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SOME FIXED POINT THEOREMS FOR SET-VALUED GENERALIZED $\alpha_\star$-(ψ, L)-WEAK CONTRACTIONS

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Abstract. The aim of this paper is to introduce a new class of mappings called set-valued generalized $\alpha_\star$-(ψ, L)-weak contractions and to establish fixed point results for such mappings on metric spaces. Some examples are given to justify and illustrate the new results.

Keywords. $\alpha_\star$-admissible mapping; $\alpha_\star$-ψ-contractive mapping; Set-valued mapping; Fixed point.

1. INTRODUCTION

In the metric fixed point theory, the contractive conditions on underlying functions play a crucial role in finding the solutions of fixed point problems. Let (X,d) be a metric space and $f\colon X \to X$ be a mapping. The mapping f is said to be a Banach contraction on X if the following contractive condition is satisfied: there exists $\lambda \in [0,1)$ such that

$$d(fx, fy) \leq \lambda d(x,y) \text{ for all } x,y \in X. \tag{1.1}$$

The contractive condition (1.1) is called the Banach contractive condition.

In 1922, Banach [1] proved that a Banach contraction on a complete metric space (X,d) has a unique fixed point in X, i.e., there is a unique point $x^* \in X$ such that $fx^* = x^*$. This theorem about the existence and uniqueness of fixed points of a mapping is known as the famous Banach contraction principle. Since the contractive condition (1.1) is satisfied for all $x,y \in X$, every Banach contraction is continuous (indeed, uniformly continuous) on X.

1968, Kannan [2] used the following contractive condition: there exists $\lambda \in [0,1/2)$ such that

$$d(fx, fy) \leq \lambda [d(x, fx) + d(y, fy)] \text{ for all } x,y \in X.$$

The mappings satisfying the above condition is called a Kannan contraction. Note that the Kannan contractions need not be even continuous. Kannan proved that the mappings on a

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complete metric space satisfying the above contractive condition have a unique fixed point, and so, the conclusion of Banach contraction principle still holds.

1972, Chatterjea [3] showed that one can use the following contractive condition: there exists $\lambda \in [0,1/2)$ such that

$$d(fx,fy) \le \lambda[d(x,fy)+d(y,fx)] \text{ for all } x,y \in X$$

and the conclusion of Banach contraction principle still holds. A mapping satisfying the above contractive condition is called a Chatterjea contraction.

1972, Zamfirescu [4] used the following condition: there exist $\lambda \in [0,1)$, $\mu \in [0,1/2)$ and $\delta \in [0,1/2)$ such that at least one of the following conditions is true:

$$\begin{aligned} d(fx,fy) &\le \lambda d(x,y) \text{ for all } x,y \in X; \\ d(fx,fy) &\le \mu[d(x,fx)+d(y,fy)] \text{ for all } x,y \in X; \\ d(fx,fy) &\le \delta[d(x,fy)+d(y,fx)] \text{ for all } x,y \in X. \end{aligned}$$

A mapping satisfying the above condition is called a Zamfirescu contraction. Again, Zamfirescu contractions on a complete metric space have a unique fixed point.

In 2004, Berinde [5] introduced the notion of weak contractions. Let (X,d) be a metric space and $f\colon X \to X$ be a mapping. Then the mapping f is called a weak contraction or (θ,L)-weak contraction if there exist two constants $\theta \in (0,1)$ and $L \ge 0$ such that

$$d(fx,fy) \le \theta d(x,y) + Ld(y,fx) \text{ for all } x,y \in X. \tag{1.2}$$

Due to the symmetry of the distance, the weak contraction condition (1.2) implicitly includes the following dual inequality:

$$d(fx,fy) \le \theta d(x,y) + Ld(x,fy) \text{ for all } x,y \in X. \tag{1.3}$$

Therefore, in order to check the weak contractiveness of a given operator, it is necessary to check both the conditions. Berinde [5] showed that the weak contractions are a generalization of Banach contractions, Kannan contractions [2], Chatterjea contractions [3] and Zamfirescu contractions [4]. Berinde showed that the weak contractions on a complete metric space have a fixed point which is not necessarily unique.

Over the years, Banach contraction principle has been generalized in various directions by several mathematicians. In particular, Ran and Reurings [6], and Nieto and Lopéz [7] considered a partial order relation defined on X and assumed that the contractive condition (1.1) is satisfied for only those x and y which can be compared by the underlying partial order. They used mappings, which are monotone with respect to the underlying partial order.

In 2012, Samet et al. [8] introduced the notions of α-admissible mappings. Such mappings are a generalization of monotone mappings. They obtained corresponding fixed point results for α-ψ-contractive mappings. The results of Samet et al. [8] are generalizations of ordered fixed point results of Ran and Reurings [6] and Nieto and Lopéz [7]. Since then, by using their idea, several authors investigated fixed point results on various spaces.

