Commun. Optim. Theory 2021 (2021), Article ID 12 https://doi.org/10.23952/cot.2021.12

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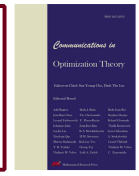

INFINITELY MANY HOMOCLINIC SOLUTIONS FOR FOURTH-ORDER DIFFERENTIAL EQUATIONS WITH LOCALLY DEFINED POTENTIALS

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Abstract. In this paper, we are concerned with the existence of homoclinic solutions for a class of nonautonomous fourth-order differential equations $u^{(4)}(x)+\omega u''(x)+a(x)u(x)=f(x,u(x))$, $x\in\mathbb{R}$. Using a symmetric mountain pass theorem established by Kajikiya, infinitely many homoclinic solutions of the equation are obtained when the function a is only steel bounded from below unnecessary coercive at infinity, and the potential $F(x,u)=\int_0^u f(x,v)dv$ is only locally defined near the origin with respect to u. Our result extends some previously known results in the literature.

Keywords. Fourth-order differential equation; Homoclinic solutions; Symmetric mountain pass theorem; Variational methods; Locally defined potentials; .

1. Introduction

Consider the following fourth-order differential equation

$$u^{(4)}(x)+\omega u''(x)+a(x)u(x)=f(x,u(x)),\quad x\in\mathbb{R} \qquad (\mathscr{F}\mathscr{E}),$$

where ω is a constant, $a\in C(\mathbb{R},\mathbb{R})$, and $f\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$. Here, as usual, we say that a solution u of $(\mathscr{F}\mathscr{E})$ is homoclinic (to 0) if $u\in C^{(4)}(\mathbb{R},\mathbb{R})$, and $u(x)\longrightarrow 0$ as $|x|\longrightarrow 0$. In addition, if $u\neq 0$, then u is called a nontrivial homoclinic solution. This equation arises from some problems associated with mathematical models for the study of the pattern formation in physics and mechanics, for example, the well-known extended Fisher-Kolmogorov equation proposed by Coullet et al. in 1987 [1], in the study of phase transitions, the fourth-order elastic beam equation in describing a large class of elastic deflection [2], and the Swift-Hohenberg equation, which is a general model for pattern-forming process derived in [3] to describe random thermal fluctuations in the Boussinesque equation and in the propagation of lasers [4]. With the aid of variational methods and the critical point theory, the existence and multiplicity of homoclinic solutions for $(\mathscr{F}\mathscr{E})$ have been extensively investigated in the literature over the past years; see, e.g., [2], [5]-[19] and the references cited therein. Many early papers treated the periodic case

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Received April 12, 2021; Accepted October 20, 2021.

that $a(x)$ and $f(x,u)$ are either independent of x or periodic in x; see, e.g., [5, 6, 7, 8] and the references therein. Compared with the periodic case, the problem is quite different in nature for the nonperiodic case due to the lack of the compactness of the Sobolev embedding. After the work of Li [9], there are many results concerning the nonperiodic case; see, e.g., [9]-[19]. For this case, the function a plays an important role. In all these results, $F(x,u)=\int_0^u f(x,v)dv$ was always required to satisfy some kinds of growth conditions at infinity with respect to u, such as, superquadratic, subquadratic or asymptotically quadratic growth. Besides, the function a is required to satisfy one of the following conditions:

(a) there exists a constant $a_0>0$ such that

$$a_0 \leq a(x) \longrightarrow +\infty \; as \; |x| \longrightarrow \infty,$$

and

$$\omega \leq 2\sqrt{a_0};$$

(b) there exists a constant $\sigma<0$ such that

$$|x|^{\sigma-1} a(x) \longrightarrow +\infty \; as \; |x| \longrightarrow \infty;$$

(c) there exist two constants $0<\tau_1<\tau_2<\infty$ such that

$$\tau_1 \leq a(x) \leq \tau_2, \; \forall x \in \mathbb{R}.$$

In the present paper, we will study the existence of infinity many homoclinic solutions for $(\mathscr{F}\mathscr{E})$ in the case that $f(x,u)$ is still only locally defined near the origin with respect to u, and a is only bounded from below and unnecessary coercive. More precisely, we make the following assumptions:

