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SOME REGULARITY OF NONLINEAR DEGENERATE PARABOLIC EQUATIONS WITH L^1 DATA

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Abstract. In this paper, we prove the existence of solutions of nonlinear anisotropic parabolic equations whose model is $\frac{\partial u}{\partial t} - \text{div}\left(\frac{|u|^{p-2}\nabla u}{(1+|u|)^{\gamma}}\right) = f$, on $\Omega \times (0,T)$, with homogeneous Cauchy-Dirichlet boundary conditions, where $\gamma+1 < p < \gamma+2$, $0 \leq \gamma < 1$, and f belongs to L^1.

Keywords. Degenerate coercivity; Existence and Regularity; L^1 data; Nolinear parabolic equations.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$. For $T > 0$, let us denote by Q the cylinder $\Omega \times (0,T)$, and by Γ the lateral surface $\partial\Omega \times (0,T)$. We will consider the following nonlinear degenerate parabolic Cauchy-Dirichlet problem:

$$\begin{cases} \frac{\partial u}{\partial t} + Au = f & \text{in } Q, \\ u = 0 & \text{on } \Gamma, \\ u(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $Au = -\text{div}(\widehat{a}(x,t,u,\nabla u))$, $f \in L^1(Q)$, or $L^m(Q)$, and $\widehat{a} : \Omega\times]0,T[\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $(x,t,s) \in Q\times\mathbb{R}$, $\forall \xi, \xi' \in \mathbb{R}^N$,

$$\widehat{a}(x,t,s,\xi).\xi \geq \frac{\alpha|\xi|^p}{(1+|s|)^{\gamma}}, \tag{1.2}$$

$$|\widehat{a}(x,t,s,\xi)| \leq \beta\left(1+|\xi|^{\frac{p-1}{\gamma+1}}\right), \tag{1.3}$$

and

$$\left(\widehat{a}(x,t,s,\xi) - \widehat{a}(x,t,s,\xi')\right).(\xi-\xi') > 0, \tag{1.4}$$

where α and β are positive real numbers, and $0 \leq \gamma < 1$.

We begin by recalling some well-known results. In the uniform ellipticity case, that is, $\theta = 0$, it was proved in [1] that

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(1) if $\frac{2N+1}{N+1} < p < 2$, and f belongs to $L^1(Q)$, then there exists a solution u of (1.1) belonging to $L^r(0,T;W_0^{1,1}(\Omega))$ for every $1 \le r < N(p-2)+p$.
(2) if $\frac{2N+1}{N+1} \le p < 2$, and f belongs to $L^1 \log L^1(Q)$, then there exists a solution u of (1.1) belonging to $L^r(0,T;W_0^{1,1}(\Omega))$ for every $r = N(p-2)+p$.

In particular, if $p = 2$, the existence and regularity results for problem (1.1) were obtained in [2] under the fact that f and γ satisfy

$$f \in L^m(Q), \quad m \ge 1 \quad \text{and} \quad 0 \le \gamma < 1 + \frac{2}{N}.$$

The main difficulty of problems (1.1) is the fact that the differential operator Au is not coercive as u becomes large. This shows that the classical methods (see [3]) cannot be applied to prove the existence of solutions of problem (1.1) even if the data is sufficiently regular (see [4]). A similar problem to elliptic equations was studied in [5] (see also [6] and [7]). The goal in this paper is to study problem (1.1) under the assumptions: (1.2)-(1.4). The proof is essentially based on the approximate problems (3.6) with some non-degenerate coercivity and priori estimates on the weak solutions of these problems.

2. Preliminaries

In this paper, we will use the following functions of one real variable defined by $T_k(s) = sgn(s)\min\{|s|,k\}$, for $k > 0$, and its primitive $S_k : \mathbb{R} \to \mathbb{R}^+$

$$S_k(x) = \int_0^x T_k(s)ds. \tag{2.1}$$

It results

$$\frac{1}{2}|T_k(s)|^2 \le S_k(s) \le k|s| \quad \forall k > 0, \ \forall s \in \mathbb{R}, \tag{2.2}$$

We define

$$G_k(s) = s - T_k(s), \quad \varphi_k(s) = T_1(G_k(s)).$$

for $k \ge 0$, and s in $\mathbb{R}$. We begin by recalling the well-known Gagliardo-Nirenberg embedding theorem.

Lemma 2.1. [1] *Let v be a function in $W_0^{1,1}(\Omega) \cap L^\rho(\Omega)$ with $q \ge 1$ and $\rho \ge 1$. Then there exists a positive constant M_1 depending on N, q, and ρ such that*

$$\|v\|_{L^q(\Omega)} \le M_1 \|\nabla v\|^{\theta}_{(L^1(\Omega))^N} \|v\|^{1-\theta}_{L^\rho(\Omega)}, \tag{2.3}$$

for every θ and q satisfying

$$0 \le \theta \le 1, \quad 1 \le q < +\infty, \quad \frac{1}{q} = \theta\left(1 - \frac{1}{N}\right) + \frac{1-\theta}{\rho}. \tag{2.4}$$

An immediate consequence of the lemma is the following embedding result.