On the other hand, Nadler [9] generalized the Banach contraction principle for the mappings under which the points of underlying space mapped into nonempty subsets of space. Such mappings are called set-valued mappings or multifunctions.

Let (X,d) be a metric space and $CB(X)$ denote the set of all nonempty closed and bounded subsets of X. The distance between a point $x \in X$ and a set A is defined by

$$d(x,A) = \inf\{d(x,a) \colon a \in A\}.$$

A function $H\colon CB(X) \times CB(X) \to [0,\infty)$ defined by

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\} \text{ for all } A,B \in CB(X)$$

is called the Pompeiu-Hausdorff metric on $CB(X)$. Note that, H is indeed a metric on $CB(X)$.

Remark 1.1. It is obvious that $d(a,B) \le d(a,c) + d(c,B)$ and $d(a,B) \le d(a,C) + H(C,B)$ for all $a,c \in X$ and $B,C \in CB(X)$.

A mapping $T\colon X \to CB(X)$ is called a Nadler contraction if it satisfies the following condition: there exists $\lambda \in [0,1)$ such that

$$H(Tx,Ty) \le \lambda d(x,y) \text{ for all } x,y \in X$$

where H is the Pompeiu-Hausdorff metric on $CB(X)$.

In [9], it was shown that the Nadler contractions on a complete metric space have at least one fixed point (i.e., a point $x^* \in X$ such that $x^* \in Tx^*$). He proved an extension of Banach contraction principle for set-valued mappings. Hasanzade et al. [10] extended the notion of α-admissible mappings to set-valued mappings by introducing the notion of $\alpha_\star$-admissible mappings. They proved some fixed point results for $\alpha_\star$-ψ-contractive mappings which generalize the fixed point results of Samet et al. [8]. In 2014, Hussain et al.[11] extended these modified notions.

Inspired by the work of Hussain et al. [11], in this paper we introduce a new class of set-valued mappings called set-valued generalized $\alpha_\star$-(ψ,L)-weak contractions. This new class generalizes the class of mappings introduced by Hussain et al. [11] and extend the class of contractions of Kannan [2], Chatterjea [3], Zamfirescu [4], Berinde [5], Ran and Reurings [6] and Nieto and Lopéz [7] and Samet et al. [8] to their set-valued versions. Fixed point results for the mappings of this new class are proved which generalize and extend fixed point results for the contractions said above. Some examples are given which justify and illustrate our claims.

2. Preliminaries

In this section, we collect some definitions about α-admissible, $\alpha_\star$-admissible and $\alpha_\star$-ψ-contractive mappings.

Samet et al. [8] defined the notion of α-admissible mappings as follows.

Definition 2.1 (Samet et al. [8])**.** Let T be a self-mapping on X and $\alpha : X \times X \to [0,\infty)$ be a function. We say that T is a α-admissible mapping if

$$x,y \in X, \alpha(x,y) \ge 1 \Rightarrow \alpha(Tx,Ty) \ge 1.$$

Denote by Ψ the family of increasing functions $\psi\colon [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ. It is well known that if $\psi \in \Psi$, then $\psi(t) = 0$ if and only if $t = 0$, and $\psi(t) < t$ for all $t \in (0,\infty)$.

If (X,d) is a complete metric space, then T is called an α-ψ-contraction if there exists $\psi \in \Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y)) \text{ for all } x,y \in X.$$

Afterwards, Hasanzade et al. [10] generalized these notions by introducing the concepts of $\alpha_\star$-ψ-contractive multifunction and $\alpha_\star$-admissible mappings. They obtained some fixed point results for this type of multifunction.

Definition 2.2 (Hasanzade et al. [10]). Let (X,d) be a metric space, $T: X \to 2^X$ be a given closed-valued multifunction. We say that T is $\alpha_\star$-ψ-contractive multifunction if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha_\star(Tx,Ty)H(Tx,Ty) \leq \psi(d(x,y))$$

for all $x,y \in X$, where $\alpha_\star(A,B) = \inf\{\alpha(a,b) : a \in A, b \in B\}$ and 2^X denotes the family of all nonempty subsets of X.

Definition 2.3 (Hasanzade et al. [10]). Let (X,d) be a metric space. Let $T : X \to 2^X$ be a given closed-valued multifunction and $\alpha : X \times X \to [0,\infty)$. We say that T is $\alpha_\star$-admissible whenever $\alpha(x,y) \geq 1$ implies that $\alpha_\star(Tx,Ty) \geq 1$.