$(\mathscr{A})$ There exists a constant $a_0>0$ such that $\omega \leq 2\sqrt{a_0}$ and

$$a_0 \leq a(x), \; \forall x \in \mathbb{R};$$

(F_1) There exists a constant $\delta>0$ such that $f \in C(\mathbb{R}\times I_\delta,\mathbb{R})$, where $I_\delta =]-\delta,\delta[$, satisfes

$$F(x,-u)=F(x,u), \; \forall (x,u) \in \mathbb{R}\times I_\delta;$$

(F_2) There exist constants $\nu \in]1,2[$, $\beta_1 \in [1,2]$, $\beta_2 \in [1,\frac{2}{2-\nu}]$ and nonnegative functions $p \in L^{\beta_1}(\mathbb{R},\mathbb{R}^+)$, $q \in L^{\beta_2}(\mathbb{R},\mathbb{R}^+)$ such that

$$|f(x,u)| \leq p(x)+q(x)|u|^{\nu-1}, \; \forall (x,u) \in \mathbb{R}\times I_\delta;$$

$$(F_3) \qquad \lim_{|u|\longrightarrow 0} \frac{F(x,u)}{|u|^2} = +\infty, \; uniformly \; in \; x \in \mathbb{R}.$$

Our main result reads as follows.

Theorem 1.1. *Suppose that $(\mathscr{A})$ and $(F_1)-(F_3)$ are satisfied. Then the fourth-order differential equation $(\mathscr{F}\mathscr{E})$ possesses a sequence of homoclinic solutions (u_k) such that*

$$\max_{x\in\mathbb{R}} |u_k(x)| \longrightarrow 0 \; as \; k \longrightarrow \infty.$$

2. PRELIMINARIES

To prove our main result via critical point theory, we need to establish the variational setting for $(\mathscr{F}\mathscr{E})$. In the following, we shall use $\|.\|_{L^s}$ to denote the norm of $L^s(\mathbb{R})$ for any $s\in[2,\infty]$. Let $H^2(\mathbb{R})$ be the Sobolev space with inner product and norm given, respectively, by

$$< u,v >_{H^2}= \int_{\mathbb{R}}[u''(x)v''(x)+u'(x)v'(x)+u(x)v(x)]dx,$$

and

$$\|u\|_{H^2} =< u,u >_{H^2}^{\frac{1}{2}}$$

for all $u,v\in H^2(\mathbb{R})$.

Lemma 2.1. [7, Lemma 8] *Assume that the function a satisfies $(\mathscr{A})$. Then there exists a constant $c_0>0$ such that*

$$\int_{\mathbb{R}}[u^{''}(x)^2-\omega u^{'}(x)^2+a(x)u(x)^2]dx\geq c_0\|u\|^2_{H^2},\ \forall u\in H^2(\mathbb{R}).$$

By Lemma 2.1, we define

$$E=\left\{u\in H^2(\mathbb{R})/\int_{\mathbb{R}}[u^{''}(x)^2-\omega u^{'}(x)^2+a(x)u(x)^2]dx<\infty\right\}$$

with inner product

$$< u,v >= \int_{\mathbb{R}}[u''(x)v''(x)-\omega u'(x)v'(x)+a(x)u(x)v(x)]dx$$

and corresponding norm

$$\|u\|=(\int_{\mathbb{R}}[u^{''}(x)^2-\omega u^{'}(x)^2+a(x)u(x)^2]dx)^{\frac{1}{2}}.$$