Lemma 2.2. [2] *Let $v \in L^q(0,T;W_0^{1,h}(\Omega)) \cap L^\infty(0,T;L^\rho(\Omega))$ with $h,\rho \ge 1$. Then v belongs to $L^\sigma(Q)$, where $\sigma = q\frac{N+\rho}{N}$, and there exists a positive constant M_2 depending only on N,q,ρ such that*

$$\int_Q |v|^\sigma \, dx\, dt \le M_2 \|v\|^{\frac{\rho q}{N}}_{L^\infty(0,T;L^\rho(\Omega))} \int_Q |\nabla v|^q \, dx\, dt. \tag{2.5}$$

Let n, ν, and k be positive integers, and let ε be a positive real number. We will denote by $\omega(n,\nu,k,\varepsilon)$ any quantity such that

$$\lim_{\varepsilon\to 0^+}\limsup_{k\to+\infty}\limsup_{\nu\to+\infty}\limsup_{n\to+\infty}|\omega(n,\nu,k,\varepsilon)|=0. \tag{2.6}$$

Sometimes we will also use a subset of parameters. For example, we will denote by $\omega^{\nu,k,\varepsilon}(n)$ a quantity such that, for any fixed ν, k, and ε, $\lim_{n\to+\infty}|\omega^{\nu,k,\varepsilon}(n)|=0$. If the quantities that we are taking into account do not depend on some parameters, we will omit the dependence of ω from them. For example, $\omega(n,k)$ is a quantity that depends only on n and k such that

$$\lim_{k\to+\infty}\limsup_{n\to+\infty}|\omega(n,k)|=0.$$

We begin with the following result.

Lemma 2.3 ([8])**.** *Let $\{v_n\}$ be a sequence in $L^p(0,T;W_0^{1,p}(\Omega))\cap C^0([0,T];L^2(\Omega))$ such that $v_n(.,0)=0$, and $v_n'\in L^{p'}(0,T;W_0^{-1,p'}(\Omega))$. Suppose that v_n converges almost everywhere in Q to a function $v\in L^1(0,T;W_0^{1,1}(\Omega))$ such that $T_k(v)\in L^p(0,T;W_0^{1,p}(\Omega))$ for every $k>0$. Then, for every choice of ε, k and ν,*

$$\langle v_n', T_\varepsilon(v_n-T_k(v)_\nu)\rangle \geq \omega^{\nu,k,\varepsilon}(n),$$

where $\omega^{\nu,k,\varepsilon}$ is defined as in (2.6).

The following result is well known.

Lemma 2.4 ([8])**.** *Let Ω be an open bounded subset of $\mathbb{R}^N$ with $N\geq 1$, and let $\{v_n\}$ be a sequence of measurable functions on Ω such that v_n converges to some function v almost everywhere in Ω. Then, for almost every h in $\mathbb{R}^+$,*

$$\chi_{\{|v_n|>h\}}\to\chi_{\{|v|>h\}} \quad \text{strongly in } L^\rho(\Omega), \text{ for every } 1\leq\rho<+\infty.$$

Here χ_E denotes the characteristic function of a set $E\subseteq\Omega$.

3. The Main Results and Approximate Solutions

3.1. The main results.

Definition 3.1. We say that a function u in $L^1(0,T;W_0^{1,1}(\Omega))$ is a weak solution of (1.1) iff $\widehat{a}(x,t,u,\nabla u)\in(L^1(Q))^N$, and

$$-\int_Q u'\varphi\,dx\,dt+\int_Q\widehat{a}(x,t,u,\nabla u).\nabla\varphi\,dx\,dt=\int_Q f\varphi\,dx\,dt, \tag{3.1}$$

for every $\varphi\in C^\infty(\overline{Q})$, which is zero in a neighborhood of $\Gamma\cup(\Omega\times\{T\})$.

Our result is described as follows.

Theorem 3.2. *Under hypotheses (1.2)-(1.4), if $f\in L^1(\Omega)$ and*

$$\frac{N(\gamma+2)+1}{N+1}<p<\gamma+2, \tag{3.2}$$

then there exists a weak solution u of (1.1) belonging to $L^r(0,T;W_0^{1,1}(\Omega))$ with

$$1\leq r<N(p-\gamma-2)+p. \tag{3.3}$$

Moreover, if $\frac{N(\gamma+2)+1}{N+1} \le p < \gamma+2$*, then there exists a solution u of* (1.1) *belonging to* $L^r(0,T;W_0^{1,1}(\Omega))$*, with* $r = N(p-\gamma-2)+p$

3.2. **Approximate solutions.** Let (f_n) be a sequence of bounded functions defined in Q, where f_n is in $\mathscr{D}(Q)$, and satisfy, for every $m \ge 1$,

$$\|f_n\|_{L^m(Q)} \le \|f\|_{L^m(Q)} \le c, \quad \forall n, \tag{3.4}$$

$$f_n \to f \quad \text{strongly in } L^m(Q). \tag{3.5}$$

We approximate problem (1.1) by the following problems

$$\begin{cases} u_n' - \operatorname{div}(\widehat{a}(x,t,T_n(u_n),\nabla u_n))) = f_n & \text{in } Q, \\ u_n = 0 & \text{on } \Gamma, \\ u_n(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{3.6}$$

For $n \in \mathbb{N}$, we define the operator A_n by $A_n = -\operatorname{div}(\widehat{a}(.,.,T_n(u),\nabla u)))$. From (1.2) and (1.4), we have

$$\int_Q \widehat{a}(x,t,T_n(u),\nabla u).\nabla u \, dx\, dt \ge g(n) \int_Q |\nabla u|^p \, dx\, dt,$$

with

$$g(n) = \frac{1}{(1+n)^\gamma},$$

so that the operator A_n from $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$ is coercive and satisfies the classical Leary-Lions conditions. From the well-known result of [9, 10], there exists at least a solution u_n in $L^p(]0,T[;W_0^{1,p}(\Omega)) \cap C([0,T];L^2(\Omega))$ to problem (3.6) such that $u_n' \in L^{p'}(]0,T[;W^{-1,p'}(\Omega))$ and satisfies

$$\int_Q u_n'\phi \, dx\, dt + \int_Q \widehat{a}(x,t,T_n(u_n),\nabla u_n)\nabla\phi \, dx\, dt = \int_Q f_n\phi \, dx\, dt,$$

for any $\phi \in L^p(]0,T[;W_0^{1,p}(\Omega))$ and $u_n(x,0) = 0$.