Lemma 2.4 (Nadler [9]). *Let A and B be nonempty, closed and bounded subsets of a metric space (X,d) and $0 < h \in \mathbb{R}$. Then, for every $b \in B$, there exists $a \in A$ such that $d(a,b) \leq H(A,B) + h$.*

Lemma 2.5 (Ali and Kamran [12]). *Let (X,d) be a metric space and B be nonempty, closed subsets of X and $q > 1$. Then, for each $x \in X$ with $d(x,B) > 0$ and $q > 1$, there exists $b \in B$ such that $d(x,b) < qd(x,B)$.*

Hussain et al. [11] proved fixed point results for $\alpha_\star$-admissible and generalized the result of Samet et al. [8].

Let (X,d) be a metric space and $T\colon X \to CB(X)$ be a mapping. The set

$$G(T) = \{(x,y)\colon x \in X, y \in Tx\} \subseteq X \times X$$

is called the graph of mapping T. The mapping T is called a closed mapping if the $G(T)$ is a closed subset of $X \times X$.

A sequence $\{x_n\}$ in a metric space (X,d) is called a trajectory of a mapping $T\colon X \to 2^X$, starting at x_1 if $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}$. By $\mathscr{T}(x,T)$, we denote the class of all trajectories of T starting at $x \in X$.

Definition 2.6 (Shukla et al. [13]). Let (X,d) be a metric space, $\alpha\colon X \times X \to [0,\infty)$ a function and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called an α-sequence if $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. By $\mathfrak{A}_\alpha$, we denote the class of all α-sequences in X. For a closed-valued mapping $T\colon X \to 2^X$, we denote by $G_\alpha(T)$ the α-graph of T and

$$G_\alpha(T) = \{(x_n,x_{n+1}) \in X \times X\colon x_1 \in X \text{ and } \{x_n\} \in \mathfrak{A}_\alpha \cap \mathscr{T}(x_1,T)\}.$$

The mapping T is called an α-closed mapping if $G_\alpha(T)$ is a closed subset of $X \times X$.

The closed mappings are a particular case of α-closed mappings, but the converse may not be true (see, [13]).

Definition 2.7 (Shukla et al. [13])**.** Let (X,d) be a metric space and $\alpha\colon X\times X\to[0,\infty)$ be a function. Then (X,d) is called α-complete if every Cauchy sequence in $\mathfrak{A}_\alpha$ is convergent in $X\times X$.

Completeness of a metric space is a particular case of α-completeness. Again, the converse may not be true (see, [13]).

3. Main results

We first introduce the set-valued generalized (ψ,L)-weak contractions on metric spaces. Let (X,d) be a metric space and $T\colon X\to 2^X$ be a mapping. Throughout the paper, we use the notation: for $x,y\in X$

$$M(x,y,T)=\max\left\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty)d(y,Tx),\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}.$$

Definition 3.1. Let (X,d) be a metric space and $T\colon X\to 2^X$ be a mapping. We say that T is a set-valued generalized (ψ,L)-weak contraction if there exist $\psi\in\Psi$ and $L\geq 0$ such that

$$H(Tx,Ty)\leq\psi(M(x,y,T))+Ld(y,Tx)\text{ for all }x,y\in X. \tag{3.1}$$

We next introduce the set-valued generalized $\alpha_\star$-(ψ,L)-weak contractions on metric spaces which unifies the contractive condition of Hussain et al. [11] and the contractive condition of Berinde [5].

Definition 3.2. Let (X,d) be a metric space, $T\colon X\to 2^X$ a mapping, and $\alpha: X\times X\to[0,\infty)$ be a function. We say that T is a set-valued generalized $\alpha_\star$-(ψ,L)-weak contraction if there exist $\psi\in\Psi$ and $L\geq 0$ such that

$$H(Tx,Ty)\leq\psi(M(x,y,T))+Ld(y,Tx)\text{ for all }x,y\in X\text{ with }\alpha_\star(Tx,Ty)\geq 1. \tag{3.2}$$

Remark 3.3. Consider the following contractive condition:

$$\alpha_\star(Tx,Ty)H(Tx,Ty)\leq\psi(M(x,y,T))+Ld(y,Tx) \tag{3.3}$$

for all $x,y\in X$. The most obvious way to unify the contractive condition of Hussain et al. [11] and the contractive condition of Berinde [5] is to define the condition (3.3), but we do not unify in this obvious manner. In fact, we see that the condition (3.3) implies the condition (3.2), but the converse is not true in general (see the example below), and so, the contractive condition (3.2) is more general than the condition (3.3).

Example 3.4. Let $X=\mathbb{R}$ and $d\colon X\times X\to\mathbb{R}$ be the usual metric on X. Define a function $\alpha\colon X\times X\to[0,\infty)$ by:

$$\begin{aligned}\alpha(0,x)&=1\text{ for all }x\in X;\\ \alpha(x,0)&=\frac{1}{x}\text{ if }x\in(1,\infty);\\ \alpha(x,y)&=0\text{ in all other cases.}\end{aligned}$$

Consider the mapping $T\colon X\to 2^X$ defined by $Tx=\{0,x\}$ for all $x\in X$.