Evidently, E is a Hilbert space and it is continuously embedded into $H^2(\mathbb{R})$. Hence E is continuously embedded in $L^s(\mathbb{R})$ for all $s\in[2,\infty]$ and compactly embedded in $L^s_{loc}(\mathbb{R})$ for all $s\in[2,\infty]$, where $L^s_{loc}(\mathbb{R})$ denotes the space of measurable functions u from $\mathbb{R}$ into $\mathbb{R}$ such that, for all compact $K\subset\mathbb{R}$, $\int_K|u(x)|^s dx<\infty$. Consequently, for all $s\in[2,\infty]$, there exists a constant $\eta_s>O$ such that

$$\|u\|_{L^s}\leq\eta_s\|u\|,\quad\forall u\in E. \tag{2.1}$$

To prove our main result via critical point theory, we shall use the following symmetric mountain pass theorem developed by Kajikiya [20]. We will first recall the notion of genus. Let E be a Banach space and let A be a subset of E. A is said to be symmetric if $u\in A$ implies $-u\in A$. For a closed symmetric set A, which does not contain the origin, we define the genus $\gamma(A)$ of A by the smallest integer k for which there exists an odd continuous mapping from $\mathbb{R}$ to $\mathbb{R}^k\setminus\{0\}$. If such a k does not exist, we define $\gamma(A)=+\infty$. Moreover, we set $\gamma(\phi)=0$. Let

$$\Gamma_k=\{A\subset E/A\ is\ a\ closed\ symmetric\ subset,\ 0\notin A,\ \gamma(A)\geq k\}.$$

The properties of genus used in the proof of our main result are summarized as follows.

Lemma 2.2. [20, Proposition 7.5] *Let A and B be closed symmetric subsets of E that do not contain the origin. Then the following hold*
(i) If $A\subset B$, then $\gamma(A)\leq\gamma(B)$;
(ii) the $N-$dimensional sphere S^N has a genus of $N+1$ by the Borsuk-Ulam theorem.

Lemma 2.3. [20, Theorem 1] *Let E be an infinite-dimensional Banach space and $\Phi \in C^1(E,\mathbb{R})$ an even functional with $\Phi(0) = 0$. Suppose that Φ satisfies*
(1) Φ is bounded from below and satisfies the (PS)-condition;
(2) *for each $k \in \mathbb{N}$, there exists $A_k \subset \Gamma_k$ such that*

$$\sup_{u\in A_k} \Phi(u) < 0.$$

Then (a) or (b) below holds
(a) there exists a critical point sequence (u_k) such that $\Phi(u_k) < 0$ and $\lim_{k\longrightarrow\infty} u_k = 0$;
(b) there exist two critical point sequences (u_k) and (v_k) such that $\Phi(u_k) = 0$, $u_k \neq 0$ and $\lim_{k\longrightarrow\infty} u_k = 0$, $\Phi(v_k) < 0$, $\lim_{k\longrightarrow\infty} \Phi(v_k) = 0$ and (v_k) converges to a non-zero limit.

3. Proof of Theorem 1.1.

In order to prove our main result via critical point theory, we need to modify $f(x,u)$ for x outside a neighborhood of the origin to obtain $\tilde{f}(x,u)$ as follows. Choose a constant $r \in]0, \frac{\delta}{2}[$, and define a cut-off function $\chi \in C^1(\mathbb{R}^+,\mathbb{R}^+)$ such that $\chi(s) = 1$ for $0 \leq s \leq r$, $\chi(s) = 0$ for $s \geq 2r$ and $-\frac{2}{r} \leq \chi'(s) < 0$ for $r < s < 2r$. Let

$$\widetilde{F}(x,u) = \chi(|x|)\widetilde{F}(x,u), \quad \forall (x,u) \in \mathbb{R}^2,$$

where $\widetilde{F}(x,u) = \int_0^u \widetilde{f}(x,v)dv$. Combining (F_1), (F_2), and the definition of χ, we obtain

$$\left|\widetilde{F}(x,u)\right| \leq p(x)\,|u| + q(x)\,|u|^{\nu}, \ \forall (x,u) \in \mathbb{R}^2, \tag{3.1}$$

and

$$\left|\widetilde{f}(x,u)\right| \leq 5\Big(p(x) + q(x)\,|u|^{\nu-1}\Big), \quad \forall (x,u) \in \mathbb{R}^2. \tag{3.2}$$