4. Priori Estimates

In this section, we will prove some a priori estimate on the sequence $\{u_n\}$ defined in (3.6), which depends on various assumptions on the sequence $\{f_n\}$.

We now prove some priori estimates, as well as convergence properties, on the sequence $\{u_n\}$ of solutions of (3.6), with $\{f_n\}$ satisfying assumption (3.4)-(3.5).

Lemma 4.1. *Let* $f \in L^1(Q)$ *and* (1.2)-(1.4) *hold. Then there exists a positive constant c such that*

$$\int_Q |\nabla T_k(u_n)|^p \le ck(k+1)^\gamma, \tag{4.1}$$

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega))} \le c, \tag{4.2}$$

and

$$\int_{B_k} \frac{|\nabla u_n|^p}{(1+|u_n|)^\gamma} \, dx\, dt \le c, \tag{4.3}$$

where $B_k = \{(x,t) \in Q : k \le |u_n(x,t)| < k+1\}$.

Proof. Choosing $T_1(u_n)\chi_{(0,\tau)}$ as a test function for problem (3.6), using (1.2), (3.4), and the fact that $|T_k(u_n)\chi_{(0,\tau)}| \le k$, we have

$$\int_\Omega S_k(u_n(x,\tau))dx + \alpha\int_0^\tau\int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^\gamma}\,dx\,dt \le k\|f\|_{L^1(Q)}. \tag{4.4}$$

By dropping the nonegative first term in (4.4), we obtain (4.1). By (4.2) ($k=1$) and dropping the nonnegative lower order term, we obtain

$$\int_\Omega S_1(u_n(x,\tau))dx \le \|f\|_{L^1(Q)}. \tag{4.5}$$

For any $s\in\mathbb{R}$, it follows from (2.1)-(2.2) that

$$|s| - \frac{1}{2} \le S_1(s) \le |s| \tag{4.6}$$

In view of (4.5) and (4.6), we have

$$ess \sup_{0\le t\le T}\int_\Omega |u_n(x,t)|dx \le \|f_n\|_{L^1(Q)} + \frac{1}{2}|\Omega|. \tag{4.7}$$

This shows that (4.7) yield (4.2). Taking $\varphi_k(u_n)\chi_{(0,\tau)}$ as a test function in (3.6), and using (1.2), (3.4), (3.5), and the fact that $|\varphi_k(s)| \le 1$, we have

$$\frac{1}{2}\int_\Omega |T_1(G_k(u_n(x,\tau)))|^2\,dx + \alpha\int_{B_k}\frac{|\nabla u_n|^p}{(1+|u_n|)^\gamma}\,dx\,dt \le \|f\|_{L^1(Q)}. \tag{4.8}$$

Dropping the nonegative first term in (4.8), we obtain (4.3). This completes the proof. □

Lemma 4.2. *Let $f\in L^1(Q)$, $\frac{N(\gamma+2)+1}{N+1} < p < \gamma+2$, and (1.2)-(1.4) hold. Then there exists a positive constant c such that*

$$\|u_n\|_{L^r(0,T;W_0^{1,1}(\Omega))} \le c, \tag{4.9}$$

where r is defined in (3.3). Furthermore, up to subsequences, u_n is weakly convergent in $L^1(0,T;W_0^{1,1}(\Omega))$ to some function u.

Proof. If $\lambda > 1$, and $h > 0$, we obtain from (4.3) that

$$\begin{aligned}\int_{\{|u_n|\ge h\}}\frac{|\nabla u_n|^p}{(1+|u_n|)^{\lambda+\gamma}}dx &= \sum_{k=h}^{+\infty}\int_{B_k}\frac{|\nabla u_n|^p}{(1+|u_n|)^{\lambda+\gamma}}dx \\ &\le \sum_{k=h}^{+\infty}\frac{1}{(1+k)^\lambda}\int_{B_k}\frac{|\nabla u_n|^p}{(1+|u_n|)^\gamma}dx \\ &\le c\sum_{k=h}^{+\infty}\frac{1}{(1+k)^\lambda}.\end{aligned} \tag{4.10}$$

Note that $\sum_{k=h}^{+\infty}\frac{1}{(1+k)^\lambda}$ converges due to $\lambda > 1$. (4.10) implies

$$\int_{\{|u_n|\ge h\}}\frac{|\nabla u_n|^p}{(1+|u_n|)^{\lambda+\gamma}}dx \le C(\lambda,h). \tag{4.11}$$

where $C(\lambda,h)\to 0$ as $h\to 0$. For almost every $t\in(0,T)$, using the Hölder inequality with exponents p and $p'=\frac{p}{p-1}$, we can write

$$\begin{aligned}\int_\Omega |\nabla u_n|dx &= \int_\Omega \frac{|\nabla u_n|}{(1+|u_n|)^{\frac{\lambda+\gamma}{p}}}(1+|u_n|)^{\frac{\lambda+\gamma}{p}}dx\\ &\le \left(\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{\lambda+\gamma}}dx\right)^{\frac{1}{p}}\left(\int_\Omega (1+|u_n|)^{\frac{\lambda+\gamma}{p-1}}dx\right)^{\frac{p-1}{p}}. \qquad (4.12)\end{aligned}$$