If $\alpha_\star(Tx,Ty)\geq 1$, then we must have $1\leq\alpha_\star(x,y)\leq\alpha(a,b)$ for all $a\in Tx,b\in Ty$, and so, $x=0$ and $y\in X$. Therefore, $H(Tx,Ty)=H(\{0\},\{0,y\})=|y|$. Also, $d(x,y)=|y|$, $d(x,Tx)=$

$d(0,\{0\})=0$, $d(y,Ty)=d(y,\{0,y\})=0$, $d(y,Tx)=d(y,\{0\})=|y|$ and $d(x,Ty)=d(0,\{0,y\})=0$. Hence

$$\psi(M(x,y,T))+Ld(y,Tx)=\psi(|y|)+L|y|.$$

Therefore, T is a set-valued generalized $\alpha_\star$-(ψ,L)-weak contraction with $\psi(t)=kt$ for all $t\in[0,\infty)$, and $L\geq 1-k$, where $k\in[0,1)$.

On the other hand, we choose $x\in X$ and $y=0$. Then $H(Tx,Ty)=H(\{0,x\},\{0\})=|x|$, $d(x,y)=|x|$, $d(x,Tx)=d(x,\{0,x\})=0$, $d(y,Ty)=d(0,\{0\})=0$, $d(y,Tx)=d(0,\{0,x\})=0$ and $d(x,Ty)=d(x,\{0\})=|x|$. Hence

$$\psi(M(x,y,T))+Ld(y,Tx)=\psi(|x|).$$

If (3.3) holds, then

$$|x|\leq\psi(|x|)\ \text{for all}\ x\in X.$$

This is a contradiction since $\psi(t)<t$ for all $t\in(0,\infty)$. Hence, (3.3) is not satisfied. Thus, the contractive condition (3.2) is more general than contractive condition (3.3).

Remark 3.5. As we have seen that due to the symmetry of the distance, the weak contractive condition (1.2) of Berinde [5] implicitly includes the dual one, i.e., the contractive condition (1.3). Consequently, in order to check the weak contractiveness of mapping, it is necessary to check both (1.2) and (1.3). Again, because of symmetry of distances, this fact is true for a set-valued generalized (ψ,L)-weak contraction as well, i.e., if the contractive condition (3.1) is satisfied, then so its dual one. In case of a set-valued generalized $\alpha_\star$-(ψ,L)-weak contraction the contractive condition (3.2) is satisfied only for those $x,y\in X$ for which $\alpha_\star(Tx,Ty)\geq 1$. The benefit of this constraint is that it is not necessary to check the contractive condition (3.2) for all $x,y\in X$. In particular, if$\alpha(x,y)$ is not symmetric in x,y, then a set-valued generalized $\alpha_\star$-(ψ,L)-weak contraction need not be a set-valued generalized (ψ,L)-weak contraction as shown in the following example.

Example 3.6. Let $X=[0,1]$ and d is the usual metric on X. Define $T\colon X\to 2^X$ by

$$Tx=\begin{cases}\left\{0,\dfrac{2}{3}\right\}, & \text{if}\quad x\in[0,1);\\ \{1\}, & \text{if}\quad x=1.\end{cases}$$

and $\alpha\colon X\times X\to[0,\infty)$ by

$$\alpha(x,y)=\begin{cases}1, & \text{if}\quad x,y\in[0,1], x\leq y;\\ 0, & \text{otherwise}.\end{cases}$$

Then, we first show that T is a set-valued generalized $\alpha_\star$-(ψ,L)-weak contraction. If $\alpha_\star(x,y)\geq 1$, then we must have $x=y$ or; $x\in[0,1], y=1$. We consider the following cases:

- If $x=y$, then $H(Tx,Ty)=0$, hence (3.2) satisfies trivially.
- If $x\in[0,1), y=1$, then by routine calculations one can see that T is a set-valued generalized $\alpha_\star$-(ψ,L)-weak contraction with arbitrary $\psi\in\Psi$ and $L\geq 3$.
- Now, to consider the dual one of contractive condition (3.1), take $x=1, y\in[0,1)$ (see previous case), then $H(Tx,Ty)=1$, $d(x,y)=1-y$, $d(x,Tx)=0$, $d(y,Ty)=\inf\{y,|y-2/3|\}$, $d(y,Tx)=1-y$ and $d(x,Ty)=1/3$. Hence

$$\psi(M(x,y,T))+Ld(y,Tx)=\psi(\max\{1-y,\inf\{y,|y-2/3|\}\})+L(1-y).$$

If (3.1) holds, then there exists $L \geq 0$ such that:

$$1 \leq \psi(\max\{1-y, \inf\{y, |y-2/3|\}\}) + L(1-y).$$

Therefore, for those values of y, which are sufficient close to 1, we must have

$$1 - L(1-y) \leq \psi(|y-2/3|) \leq \psi(1).$$

Letting $y \to 1-$, we have the above inequality yields $1 \leq \psi(1)$, a contradiction. Hence, the dual one of contractive condition (3.1) is not satisfied. So, T is not a set-valued generalized (ψ, L)-weak contraction.