Now, we introduce the following modified system

$$u^{(4)}(x) + \omega u''(x) + a(x)u(x) = \widetilde{f}(x,u(x)), \ x \in \mathbb{R}, \tag{$\widetilde{\mathscr{F}\mathscr{E}}$}$$

and define the variational functional Φ associated with $(\widetilde{\mathscr{F}\mathscr{E}})$ by

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}} \left[u''(x)^2 - \omega u'(x)^2 + a(x)u(x)^2\right]dx - \int_{\mathbb{R}} \widetilde{F}(x,u(x))dx \\ &= \frac{1}{2}\|u\|^2 - \varphi(u),\end{aligned}$$

where $\varphi(u) = \int_{\mathbb{R}} \widetilde{F}(x,u(x))dx$.

Lemma 3.1. *Assume that $(\mathscr{A})$, (F_1) and (F_2) are satisfied. Then $\varphi \in C^1(E,\mathbb{R})$ and $\varphi' : E \longrightarrow E'$ is compact, and hence $\Phi \in C^1(E,\mathbb{R})$. Moreover*

$$\varphi'(u)v = \int_{\mathbb{R}} \widetilde{f}(x,u(x))v(x)dx, \tag{3.3}$$

and

$$\Phi'(u)v = < u,v > - \int_{\mathbb{R}} \widetilde{f}(x,u(x))v(x)dx \tag{3.4}$$

for all $u,v \in E$, and nontrivial critical points of Φ on E are homoclinic solutions of $(\mathscr{F}\mathscr{E})$.

Proof. In the following, we set

$$\overline{\beta}_1=\frac{\beta_1}{\beta_1-1},\ \overline{\beta}_2=\frac{\nu\beta_2}{\beta_2-1},(\overline{\beta}_1=\infty,\overline{\beta}_2=\infty,\ if\ \beta_1=1\ or\ \beta_2=1).$$

It is easy to see that $\overline{\beta}_1,\overline{\beta}_2\in[2,\infty]$. By (2.1), (3.1) and Hölder's inequality, we have for $u\in E$

$$\begin{aligned}\int_{\mathbb{R}}\left|\widetilde{F}(x,u(x))\right|dx&\le\int_{\mathbb{R}}p(x)\,|u(x)|\,dx+\int_{\mathbb{R}}q(x)\,|u(x)|^{\nu}\,dx\\&\le\|p\|_{L^{\beta_1}}\|u\|_{L^{\overline{\beta}_1}}+\|q\|_{L^{\beta_2}}\|u\|^{\nu}_{L^{\overline{\beta}_2}}\\&\le\eta_{\overline{\beta}_1}\|p\|_{L^{\beta_1}}\|u\|+\eta^{\nu}_{\overline{\beta}_2}\|q\|_{L^{\beta_2}}\|u\|^{\nu}<\infty,\end{aligned}\tag{3.5}$$

which implies that φ and Φ are both well defined. Now, we prove that $\varphi\in C^1(E,\mathbb{R})$ and $\varphi':E\longrightarrow E'$ is compact. By (3.2), for any $u,v\in E$ and $s\in[0,1]$, there holds

$$\begin{aligned}\left|\widetilde{f}(x,u+sv)v\right|&\le5\left[p(x)+q(x)\,|u+sv|^{\nu-1}\right]|v|\\&\le5\left[p(x)+q(x)\left(|u|^{\nu-1}+|v|^{\nu-1}\right)\right]|v|\\&\le5\left[p(x)+q(x)\left(|u|^{\nu-1}|v|+|v|^{\nu}\right)\right]|v|.\end{aligned}$$

Hence, by the Mean Value Theorem and Lebesgue's Dominated Convergence Theorem, we have, for all $u,v\in E$,

$$\begin{aligned}\lim_{s\longrightarrow0}\frac{\varphi(u+sv)-\varphi(u)}{s}&=\lim_{s\longrightarrow0}\int_{\mathbb{R}}\int_0^1\widetilde{f}(x,u+rsv)vdrdx\\&=\int_{\mathbb{R}}\widetilde{f}(x,u)vdx=\mathscr{L}(u)v.\end{aligned}$$