Raising to the power $r<p$, integrating on t, and using again the Hölder's inequality with exponents $\frac{p}{r}$ and $\frac{p}{p-r}$, (4.11) ($h=0$), we conclude from (4.12) that

$$\begin{aligned}&\int_0^T \|\nabla u_n\|_{(L^1(\Omega))^N}dt\\ &\le \int_0^T\left[\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{\lambda+\gamma}}dx\right]^{\frac{r}{p}}\left[\int_\Omega (1+|u_n|^{\frac{\lambda+\gamma}{p-1}})dx\right]^{\frac{(p-1)r}{p}}dt\\ &\le \left(\int_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^{\lambda+\gamma}}\,dx\,dt\right)^{\frac{r}{p}}\left(\int_0^T\left[\int_\Omega (1+|u_n|^{\frac{\lambda+\gamma}{p-1}})dx\right]^{\frac{(p-1)r}{p-r}}dt\right)^{\frac{p-r}{p}}\\ &\le c\left[1+\left(\int_0^T \|u_n\|^{\frac{(\lambda+\gamma)r}{p-r}}_{L^{\frac{\lambda+\gamma}{p-1}}(\Omega)}dt\right)^{\frac{p-r}{p}}\right]. \qquad (4.13)\end{aligned}$$

Applying Lemma 2.1 with $\rho=1$ and $q=\frac{\lambda+\gamma}{p-1}$, and recalling (4.2), we obtain, for almost every t in $[0,T]$,

$$\begin{aligned}\|u_n\|_{L^{\frac{\lambda+\gamma}{p-1}}(\Omega)} &\le M_1\|\nabla u_n\|^\theta_{(L^1(\Omega))^N}\|u_n\|^{1-\theta}_{L^1(\Omega)}\\ &\le c\|\nabla u_n\|^\theta_{(L^1(\Omega))^N}, \qquad (4.14)\end{aligned}$$

where θ is such that

$$\frac{p-1}{\lambda+\gamma}=1-\frac{\theta}{N}. \qquad (4.15)$$

Raising (4.14) to the power $\frac{r}{\theta}$ and integrating on $(0,T)$, we obtain

$$\int_0^T \|u_n\|^{\frac{r}{\theta}}_{L^{\frac{\lambda+\gamma}{p-1}}(\Omega)}dt \le c\int_0^T \|\nabla u_n\|^r_{(L^1(\Omega))^N}dt. \qquad (4.16)$$

Now, we assume that

$$\frac{r}{\theta}=\frac{(\lambda+\gamma)r}{p-r}. \qquad (4.17)$$

Thus, (4.13) and (4.16) imply

$$\int_0^T \|\nabla u_n\|_{(L^1(\Omega))^N}dt \le c\left[1+\left(\int_0^T \|\nabla u_n\|^r_{(L^1(\Omega))^N}\right)^{\frac{p-r}{p}}\right].$$

In view of $\frac{p-r}{p}<1$, one easily obtains from the above inequality a priori estimate on the norm of u_n in $L^r(0,T;W_0^{1,1}(\Omega))$. Putting (4.15) and (4.17) together, we obtain

$$\lambda=\frac{N(p-1-\gamma)+p-r}{N},\quad \theta=\frac{p-r}{\lambda+\gamma},$$

with $1\le r<p$, $\lambda>1$, $\lambda\ge p-1-\gamma$, $0\le\theta\le 1$, and $0\le\gamma<1$. These inequalities with the values of λ and θ given above are satisfied if $1\le r<N(p-2-\gamma)+p$, which is our assumption. Note that the condition on r is not empty if $p>\frac{N(\gamma+2)+1}{N+1}$. Thus, one has an estimate on u_n in $L^r(0,T;W_0^{1,1}(\Omega))$ for every $1\le r<N(p-2-\gamma)+p$. Now we turn to the weak convergence of u_n. Repeating the steps above with $r=1$, the values of θ, and (4.16), we have

$$\int_Q |u_n|^s\,dx\,dt\le c, \tag{4.18}$$

where

$$s=\frac{\lambda+\gamma}{p-1}=1+\frac{p-r}{(p-1)N}>1.$$

Therefore, for every $h>0$ we have by repeating the steps above

$$\begin{aligned}&\int_{\{|u_n|\ge h\}}|\nabla u_n|\,dx\,dt\\&=\int_{\{|u_n|\ge h\}}\frac{|\nabla u_n|}{(1+|u_n|)^{\frac{\lambda+\gamma}{p}}}(1+|u_n|)^{\frac{\lambda+\gamma}{p}}dx\,dt\\&\le\left(\int_{\{|u_n|\ge h\}}\frac{|\nabla u_n|^p}{(1+|u_n|)^{\lambda+\gamma}}dx\right)^{\frac{1}{p}}\left(\int_Q(1+|u_n|)^{\frac{\lambda+\gamma}{p-1}}dx\,dt\right)^{\frac{p-1}{p}}\\&\le C(\lambda,h).\end{aligned}\tag{4.19}$$

Let E be a measurable subset of Q, and $h>0$. We have

$$\int_E|\nabla u_n|\,dx\,dt=\int_E|\nabla T_h(u_n)|\,dx\,dt+\int_{E\cap\{|u_n|\ge h\}}|\nabla u_n|\,dx\,dt. \tag{4.20}$$