We next prove the main results of this paper.

Theorem 3.7. Let (X,d) be a α-complete metric space. Suppose that $T\colon X \to CB(X)$ is a set-valued generalized $\alpha_\star$-(ψ,L)-weak contraction and the following conditions are satisfied:

(I) T is $\alpha_\star$-admissible.
(II) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0,x_1) \geq 1$;
(III) for a trajectory $\{x_n\} \in \mathfrak{A}_\alpha$, starting at x_1 and converging to $x \in X$, we have $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. By (II), there is $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$. If $x_0 = x_1$, then we are done. Let $x_0 \neq x_1$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T. Assume that $x_1 \notin Tx_1$. Since T is $\alpha_\star$-admissible and $\alpha(x_0,x_1) \geq 1$, therefore, we have $\alpha_\star(Tx_0,Tx_1) \geq 1$. Hence

$$0 < d(x_1,Tx_1) \leq H(Tx_0,Tx_1) \leq \psi(M(x_0,x_1,T)) + Ld(x_1,Tx_0),$$

where

$$\begin{aligned} M(x_0,x_1,T) &= \max\Big\{d(x_0,x_1), d(x_0,Tx_0), d(x_1,Tx_1), d(x_0,Tx_1)d(x_1,Tx_0), \\ &\qquad \frac{d(x_0,Tx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)}\Big\} \\ &\leq \max\left\{d(x_0,x_1), d(x_0,x_1), d(x_1,Tx_1), \frac{d(x_0,x_1)d(x_1,Tx_1)}{1+d(x_0,x_1)}\right\} \\ &= \max\{d(x_0,x_1), d(x_1,Tx_1)\}. \end{aligned}$$

If $\max\{d(x_0,x_1), d(x_1,Tx_1)\} = d(x_1,Tx_1)$, then we find from $\psi(t) < t$ for all $t > 0$ that

$$0 < d(x_1,Tx_1) \leq \psi(d(x_1,Tx_1)).$$

This is a contradiction. Therefore, $\max\{d(x_0,x_1), d(x_1,Tx_1)\} = d(x_0,x_1)$, and so

$$d(x_1,Tx_1) \leq \psi(d(x_0,x_1)).$$

Choosing $\mu > 1$, we find from Lemma 2.5 that there exist $x_2 \in Tx_1$ such that

$$d(x_1,x_2) < \mu d(x_1,Tx_1) \leq \mu\psi(d(x_0,x_1)).$$

Note that, if $x_1 = x_2$, then we are done. If $x_1 \neq x_2$, then $d(x_1,x_2) > 0$. Let $\mu_1 = \dfrac{\psi(\mu\psi(d(x_0,x_1)))}{\psi(d(x_1,x_2))}$. Then, $\mu_1 > 1$ and since $x_1 \in Tx_0, x_2 \in Tx_1$, $\alpha_\star(Tx_0,Tx_1) \geq 1$, we have $\alpha(x_1,x_2) \geq 1$. Since T is $\alpha_\star$-admissible, then $\alpha_\star(Tx_1,Tx_2) \geq 1$.

If $x_2 \in Tx_2$, then x_2 is a fixed point of T. Assume that $x_2 \notin Tx_2$. Hence

$$0 < d(x_2, Tx_2) \leq H(Tx_1, Tx_2) \leq \psi(M(x_1, x_2, T)) + Ld(x_2, Tx_1),$$

where

$$\begin{aligned} M(x_1, x_2, T) &= \max\Big\{d(x_1, x_2), d(x_1, Tx_1), d(x_2, Tx_2), d(x_1, Tx_2)d(x_2, Tx_1), \\ &\quad \frac{d(x_1, Tx_1)d(x_2, Tx_2)}{1 + d(x_1, x_2)}\Big\} \\ &\leq \max\left\{d(x_1, x_2), d(x_1, x_2), d(x_2, Tx_2), \frac{d(x_1, x_2)d(x_2, Tx_2)}{1 + d(x_1, x_2)}\right\} \\ &= \max\{d(x_1, x_2), d(x_2, Tx_2)\}. \end{aligned}$$