Moreover, it follows from (2.1), (3.2), and Hölder's inequality that

$$\begin{aligned}|\mathscr{L}(u)v|&\le\int_{\mathbb{R}}\left|\widetilde{f}(x,u)\right||v|\,dx\\&\le5\left[\int_{\mathbb{R}}p(x)\,|v|\,dx+\int_{\mathbb{R}}q(x)\,|u|^{\nu-1}|v|\,dx\right.\\&\le5\left[\|p\|_{L^{\beta_1}}\|v\|_{L^{\overline{\beta}_1}}+\|q\|_{L^{\beta_2}}\|u\|^{\nu-1}_{L^{\overline{\beta}_2}}\|v\|_{L^{\overline{\beta}_2}}\right]\\&\le5\left[\eta_{\overline{\beta}_1}\|p\|_{L^{\beta_1}}+\eta^{\nu}_{\overline{\beta}_2}\|q\|_{L^{\beta_2}}\|u\|^{\nu-1}\right]\|v\|,\ \forall v\in E,\end{aligned}$$

which means that $\mathscr{L}(u)$ is bounded. This means that φ is Gâteaux-differentiable on E, and its Gâteaux-derivative at u is $\mathscr{L}(u)$. Let $u_n\rightharpoonup u$ in E as $n\longrightarrow\infty$. Then (u_n) is bounded in E and

$$u_n\longrightarrow u\ in\ L^{\infty}_{loc}(\mathbb{R})\ as\ n\longrightarrow\infty.\tag{3.6}$$

Therefore, there exists a constant $c_1>0$ such that

$$\|u_n\|^{\nu-1}+\|u\|^{\nu-1}\le c_1,\ \forall n\in\mathbb{N}.\tag{3.7}$$

By (F_2), for any $\varepsilon>0$, there exists $R_\varepsilon>0$ such that

$$\left(\int_{|x|\ge R_\varepsilon}(p(x))^{\beta_1}dx\right)^{\frac{1}{\beta_1}}\le\frac{\varepsilon}{40\eta_{\overline{\beta}_1}},\tag{3.8}$$

and

$$\Big(\int_{|x|\geq R_\varepsilon}(q(x))^{\beta_2}dx\Big)^{\frac{1}{\beta_2}}\leq\frac{\varepsilon}{20c_1\eta^{\nu}_{\overline{\beta}_2}}.\tag{3.9}$$

Combining (3.2) with (3.7)-(3.9), the Hölder's inequality implies

$$\begin{aligned}
&\int_{|x|\geq R_\varepsilon}\left|\widetilde{f}(x,u_n)-\widetilde{f}(x,u)\right||v|\,dx\\
&\leq 5\int_{|x|\geq R_\varepsilon}\left[2p(x)+q(x)(|u_n|^{\nu-1}+|u|^{\nu-1})\right]|v|\,dx\\
&\leq 10\Big(\int_{|x|\geq R_\varepsilon}(p(x))^{\beta_1}dx\Big)^{\frac{1}{\beta_1}}\|v\|_{L^{\overline{\beta}_1}}\\
&+5\Big(\int_{|x|\geq R_\varepsilon}(q(x))^{\beta_2}dx\Big)^{\frac{1}{\beta_2}}\left(\|u_n\|^{\nu-1}_{L^{\overline{\beta}_2}}+\|u\|^{\nu-1}_{L^{\overline{\beta}_2}}\right)\|v\|_{L^{\overline{\beta}_2}}\\
&\leq 10\eta_{\overline{\beta}_1}\Big(\int_{|x|\geq R_\varepsilon}(p(x))^{\beta_1}dx\Big)^{\frac{1}{\beta_1}}\\
&+5\eta^{\nu}_{\overline{\beta}_2}\Big(\int_{|x|\geq R_\varepsilon}(q(x))^{\beta_2}dx\Big)^{\frac{1}{\beta_2}}\left(\|u_n\|^{\nu-1}+\|u\|^{\nu-1}\right)\\
&\leq\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2},\ \forall n\in\mathbb{N},\ \textit{and}\ \|v\|=1.
\end{aligned}\tag{3.10}$$