From (4.1), and the Hölder's inequality, we have

$$\int_E|\nabla T_h(u_n)|\,dx\,dt\le c|E|^{\frac{1}{p'}}h^{\frac{1}{p}}(h+1)^{\frac{\gamma}{p}}. \tag{4.21}$$

Letting $\varepsilon>0$, we use (4.21) to choose $|E|$ small enough, and choose h large enough so that

$$\int_E|\nabla T_h(u_n)|\,dx\,dt<\frac{\varepsilon}{2}. \tag{4.22}$$

It follows from (4.19) that

$$\int_{E\cap\{|u_n|\ge h\}}|\nabla u_n|\,dx\,dt\le\int_{\{|u_n|\ge h\}}|\nabla u_n|\,dx\,dt\le c.$$

Thus

$$\int_{E\cap\{|u_n|\ge h\}}|\nabla u_n|\,dx\,dt<\frac{\varepsilon}{2} \tag{4.23}$$

In view of (4.20), (4.22), and (4.23), for every $\varepsilon>0$, we have that there exists $\delta>0$ such that if $|E|<\delta$, then $\int_E|\nabla u_n|\,dx\,dt<\varepsilon$, which proves that $|\nabla u_n|$ is equi-integrable. Arguing as in [1], we can prove that u_n weakly converges to u in $L^1(0,T;W_0^{1,1}(\Omega))$. □

Lemma 4.3. *Let $f \in L^1(Q)$, $\frac{N(\gamma+2)+1}{N+1} \le p < \gamma+2$, and (1.2)-(1.4) hold. Then there exists a positive constant c such that*

$$\|u_n\|_{L^r(0,T;W_0^{1,1}(\Omega))} \le c, \quad \textit{with} \quad r = N(p-\gamma-2)+p. \tag{4.24}$$

Furthermore, up to subsequences, u_n is weakly convergent in $L^1(0,T;W_0^{1,1}(\Omega))$ to some function u.

Proof. The key estimate for the proof is inequality (4.10) with $\lambda = 1$, that is,

$$\int_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^{\gamma+1}}\, dx\, dt \le c. \tag{4.25}$$

Estimate (4.25) was proved in [11, Lemma 2.2]. From (4.25) and (4.2) (which still holds since f belongs to $L^1(Q)$), one obtains from Lemma 2.1 the desired estimate immediately. The weak convergence of u_n can be proved as in the proof of Lemma 4.2 by using (4.18), which still holds in this case. ☐

Theorem 4.4. *Let f and p be such that one of Theorem 3.2 holds. Let u_n be a sequence of solutions of (3.6) with data f_n satisfying (3.4)-(3.5). Then there exists a function u in $L^1(0,T;W_0^{1,1}(\Omega))$ such that u_n weakly converges to u in the same space (up to subsequences), and strongly in $L^1(Q)$. Furthermore, $\nabla u_n \to \nabla u$ a.e in Ω.*

Proof. Up to the subsequences of $\{u_n\}$, we still denote by u_n (for simplicity of notation, we will omit the dependence of $\widehat{a}$ and $\widehat{a}$ on x and t),

$$[\widehat{a}(u_n,\nabla u_n)-\widehat{a}(u_n,\nabla u)]\nabla(u_n-u) \quad \text{a.e. in } Q. \tag{4.26}$$

Furthermore, (4.26) is true for some subsequence if we show that

$$\lim_{n\to+\infty}\int_Q \{[\widehat{a}(u_n,\nabla u_n)-\widehat{a}(u_n,\nabla u)]\nabla(u_n-u)\}^\theta dxdt = 0, \tag{4.27}$$

for some $\theta > 0$. To do this, we will prove that

$$0 \le \int_Q \{[\widehat{a}(u_n,\nabla u_n)-\widehat{a}(u_n,\nabla u)]\nabla(u_n-u)\}^\theta dxdt \le \omega(n,\nu,k,\varepsilon). \tag{4.28}$$

Since u belongs to $L^1(Q)$, the following estimate holds

$$meas(\{(x,t)\in Q : |u(x,t)| \ge k\}) = w(k). \tag{4.29}$$

We can write

$$\begin{aligned}
&\int_Q \{[\widehat{a}(u_n,\nabla u_n)-\widehat{a}(u_n,\nabla u)]\nabla(u_n-u)\}^\theta dxdt\\
&\quad = \int_{\{|u|\ge k\}} \{[\widehat{a}(u_n,\nabla u_n)-\widehat{a}(u_n,\nabla u)]\nabla(u_n-u)\}^\theta dxdt\\
&\qquad + \int_{\{|u|<k\}} \{[\widehat{a}(u_n,\nabla u_n)-\widehat{a}(u_n,\nabla u)]\nabla(u_n-u)\}^\theta dxdt\\
&\quad = I_{n,k} + J_{n,k}.
\end{aligned}$$

