definition of $C(f)$-strongly pseudomonotonicity reduces to that obtained in [15].

Definition 2.8. Let X and Y be two Hausdorff topological vector spaces and let $K \subset L(X,Y)$ be a closed convex set and K_0 is a compact subset of K. Then a multifunction $F : K \times K \to 2^Y$ is said to satisfying coercive condition on K_0, if there exists $g_0 \in K_0$ such that

$$F(g_0, f) \subset -C(f) \setminus \{0\}, \quad \forall f \in K \setminus K_0.$$

3. Main Results

First, we prove the Minty type lemma for problem (1.1) to obtain the existence results in both compact and noncompact settings.

Lemma 3.1. *Let K be a nonempty closed and convex subset of $L(X,Y)$. Let the multifunction $F : K \times K \to 2^Y$ be generalized hemicontinuous in the first argument, P-convex in second argument, and $C(f)$-strongly pseudomonotone. Then the following statements are equivalent:*

(i) *There exists $f_0 \in K$ such that $F(f_0, g) \nsubseteq -C(f_0) \setminus \{0\}$, for all $g \in K$.*
(ii) *The exists $f_0 \in K$ such that $F(g, f_0) \subseteq -C(f_0)$, for all $g \in K$.*

Proof. $(i) \Rightarrow (ii)$. It is a immediate consequence of $C(f)$-strongly pseudomonotonicity of F.

$(ii) \Rightarrow (ii)$. Suppose that (ii) is true. Then there exists $f_0 \in K$ such that

$$F(g, f_0) \subseteq -C(f_0), \quad \forall g \in K.$$

Let $f_\lambda := (1-\lambda) f_0 + \lambda g$, $\forall \lambda \in [0,1]$. Then $f_\lambda \in K$ and hence

$$F(f_\lambda, f_0) \subseteq -C(f_0).$$

Since F is P-convex in second argument, we have

$$\begin{aligned} 0 \in & F(f_\lambda, f_\lambda) \subseteq (1-\lambda) F(f_\lambda, f_0) + \lambda F(f_\lambda, g) - P \\ & \subseteq \lambda F(f_\lambda, g) - (1-\lambda) C(f_0) - C(f_0) \\ & \subseteq \lambda F(f_\lambda, g) - \lambda C(f_0), \end{aligned} \tag{3.1}$$

which implies $F(f_\lambda, g) \subseteq C(f_0)$. Since F is generalized hemicontinuous in first argument and $f_\lambda \to f_0$ w.r.to pointwise convergence, it follows from (3.1) that

$$F(f_0, g) \subseteq C(f_0), \quad \forall g \in K.$$

Hence there exists $f_0 \in K$ such that

$$F(f_0, g) \nsubseteq -C(f_0) \setminus \{0\}, \quad \forall g \in K.$$

This complete the proof. □

Next, we give the existence of solutions of problem (1.1) in a compact setting.

Theorem 3.1. *Let $K \subseteq L(X,Y)$ be a nonempty convex compact set. Let the multifunction $F : K \times K \to 2^Y$ be generalized hemicontinuous in first argument, P-convex, and lower semicontinuous in second argument. Assume that F is C(f)-strongly pseudomonotone. Then there exist $f_0 \in K$ such that*

$$F(f_0, g) \nsubseteq -C(f_0) \setminus \{0\}, \quad \forall g \in K. \tag{3.2}$$

Proof. Let us define a multifunction $\mathscr{Q} : K \to 2^K$ by

$$\mathscr{Q}(g) := \{f \in K : F(f,g) \nsubseteq -C(f) \setminus \{0\}\}, \quad \forall g \in K.$$

We first show that F is a KKM-mapping. Let us suppose that there exists a finite set

$$\{g_1, g_2, \cdots, g_n\} \subset K,$$

and $t_i \geq 0$ with $\sum_{i=1}^n t_i = 1$ such that

$$co\{g_1, g_2, \cdots, g_n\} \nsubseteq \bigcup_{i=1}^n \mathscr{Q}(g_i).$$

Then, there exists $f = \sum_{i=1}^n t_i g_i \in co\{g_1, g_2, \cdots, g_n\}$ and $f \notin \bigcup_{i=1}^n \mathscr{Q}(g_i)$. Observe that $f \notin \bigcup_{i=1}^n \mathscr{Q}(g_i)$, which means that $f \notin \mathscr{Q}(g_i)$, for all $i = 1, 2, \cdots, n$. Hence $F(f, g_i) \subseteq -C(f) \setminus \{0\}$, for all $i = 1, 2, \cdots, n$. Observe that

$$\begin{aligned} 0 \in F(f,f) = F(f, \sum_{i=1}^n t_i g_i) &\subseteq \sum_{i=1}^n F(f, g_i) - P \\ &\subseteq -C(f) \setminus \{0\} - C(f) \\ &\subseteq -C(f) \setminus \{0\}, \end{aligned}$$

which is a contradiction. Thus, $\mathscr{Q}$ is a KKM-mapping.