For the R_ε given above, by (2.1), (3.6), and the continuity of $\widetilde{f}$, there exists $n_0\in\mathbb{N}$ such that, for all $n\geq n_0$ and $\|v\|=1$

$$\int_{-R_\varepsilon}^{R_\varepsilon}\left|\widetilde{f}(x,u_n)-\widetilde{f}(x,u)\right||v|\,dx\leq\eta_\infty\int_{-R_\varepsilon}^{R_\varepsilon}\left|\widetilde{f}(x,u_n)-\widetilde{f}(x,u)\right|dx<\frac{\varepsilon}{2}.\tag{3.11}$$

Combining (3.10) with (3.11), we obtain

$$\begin{aligned}
\|\mathscr{L}(u_n)-\mathscr{L}(u)\|_{E'}&=\sup_{\|v\|=1}|(\mathscr{L}(u_n)-\mathscr{L}(u))v|\\
&=\sup_{\|v\|=1}\left|\int_{\mathbb{R}}(\widetilde{f}(x,u_n)-\widetilde{f}(x,u))v\,dx\right|\\
&\leq\sup_{\|v\|=1}\int_{-R_\varepsilon}^{R_\varepsilon}\left|\widetilde{f}(x,u_n)-\widetilde{f}(x,u)\right||v|\,dx\\
&+\sup_{\|v\|=1}\int_{|x|\geq R_\varepsilon}\left|\widetilde{f}(x,u_n)-\widetilde{f}(x,u)\right||v|\,dx\\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\forall n\geq n_0.
\end{aligned}$$

This implies that $\mathscr{L}$ is continuous. Thus $\varphi\in C^1(E,\mathbb{R})$ and (3.3) holds with $\varphi'=\mathscr{L}$. This together with the reflexivity of E implies that φ' is compact. In addition, due to the form of Φ, we see that $\Phi\in C^1(E,\mathbb{R})$ and (3.4) also holds. The proof of Lemma 3.1 is completed. □

Lemma 3.2. *Assume that $(\mathscr{A})$, (F_1) and (F_2) hold. Then Φ is bounded from below and satisfies the (PS)-condition.*

Proof. First, we prove that Φ is bounded from below. By (3.5), it follows

$$\begin{aligned}\Phi(u) &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \left|\widetilde{F}(x,u)\right| dx \\ &\geq \frac{1}{2}\|u\|^2 - \eta_{\overline{\beta}_1}\|p\|_{L^{\beta_1}}\|u\| - \eta_{\overline{\beta}_2}^{\nu}\|q\|_{L^{\beta_2}}\|u\|^{\nu}.\end{aligned} \tag{3.12}$$

Since $\nu < 2$, it follows that Φ is bounded from below. Next, we show that Φ satisfies the (PS)-condition. Let (u_n) be a $(PS)-$sequence, that is,

$$|\Phi(u_n)| \leq M,\ \forall n \in \mathbb{N},\ \Phi'(u_n) \longrightarrow 0 \ as\ n \longrightarrow \infty, \tag{3.13}$$

for some constant $M > 0$. From (3.12) and (3.13), we have

$$M \geq \frac{1}{2}\|u_n\|^2 - \eta_{\overline{\beta}_1}\|p\|_{L^{\beta_1}}\|u_n\| - \eta_{\overline{\beta}_2}^{\nu}\|q\|_{L^{\beta_2}}\|u_n\|^{\nu}$$

which implies that (u_n) is bounded in E due to $\nu < 2$. Hence, up to a subsequence if necessary, we can assume that $u_n \rightharpoonup u$ in E as $n \longrightarrow \infty$ for some $u \in E$. By virtue of the Riez Representation Theorem, $\varphi : E \longrightarrow E'$ and $\Phi' : E \longrightarrow E'$ can be viewed as $\varphi : E \longrightarrow E$ and $\Phi' : E \longrightarrow E$, respectively. This together with (3.3) and (3.4) yields

$$u_n = \Phi'(u_n) + \varphi'(u_n),\ \forall n \in \mathbb{N}. \tag{3.14}$$

By Lemma 3.1, φ' is compact. Combining this with (3.13)-(3.14), the right side of (3.14) converges strongly in E and hence $u_n \longrightarrow u$ in E as $n \longrightarrow \infty$. Then Φ satisfies the (PS)-condition. The proof of Lemma 3.2 is completed. ☐