If $\max\{d(x_1, x_2), d(x_2, Tx_2)\} = d(x_2, Tx_2)$, then we find from $\psi(t) < t$ for all $t > 0$ that

$$0 < d(x_2, Tx_2) \leq \psi(d(x_2, Tx_2)).$$

This is a contradiction. Therefore, $\max\{d(x_1, x_2), d(x_2, Tx_2)\} = d(x_1, x_2)$, and so

$$d(x_2, Tx_2) \leq \psi(d(x_1, x_2)).$$

Since $\mu_1 > 1$, we find from Lemma 2.5 that there exist $x_3 \in Tx_2$ such that

$$d(x_2, x_3) < \mu_1 d(x_2, Tx_2) \leq \mu_1 \psi(d(x_1, x_2)) = \psi(\mu\psi(d(x_0, x_1))).$$

Continuing in this way, we generate a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, $x_n \neq x_{n-1}$, $\alpha(x_{n-1}, x_n) \geq 1$ for all $n \in \mathbb{N}$ and

$$d(x_n, x_{n+1}) \leq \psi^{n-1}(\mu\psi(d(x_0, x_1))) \text{ for all } n \in \mathbb{N}.$$

We next show that the sequence $\{x_n\}$ is a Cauchy sequence. Then, for $n, m \in \mathbb{N}$ and $m > n$,

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \psi^{i-1}(\mu\psi(d(x_0, x_1))).$$

Since $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, the above inequality implies that $\{x_n\}$ is a Cauchy sequence in X. Thus, $\{x_n\} \in \mathfrak{A}_\alpha \cap \mathscr{T}(x_0, T)$ is a Cauchy sequence. Since X is α-complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$.

We now show that $u \in Tu$. If $d(u, Tu) = 0$, then we are done. Suppose that $d(u, Tu) > 0$. Since $\{x_n\} \in \mathfrak{A}_\alpha \cap \mathscr{T}(x_0, T)$ and $x_n \to u$ as $n \to \infty$. From (III), we have $\alpha(x_n, u) \geq 1$ for all $n \in \mathbb{N}$. From the $\alpha_\star$-admissibility of T, we have $\alpha_\star(Tx_n, Tu) \geq 1$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} d(u, Tu) &\leq d(x_{n+1}, u) + d(x_{n+1}, Tu) \leq H(Tx_n, Tu) + d(x_{n+1}, u) \\ &\leq \psi(M(x_n, u, T)) + Ld(u, Tx_n) + d(x_{n+1}, u) \\ &\leq \psi(M(x_n, u, T)) + Ld(u, x_{n+1}) + d(x_{n+1}, u), \end{aligned}$$

where

$$\begin{aligned} M(x_n,u,T) &= \max\Big\{d(x_n,u),d(x_n,Tx_n),d(u,Tu),d(x_n,Tu)d(u,Tx_n),\\ &\quad \frac{d(x_n,Tx_n)d(u,Tu)}{1+d(x_n,u)}\Big\}\\ &\le \max\Big\{d(x_n,u),d(x_n,x_{n+1}),d(u,Tu),d(x_n,Tu)d(u,x_{n+1}),\\ &\quad \frac{d(x_n,x_{n+1})d(u,Tu)}{1+d(x_n,u)}\Big\}. \end{aligned}$$

Since $x_n \to u$ as $n\to\infty$, we have that there exists $n_0\in\mathbb{N}$ such that

$$\max\left\{d(x_n,u),d(x_n,x_{n+1}),d(u,Tu),d(x_n,Tu)d(u,x_{n+1}),\frac{d(x_n,x_{n+1})d(u,Tu)}{1+d(x_n,u)}\right\}=d(u,Tu)$$

for all $n\ge n_0$. Therefore,

$$d(u,Tu)\le\psi(d(u,Tu))+Ld(u,x_{n+1})+d(x_{n+1},u) \text{ for all } n\ge n_0.$$

Letting $n\to\infty$, we obtain $d(u,Tu)\le\psi(d(u,Tu))$. Since $d(u,Tu)>0$, the above inequality yields a contradiction. Hence, $u\in Tu$, i.e., $u\in X$ is a fixed point T. □

The following example illustrates the above theorem and shows that the above theorem is a proper generalization of the known results.

Example 3.8. Let $X=[0,2]$ and let the metric d on X be defined by $d(x,y)=\max\{x,y\}$ when $x\neq y$ and $d(x,x)=0$ for all $x\in X$. Define $T: X\to 2^X$ by

$$Tx=\begin{cases}[0,x/2], & \text{if } x\in[0,1);\\ \{1\}, & \text{if } x=1;\\ [0,\sqrt{x}], & \text{otherwise.}\end{cases}$$

and

$$\alpha(x,y)=\begin{cases}1, & \text{if } x,y\in[0,1];\\ |x-y|, & \text{otherwise.}\end{cases}$$

We show that all the conditions of Theorem 3.7 are satisfied. Then:

(I) If $\alpha(x,y)\ge 1$, then by definition we must have $x,y\in[0,1]$. In this case, $Tx,Ty\subseteq[0,1]$, and so $\alpha_\star(Tx,Ty)=\inf\{\alpha(a,b): a\in Tx, b\in Ty\}=1$. Hence, T is $\alpha_\star$-admissible.