Define a multifunction $\mathscr{X} : K \to 2^K$ as

$$\mathscr{X}(g) := \{f \in K : F(g,f) \subseteq -C(f)\}, \quad \forall g \in K.$$

We claim that $\mathscr{X}(g)$ is closed, for all $g \in K$. To this end, let $\{f_j\}_{j=1}^\infty$ be a net in $\mathscr{X}(g)$ such that $f_j \to f_0$ w.r.t point-wise convergence as $j \to \infty$. We need to show that $f_0 \in \mathscr{X}$, for all $g \in K$. Let us assume that $f_0 \notin \mathscr{X}(g)$. Then $F(g, f_0) \nsubseteq -C(f_0)$ or $F(g, f_0) \cap (-C(f_0))^c \neq \phi$. Since F is l.s.c. in second argument and $(-C(f_0))^c$ is open with $F(g, f_0) \cap (-C(f_0))^c \neq \phi$, we see that there exists an open neighborhood U of f_0 such that

$$F(g,f) \cap (-C(f_0))^c \neq \phi, \quad \forall f \in U.$$

Since U is a neighborhood of f_0, we conclude that there exists some positive integer, say λ, such that $f_i \in U$, for all $i \geq \lambda$. This implies that $F(g, f_i) \cap (-C(f_0))^c \neq \phi$, which is contradiction to the fact that $f_j \in \mathscr{X}(g)$, for all $g \in K$. Thus $\mathscr{X}(g)$ is closed, for all $g \in K$.

Since F is $C(f)$-strongly pseudomonotone, it follows that $\mathscr{Q}(f) \subseteq \mathscr{X}(f)$, for all $f \in K$. Also $\mathscr{Q}(f)$ is a KKM-mapping, and $\mathscr{Q}(f) \subseteq \mathscr{X}(f)$, for all $f \in K$. Therefore, $\mathscr{X}(f)$ is also a KKM-mapping. Since $\mathscr{X}(f) \subseteq K$ and $\mathscr{X}(g)$ is closed, it follows that $\mathscr{X}(f)$ is compact, for all $f \in K$. Thus, by Ky-Fan Lemma 2.1, we have that $\bigcap_{f \in K} \mathscr{X}(f) \neq \phi$. In view of Lemma 3.1, we have $\bigcap_{f \in K} \mathscr{X}(f) = \bigcap_{f \in K} \mathscr{Q}(f)$. Hence, $\bigcap_{f \in K} \mathscr{Q}(f) \neq \phi$. Thus, there exists $f_0 \in K$ such that

$$F(f_0, g) \subseteq -C(f_0) \setminus \{0\}, \quad \forall g \in K.$$

This completes the proof. □

Let $F(f,g) := \langle T(f), g - f \rangle$ for all $f, g \in K$, where $T : K \to 2^{L(X,Y)}$. As a consequences of the above theorem, we have following corollary.

Corollary 3.1. *Let $K \subset L(X,Y)$ be a nonempty convex compact set. Let $T : K \to L(X,Y)$ be a C(f)-strongly pseudomonotone and hemicontinuous mapping with nonempty compact values where $L(X,Y)$ is equipped with topology of point-wise convergence. Then the following strong operator variational inequality has a solution $f_0 \in K$ such that*

$$\langle T(f_0), g - f_0 \rangle \notin -C \setminus \{0\}, \quad \forall g \in K.$$

Next, we prove the existence of solutions of problem (1.1) in the noncompact setting by using the coercive condition.

Theorem 3.2. *Let $K \subseteq L(X,Y)$ be a nonempty closed and convex set. Let $F : K \times K \to 2^Y$ be $C(f)$-strongly pseudomonotone, generalized hemicontinuous in first argument, P-convex, and lower semicontinuous in second argument, and satisfy the coercive condition on compact subset K_0 of K. Then there exists $f_0 \in K$ such that*

$$F(f_0, g) \nsubseteq -C(f_0) \setminus \{0\}, \quad \forall g \in K. \tag{3.3}$$

Proof. Define the multifunction $\mathscr{Q} : K \to 2^K$ by

$$\mathscr{Q}(g) := \{f \in K : F(f,g) \nsubseteq -C(f) \setminus \{0\}\}, \quad \forall g \in K.$$