Lemma 3.3. *Suppose that* $(\mathscr{A})$ *and* (F_3) *hold. Then, for each* $k \in \mathbb{N}$*, there exists an* $A_k \subset E$ *with genus* $\gamma(A_k) \geq k$ *such that* $\sup_{u \in A_k} \Phi(u) < 0$.

Proof. Let (e_n) be an orthonormal basis of E. For each $k \in \mathbb{N}$, let

$$E_k = \oplus_{m=1}^{k} span\{e_m\}.$$

Since E_k is finite dimensional, there exists a constant $\tau_k > 0$ such that

$$\|u\| \leq \tau_k \|u\|_{L^2},\ \forall u \in E_k. \tag{3.15}$$

By (F_3), there exists a constant $R_k > 0$ such that

$$\widetilde{F}(x,u) \geq \tau_k^2 |u|^2,\ \forall x \in \mathbb{R},\ |u| \leq R_k. \tag{3.16}$$

Let $u \in E$ such that $\|u\| \leq \frac{R_k}{\eta_\infty}$. By (2.1), we know that $|u(x)| \leq R_k$ for all $x \in \mathbb{R}$. It follows from (3.16) that

$$\widetilde{F}(x,u(x)) \geq \tau_k^2 |u(x)|^2,\ \forall x \in \mathbb{R}. \tag{3.17}$$

In view of (3.15) and (3.17), for all $u \in E_k \setminus \{0\}$ with $0 < \|u\| = \frac{\min\{r,R_k\}}{\eta_\infty} = \rho_k$, we have

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \widetilde{F}(x,u)dx \\ &\leq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \tau_k^2 |u(x)|^2 dx \\ &\leq \frac{1}{2}\|u\|^2 - \|u\|^2 \\ &= -\frac{1}{2}\rho_k^2,\end{aligned}$$

which implies

$$\{u \in E_k \setminus \{0\} \,/\, \|u\| = \rho_k\} \subset A_k = \left\{ u \in E_k / \Phi(u) \leq -\frac{1}{2}\rho_k^2 \right\}. \tag{3.18}$$

With the aid of Lemma 2.2, (3.18) implies

$$\gamma(A_k) \geq \gamma\Big(\{u \in E_k \setminus \{0\} \,/\, \|u\| = \rho_k\} \Big) \geq k.$$

Hence, by the definition of Γ_k, we have $A_k \subset \Gamma_k$. Moreover, the definition of A_k implies

$$\sup_{u \in A_k} \Phi(u) \leq -\frac{1}{2}\rho_k^2 < 0.$$

The proof of Lemma 3.3 is completed. □

Lemmas 3.2, 3.3 imply that the functional Φ satisfies all the conditions of Lemma 2.3. Consequently, Φ possesses a sequence of nontrivial critical points (u_k) satisfying $u_k \longrightarrow 0$ in E as $k \longrightarrow \infty$. By virtue of Lemma 3.1, (u_k) is a sequence of homoclinic solutions of $(\widetilde{\mathscr{F}\mathscr{E}})$. By (2.1), it follows that $\max_{x\in\mathbb{R}} |u_k(x)| \longrightarrow 0$ as $k \longrightarrow \infty$. Therefore, there exists a positive constant $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, u_k is a homoclinic solution of $(\mathscr{F}\mathscr{E})$. This ends the proof of Theorem 1.1.

Acknowledgments

The author would like to thank the editor and the referees for interesting comments and valuable suggestions.

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