(II) Letting $x_0\in[0,1)$, we have $Tx_0=[0,x_0/2]$. Suppose, $x_1\in Tx_0$. Then $x_1\in[0,x_0/2]\subseteq[0,1]$ and $\alpha(x_0,x_1)=1$. Hence, condition (II) of Theorem 3.7 is satisfied.

(III) Assume that $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\ge 1$ and $\{x_n\}$ converges to $x\in X$. From the definition, we must have $x_n\in[0,1]$. So, $x\in[0,1]$. This shows that $\alpha(x_n,x)\ge 1$ for all $n\in\mathbb{N}$. Hence, condition (III) of Theorem 3.7 is satisfied.

Finally, we show that T is a set-valued generalized $\alpha_\star$-(ψ,L)-weak contraction with $\psi(t)=\frac{t}{2}$ for all $t\in[0,\infty)$ and $L\ge\frac{1}{2}$. If $\alpha_\star(x,y)\ge 1$, then $x,y\in[0,1]$. So, we consider the following cases:

- If $x=y$, then $H(Tx,Ty)=0$. Hence (3.2) satisfies trivially.

- If $x,y \in [0,1)$ and $x<y$, then $H(Tx,Ty) = H([0,x/2],[0,y/2]) = y/2$.
  Also, $d(x,y)=y, d(x,Tx)=x, d(y,Ty)=y$, $d(y,Tx)=d(y,[0,x/2])=y$ and
$$d(x,Ty)=d(x,[0,y/2])=\begin{cases} 0, & \text{if } x\le y/2;\\ x, & \text{if } x>y/2.\end{cases}$$
  Hence (3.2) satisfies for $L \geq \frac{1}{2}$ and $\psi(t)=\frac{t}{2}$ for all $t\in[0,\infty)$.
- If $x,y \in [0,1)$ and $x>y$, then $H(Tx,Ty) = H([0,x/2],[0,y/2]) = x/2$.
  Also, $d(x,y)=x, d(x,Tx)=x, d(y,Ty)=y$, $d(x,Ty)=d(x,[0,y/2])=x$ and
$$d(y,Tx)=d(y,[0,x/2])=\begin{cases} 0, & \text{if } y\le x/2;\\ y, & \text{if } y>x/2.\end{cases}$$
  Hence (3.2) satisfies for $L \geq \frac{1}{2}$ and $\psi(t)=\frac{t}{2}$ for all $t\in[0,\infty)$.
- If $x\in[0,1)$, $y=1$, then $H(Tx,Ty)=H([0,x/2],\{1\})=1$.
  Also, $d(x,y)=1, d(x,Tx)=x, d(y,Ty)=1$, $d(y,Tx)=d(1,[0,x/2])=1$ and $d(x,Ty)=d(x,\{1\})=1$. Hence (3.2) satisfies for $L \geq \frac{1}{2}$ and $\psi(t)=\frac{t}{2}$ for all $t\in[0,\infty)$.
- If $y\in[0,1)$, $x=1$, then $H(Tx,Ty)=H(\{1\},[0,y/2])=1$.
  Also, $d(x,y)=1, d(x,Tx)=1, d(y,Ty)=y$, $d(y,Tx)=d(y,\{1\})=1$ and $d(x,Ty)=d(1,[0,y/2])=1$. Hence (3.2) satisfies for $L \geq \frac{1}{2}$ and $\psi(t)=\frac{t}{2}$ for all $t\in[0,\infty)$.

Hence, T is a set-valued generalized $\alpha_\star$-(ψ,L)-weak contraction with $\psi(t)=\frac{t}{2}$ for all $t\in[0,\infty)$ and $L\geq\frac{1}{2}$.

Thus, all the conditions of Theorem 3.7 are satisfied therefore we can conclude the existence of fixed point of T. Note that, the set of all fixed points of T, i.e., $\text{Fix}(T)=\{0,1\}$.

On the other hand, it is easy to see that T does not satisfy the contractive condition of Hussain et al. [11], hence result of Hussain et al. [11] is not applicable here.

Corollary 3.9. *Let (X,d) be a α-complete metric space. Suppose that $T\colon X\to CB(X)$ is a mapping and the following conditions are satisfied:*

(I) there exists $k\in(0,1)$ and $L\geq 0$ such that
$$H(Tx,Ty)\le kd(x,y)+Ld(y,Tx) \text{ for all } x,y\in X.$$
(II) T is $\alpha_\star$-admissible.
(III) there exist $x_0\in X$ and $x_1\in Tx_0$ with $\alpha(x_0,x_1)\geq 1$;
(IV) for a trajectory $\{x_n\}\in\mathfrak{A}_\alpha$, starting at x_1 and converging to $x\in X$, we have $\alpha(x_n,x)\geq 1$ for all $n\in\mathbb{N}$.