Since $\{u_n\}$ is bounded in $L^1(0,T;W_0^{1,1}(\Omega))$ (Lemma 4.2) for some $\gamma+1<p$, we can choose $\theta<\frac{1}{p}<\frac{1}{\gamma+1}<1$ so that

$$|I_{n,k}|\leq c\left(\int_Q[1+|\nabla u_n|+|\nabla u|]dxdt\right)^{\theta p}(meas(\{|u(x,t)|\geq k\}))^{1-\theta p}$$
$$\leq c|\{|u(x,t)|\geq k\})|^{1-\theta p},$$

and then $I_{n,k}=\omega(n,k)$ due to (4.29). On the other hand,

$$J_{n,k}=\int_{\{|u|<k\}}\{[\widehat{a}(u_n,\nabla u_n)-\widehat{a}(u_n,\nabla T_k(u))]\nabla(u_n-T_k(u))\}^{\theta}dxdt$$
$$\leq\int_Q\{[\widehat{a}(u_n,\nabla u_n)-\widehat{a}(u_n,\nabla T_k(u))]\nabla(u_n-T_k(u))\}^{\theta}dxdt. \tag{4.30}$$

Assume that $\varepsilon>0$ satisfies

$$\lim_{n\to+\infty}\chi_{\{|u_n-T_k(u)\nu|>\varepsilon\}}=\chi_{\{|u-T_k(u)\nu|>\varepsilon\}}\quad\forall\nu,k\in\mathbb{N}, \tag{4.31}$$
$$\lim_{\nu\to+\infty}\chi_{\{|u-T_k(u)\nu|>\varepsilon\}}=\chi_{\{|u-T_k(u)|>\varepsilon\}}\quad\forall k\in\mathbb{N}, \tag{4.32}$$

and the limit is meant in $L^{\rho}(Q)$, for every finite $\rho\geq1$. By Lemma 2.4, almost every ε satisfies (4.31)-(4.32). From now on, we will assume that ε tends to 0 along a sequence which satisfies (4.31)-(4.32). We split the last integral of (4.30) on the sets

$$\{(x,t)\in Q:|u_n-T_k(u)_\nu|\leq\varepsilon\},\quad\{(x,t)\in Q:|u_n-T_k(u)_\nu|>\varepsilon\},$$

and we define $\Psi_{n,k}=[\widehat{a}(u_n,\nabla u_n)-\widehat{a}(u_n,\nabla T_k(u))].\nabla(u_n-T_k(u))$. Then

$$\int_Q\{[\widehat{a}(u_n,\nabla u_n)-\widehat{a}(u_n,\nabla u)]\nabla(u_n-u)\}^{\theta}dxdt$$
$$\leq\int_Q\Psi_{n,k}^{\theta}\chi_{\{|u_n-T_k(u)_\nu|\leq\varepsilon\}}dxdt$$
$$+\int_Q\Psi_{n,k}^{\theta}\chi_{\{|u_n-T_k(u)_\nu|>\varepsilon\}}dxdt+\omega(n,k). \tag{4.33}$$

Since $\{\Psi_{n,k}^{\theta}\}$ is bounded in $L^{\frac{1}{\theta p}}$, and $\chi_{\{|u_n-T_k(u)_\nu|>\varepsilon\}}$ converges to zero almost everywhere in Q as k tends to infinity, we conclude from (4.31) that $\int_Q\Psi_{n,k}^{\theta}\chi_{\{|u_n-T_k(u)_\nu|>\varepsilon\}}dxdt=\omega^{\varepsilon}(n,\nu,k)$. Thus, (4.33) becomes

$$\int_Q\{[\widehat{a}(u_n,\nabla u_n)-\widehat{a}(u_n,\nabla u)]\nabla(u_n-u)\}^{\theta}dxdt$$
$$\leq\int_Q\Psi_{n,k}^{\theta}\chi_{\{|u_n-T_k(u)_\nu|\leq\varepsilon\}}dxdt+\omega^{\varepsilon}(n,\nu,k).$$

Using the Hölder inequality (with exponents $\frac{1}{\theta}$ and $\frac{1}{1-\theta}$), the last integral is smaller than

$$|Q|^{1-\theta}\left(\int_Q\Psi_{n,k}\chi_{\{|u_n-T_k(u)_\nu|\leq\varepsilon\}}dxdt\right)^{\theta}.$$

To show (4.28), we consider

$$\int_Q\Psi_{n,k}\chi_{\{|u_n-T_k(u)_\nu|\leq\varepsilon\}}dxdt=\omega(n,\nu,k,\varepsilon). \tag{4.34}$$

Recalling the definition of $\Psi_{n,k}$, we can write

$$\begin{aligned}
&\int_Q \Psi_{n,k}\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}dxdt\\
&\quad=\int_Q \widehat{a}(u_n,\nabla u_n).\nabla(u_n-T_k(u))\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}dxdt\\
&\qquad-\int_Q \widehat{a}(u_n,\nabla T_k(u)).\nabla(u_n-T_k(u))\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}dxdt.
\end{aligned}\tag{4.35}$$

By the properties of u_n, and $|T_k(u)_\nu|\le k$, we can easily deal with the latter integral

$$\begin{aligned}
&\int_Q \widehat{a}(u_n,\nabla T_k(u)).\nabla(u_n-T_k(u))\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}dxdt\\
&=\int_Q \widehat{a}(T_{\varepsilon+k}(u),\nabla T_k(u)).\nabla(T_{\varepsilon+k}(u)-T_k(u))\chi_{\{|u-T_k(u)_\nu|\le\varepsilon\}}+\omega^{\nu,k,\varepsilon}(n)\\
&=\int_Q \widehat{a}(u,\nabla T_k(u)).\nabla(u-T_k(u))\chi_{\{|u-T_k(u)_\nu|\le\varepsilon\}}+\omega^{\nu,k,\varepsilon}(n)\\
&=\omega^{\nu,k,\varepsilon}(n),
\end{aligned}\tag{4.36}$$

due to $\widehat{a}(u,\nabla T_k(u)).\nabla(u-T_k(u))=0$. On the other hand,

$$\begin{aligned}
&\int_Q \widehat{a}(u_n,\nabla u_n).\nabla(u_n-T_k(u))\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}dxdt\\
&\quad=\int_Q \widehat{a}(u_n,\nabla u_n).\nabla(u_n-T_k(u)_\nu)\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}dxdt\\
&\qquad+\int_Q \widehat{a}(u_n,\nabla u_n).\nabla(T_k(u)_\nu-T_k(u))\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}dxdt\\
&\quad=\int_Q \widehat{a}(u_n,\nabla u_n).\nabla(u_n-T_k(u)_\nu)\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}dxdt\\
&\qquad+\omega^{k,\varepsilon}(n,\nu).
\end{aligned}\tag{4.37}$$