We first shows that $\mathscr{Q}$ is a KKM-mapping. For this, let us assume that there exists $\{f_1, f_2, \cdots, f_n\} \subset K$ and $t_i \geq 0, 1 \leq i \leq n$ with $\sum_{i=1}^n t_i = 1$ such that

$$co\{f_1, f_2, .., f_n\} \nsubseteq \bigcup_{i=1}^n \mathscr{Q}(f_i).$$

Then, there exists $g = \sum_{i=1}^n t_i f_i \in co\{f_1, f_2, \cdots f_n\}$ such that $g \notin \bigcup_{i=1}^n \mathscr{Q}(f_i)$. Since $g \notin \cup_{i=1}^n Q(f_i)$. which implies that $g \notin \mathscr{Q}(f_i)$, for all$i = 1, 2, \cdots, n$. Hence, $F(g, f_i) \subseteq -C(f_i) \setminus \{0\}$, for all $i = 1, 2, \cdots, n$. On the other hand, since F is P-convex in the second argument, one concludes that

$$\begin{aligned} 0 \in F(g,g) &= F(g, \sum_{i=1}^n t_i f_i) \\ &\subseteq \sum_{i=1}^n t_i F(g, f_i) - C(f_i) \\ &\subseteq -C(f_i) \setminus \{0\} - C(f_i) \subseteq -C(f_i) \setminus \{0\}, \end{aligned}$$

which is a contradiction. Thus $\mathscr{Q}$ is a KKM-mapping.

Define a multifunction $\mathscr{Z} : K \to 2^K$ as

$$\mathscr{Z}(g) := \{f \in K : F(g,f) \subseteq -C(f)\}, \quad \forall g \in K.$$

As F is a $C(f)$-strongly pseudomonotone mapping, it follows that $\mathscr{Q}(f) \subseteq \mathscr{Z}(f)$. Also, $\mathscr{Q}$ is a KKM mapping and $\mathscr{Q}(f) \subseteq \mathscr{Z}(f)$. Hence, $\mathscr{Z}(f)$ is a KKM-mapping. Since F is lower semicontinuous in the second argument and $-C(f)$ is closed set, we have that $\mathscr{Z}(f)$ is closed for all $f \in K$. Since F is coercive on compact subset $K_0 \subseteq K$, we find that exists $g_1 \in K_0$ such that

$$F(g_1, f_1) \subseteq C(f_1) \setminus \{0\}, \quad \forall f_1 \in K \setminus K_0.$$

We claim that $\mathscr{Z}(g_1) \subseteq K_0$. Let us assume that $\mathscr{Z}(g_1) \nsubseteq K_0$. Then there exists $f_1 \in \mathscr{Z}(g_1)$ and $f_1 \notin K_0$. Now, $f_1 \in \mathscr{Z}(g_1)$ implies $F(g_1, f_1) \subseteq -C(f_1)$ and $f_1 \notin K_0$. This is a contradiction. From Lemma 2.1, we have $\bigcap_{f \in K} \mathscr{Z}(f) \neq \phi$. In view of Lemma 3.1, we conclude that

$\bigcap_{f\in K}\mathscr{Z}(f)=\bigcap_{f\in K}\mathscr{Q}(f)$. Therefore, $\bigcap_{f\in K}\mathscr{Q}(f)\neq\phi$. Thus, there exists $f_0\in K$ such that $F(f_0,g)\nsubseteq -C(f_0)\setminus\{0\}$, for all $g\in K$. This completes the proof. □

Let $K\subseteq X$ and $C(f):=C$. As a consequences of Theorem 3.2, we have followings corollary.

Corollary 3.2. *Let $K\subseteq X$ be a nonempty convex set. Let the multifunction $F:K\times K\to 2^Y$ be generalized hemicontinuous in the first argument, P-convex and l.s.c. in the second argument, and F satisfies the coercive condition on compact subset K_0 of K. Assume that F is C-strongly pseudomonotone. Then there exists $x_0\in K$ such that $F(x_0,y)\nsubseteq -C\setminus\{0\}$, for all $y\in K$.*

Let $K\subseteq X$, F is a single valued function and $C(f):=C$. As a consequences of Theorem 3.2, we have following corollary.

Corollary 3.3. *Let $K\subseteq X$ be a nonempty convex compact set. Let $F:K\times K\to Y$ be a vector valued map, which is hemicontinuous in first argument, P-convex and l.s.c. in second argument. Assume that F is C-strongly pseudomonotone. Then the following SVEP has a solution, that is, there exists $x_0\in K$ such that $F(x_0,y)\notin -C\setminus\{0\}$, for all $y\in K$.*

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