Observe that $|\widehat{a}(T_{\varepsilon+k}(u_n),\nabla T_{\varepsilon+k}(u_n))|$ is bounded in $L^{p'}(Q)$ and $T_k(u)_\nu$ converges to $T_k(u)$ strongly in $L^p(0,T;W_0^{1,p}(\Omega))$. By hypothesis (1.3) and the Hölder inequality, we have

$$\begin{aligned}
&\left|\int_Q \widehat{a}(u_n,\nabla u_n).\nabla(T_k(u)_\nu-T_k(u))\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}dxdt\right|\\
&\quad\le c\|\widehat{a}(T_{\varepsilon+k}(u_n),\nabla T_{\varepsilon+k}(u_n))\|_{(L^{p'}(Q))^N}\|\nabla(T_k(u)_\nu-T_k(u))\|_{(L^p(Q))^N}\\
&\quad=\omega_{k,\varepsilon}(n,\nu),
\end{aligned}$$

Thus (4.35), (4.36), and (4.37) imply that

$$\begin{aligned}
&\int_Q \Psi_{n,k}\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}dxdt\\
&\quad=\int_Q \widehat{a}(u_n,\nabla u_n).\nabla(u_n-T_k(u)_\nu)\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}dxdt+\omega^{k,\varepsilon}(n,\nu).
\end{aligned}\tag{4.38}$$

Now we use the equation solved by u_n. Taking $T_\varepsilon(u_n - T_k(u)_\nu)$ as the test function in (3.6), we obtain that

$$\langle u_n', T_\varepsilon(u_n - T_k(u)_\nu)\rangle + \int_Q \widehat{a}(u_n, \nabla u_n).\nabla T_\varepsilon(u_n - T_k(u)_\nu)dxdt$$
$$= \int_Q f_n T_\varepsilon(u_n - T_k(u)_\nu)dxdt.$$

By Lemma 2.3, whose hypotheses are satisfied by u_n, we have $\langle u_n', T_\varepsilon(u_n - T_k(u)_\nu)\rangle \geq \omega^{\nu,k,\varepsilon}(n)$. From the properties of f_n, we have $\int_Q f_n T_\varepsilon(u_n - T_k(u)_\nu)dxdt \leq c\varepsilon$. Thus $\int_Q \widehat{a}(u_n, \nabla u_n).\nabla T_\varepsilon(u_n - T_k(u)_\nu)dxdt \leq \omega(n,\nu,k,\varepsilon)$, which, together with (4.38), implies (4.34) and (4.28). □

5. Proof of Theorems 3.2

The only difficult part is the one with the operator since it is nonlinear. We start from the weak form of approximating problems (3.1), that is,

$$-\int_Q u'\varphi\,dx\,dt + \int_Q \widehat{a}(x,t,u,\nabla u).\nabla\varphi\,dx\,dt = \int_Q f\varphi\,dx\,dt, \tag{5.1}$$

for every ϕ as in Definition 3.1. From Lemma 4.2 and 4.3, we see that u_n is bounded in $L^1(0,T;W_0^{1,1})(Q)$ and u_n' is bounded in $L^1(0,T;W^{-1,1}(\Omega)) + L^1(Q)$. Thanks to (1.3) and the fact that $p-1 < 1+\gamma$, we can use [12, Corollary 4] to find that u_n is relatively compact in $L^1(Q)$. This implies that we can extract a subsequence (denote again by u_n) such that u_n converges to u strongly in $L^1(Q)$ and

$$u_n \to u \quad \text{a.e. in } Q. \tag{5.2}$$

From Theorem 4.4, we see that there exists a subsequence (still denoted u_n) such that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } Q. \tag{5.3}$$

From (5.2), (5.3) and the fact that $\widehat{a}$ is Carathéodory function, we obtain

$$\widehat{a}(x,t,u_n,\nabla u_n) \to \widehat{a}(x,t,u,\nabla u) \quad \text{a.e. in } Q. \tag{5.4}$$

Observe that $p < \gamma + 2$. If E is any measurable subset of Q, we see from (1.3) that

$$\begin{aligned}\int_E |\widehat{a}(x,t,u_n,\nabla u_n)|dx\,dt &\leq \beta\int_E [1+|\nabla u_n|^{p-1}]dx\,dt \\ &\leq c\left(\int_E [1+|\nabla u_n|]^{\gamma+1}dx\,dt\right)^{\frac{p-1}{\gamma+1}} |E|^{\frac{\gamma+2-p}{\gamma+1}} \\ &\leq +c\left(\int_E |\nabla u_n|^{\gamma+1}dx\,dt\right)^{\frac{p-1}{\gamma+1}} |E|^{\frac{\gamma+2-p}{\gamma+1}} \\ &\quad c|E|^{\frac{p-1}{\gamma+1}}|E|^{\frac{\gamma+2-p}{\gamma+1}}.\end{aligned} \tag{5.5}$$

Let E be a measurable subset of Q, and $h > 0$. It follows that

$$\begin{aligned}\int_E |\nabla u_n|^{\gamma+1}\,dx\,dt &= \int_E |\nabla T_h(u_n)|^{\gamma+1}\,dx\,dt \\ &\quad + \int_{E\cap\{|u_n|\geq h\}} |\nabla u_n|^{\gamma+1}\,dx\,dt.\end{aligned} \tag{5.6}$$

By (4.1), and the Hölder's inequality (due to $p > \gamma + 1$), we have

$$\int_E |\nabla u_n|^{\gamma+1}\,dx\,dt \le c|E|^{\frac{p-\gamma-1}{p}} \left(\int_E |\nabla T_h(u_n)|^p dx\,dt\right)^{\frac{\gamma+1}{p}}$$
$$\le c|E|^{\frac{p-\gamma-1}{p}} h^{\frac{\gamma+1}{p}} (h+1)^{\frac{\gamma(\gamma+1)}{p}}. \tag{5.7}$$

Therefore, for every $h > 0$, we have

$$\begin{aligned}
&\int_{\{|u_n|\ge h\}} |\nabla u_n|^{\gamma+1}\,dx\,dt \\
&= \int_{\{|u_n|\ge h\}} \frac{|\nabla u_n|^{\gamma+1}}{(1+|u_n|)^{\frac{(\lambda+\gamma)(\gamma+1)}{p}}} (1+|u_n|)^{\frac{(\lambda+\gamma)(\gamma+1)}{p}} dx\,dt \\
&\le \left(\int_{\{|u_n|\ge h\}} \frac{|\nabla u_n|^p}{(1+|u_n|)^{\lambda+\gamma}} dx\right)^{\frac{1}{p}} \left(\int_Q (1+|u_n|)^{\frac{\lambda+\gamma}{p-1}} dx\,dt\right)^{\frac{p-1}{p}} \\
&\le C(\lambda, h),
\end{aligned} \tag{5.8}$$

which further implies that

$$\int_{E\cap\{|u_n|\ge h\}} |\nabla u_n|^{\gamma+1}\,dx\,dt \le \int_{\{|u_n|\ge h\}} |\nabla u_n|^{\gamma+1}\,dx\,dt \le C(\lambda, h). \tag{5.9}$$

In view of (5.5)-(5.7), and (5.9), we have

$$\int_E |\widehat{a}(x,t,u_n,\nabla u_n)|dx\,dt \le c\left[|E| + |E|^{\frac{(\gamma+1)(p-\gamma-1)+(\gamma+2-p)p}{p(\gamma+1)}} h^{\frac{\gamma+1}{p}} (h+1)^{\frac{\gamma(\gamma+1)}{p}}\right]$$
$$+ c|E|^{\frac{\gamma+2-p}{\gamma+1}} C(\lambda, h). \tag{5.10}$$

We use (5.10) to choose $|E|$ small enough and choose h large enough such that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|E| < \delta$, then $\int_E |\widehat{a}(x,t,u_n,\nabla u_n)|dx\,dt \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. Thus, the sequence $\{\widehat{a}(x,t,u_n,\nabla u_n)\}$ is equi-integrable. From the Vitali theorem, it converges strongly in $L^1(Q)$ to $\widehat{a}(x,t,u,\nabla u)$. From the strong $L^1(Q)$ convergence proved in Theorem 4.4, this allows to pass to the limit in the approximate equations with thetest functions as in the statement of Theorem 3.2, and the result is proved.

References

[1] L. Boccardo, L. Orsina, Some borderline cases of nonlinear parabolic equations with irregular data, J. Evolution Equ. 16 (2015), 51-64.

[2] F. Li, Existence and regularity results for some parabolic equations with degenerate coercivity, Annales Academiae Scientiarum Fennicae, 37 (2012), 605-633.

[3] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, 1969.

[4] A. Porretta, Uniqueness and homogeneization for a class of noncoercive operators in divergence form, Atti Sem. Mat. Fis. Univ. Modena. 46 (1998), 915-936.

[5] L. Boccardo, A. Dall'Aglio, L. Orsina, Existence and regularity results for some elliptic equations with degenerate coercivity, Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia 46 (1998), 51-81.

[6] A. Alvino, L. Boccardo, V. Ferone, L. Orsina, G. Trombetti, Existence results for nonlinear elliptic equations with degenerate coercivity, Ann. Mat. Pura Appl. 182 (2003), 53-79.

[7] L. Boccardo, H. Brezis, Some remarks on a class of elliptic equations with degenerate Boll, Unione Mat. Ital. Sez. B Artic. Ric. Mat. 6 (2003), 521-530.

[8] L. Boccardo, A. Dall'Aglio, T. Gallouët, L. Orsina, Nonlinear parabolic equations with measure data, J Funct. Anal. 147 (1997), 237 258.

[9] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites nonlinéaires, Dunod, Paris, 1969.

[10] J.M. Rakotoson, A compactness lemma for quasilinear problems application to parabolic equations, J. Funct. Anal. 106 (1992), 358-374.

[11] A. Dall'Aglio, L. Orsina, Existence results for some nonlinear parabolic equations with non-regular data, Diff. Int. Equ. 5 (1992), 1335-1354.

[12] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura. Appl. 146 (1987), 65-96.