Then T has a fixed point.

Proof. Letting $\psi(t)=\frac{t}{2}$, we find Theorem 3.7 the desired conclusion immediately. □

Remark 3.10. Following the same arguments given by Berinde [5], and using the Remark 1.1 and Corollary 3.9, one can obtain the generalized set-valued versions of the results of Kannan [2], Chatterjea [3] and Zamfirescu [4] for $\alpha_\star$-admissible mappings.

In the next theorem, we show that the condition (III) of Theorem 3.7 can be replaced by α-closedness of the mapping T.

Theorem 3.11. *Let (X,d) be a α-complete metric space. Suppose that $T\colon X\to CB(X)$ is a set-valued generalized $\alpha_\star$-(ψ,L)-weak contraction and the following conditions are satisfied:*

(I) T is $\alpha_\star$-admissible.
(II) there exist $x_0\in X$ and $x_1\in Tx_0$ with $\alpha(x_0,x_1)\geq 1$;
(III) T is a α-closed mapping.

Then T has a fixed point.

Proof. Following the proof of Theorem 3.7, we obtain a sequence $\{x_n\}\in\mathfrak{A}_\alpha\cap\mathscr{T}(x_0,T)$ such that $x_n\to u\in X$ as $n\to\infty$. We shall show that u is a fixed point of T. Since $\{x_n\}\in\mathfrak{A}_\alpha\cap\mathscr{T}(x_0,T)$, therefore by definition $G_\alpha(T)$ we have

$$(x_{n-1},x_n)\in G_\alpha(T)\text{ for all }n\in\mathbb{N}.$$

Since T is a α-closed mapping, we have that the α-graph $G_\alpha(T)$ is a closed subset of $X\times X$. Hence, letting $n\to\infty$ and using the fact that $x_n\to u$ as $n\to\infty$ in the above inclusion, we obtain $\lim_{n\to\infty}(x_{n-1},x_n)\in G_\alpha(T)\implies(u,u)\in G_\alpha(T)$. Since $(u,u)\in G_\alpha(T)$, we have $u\in Tu$. Thus, u is a fixed point of T. ☐

We next derive the ordered versions of our main results. First we introduce some definitions on metric spaces endowed with a partial order.

Let $(X,\preceq)$ be a poset, $A,B\subseteq X$ be nonempty and $T\colon X\to 2^X$. Then, we write $A\sqsubseteq B$ if and only if $a\preceq b$ for all $a\in A$ and $b\in B$. We say that the mapping T is order preserving if $x\preceq y$ implies $Tx\sqsubseteq Ty$ for all $x,y\in X$.

Definition 3.12. Let $(X,\preceq)$ be a poset such that d be a metric on X and $T\colon X\to 2^X$ a mapping. We say that T is a set-valued generalized ordered-(ψ,L)-weak contraction if there exist $\psi\in\Psi$ and $L\geq 0$ such that

$$H(Tx,Ty)\leq\psi(M(x,y,T))+Ld(y,Tx)\tag{3.4}$$

for all $x,y\in X$ with $Tx\sqsubseteq Ty$.

Let $\{x_n\}$ be a sequence in X. Then the sequence $\{x_n\}$ is called an ordered sequence if $x_n\preceq x_{n+1}$ for all $n\in\mathbb{N}$. By $\mathfrak{A}_\preceq$ we denote the class of all ordered sequences in X. For the mapping T, we denote by $G_\preceq(T)$ the $\preceq$-graph of T and $G_\preceq(T)=\{(x_n,x_{n+1})\in X\times X\colon x_1\in X\text{ and }\{x_n\}\in\mathfrak{A}_\preceq\cap\mathscr{T}(x_1,T)\}\}$. The mapping T is called $\preceq$-closed if $G_\preceq(T)$ is a closed subset of $X\times X$. The space (X,d) is called $\preceq$-complete if every Cauchy sequence in $\mathfrak{A}_\preceq$ is convergent in $X\times X$ (see, Shukla et al. [13]).

The next corollary is an improved set-valued version of result of Ran and Reurings [6] and Nieto and Rodríguez Lopéz [7].

Corollary 3.13. Let (X,d) be a $\preceq$-complete metric space. Suppose that $T\colon X\to CB(X)$ is a set-valued generalized ordered (ψ,L)-weak contraction and the following conditions are satisfied:

(I) T is order preserving.
(II) there exist $x_0\in X$ and $x_1\in Tx_0$ with $x_0\preceq x_1$;
(III) at least one of the following conditions is satisfied:
 (a) for a trajectory $\{x_n\}\in\mathfrak{A}_\preceq$, starting at x_1 and converging to $x\in X$, we have $x_n\preceq x$ for all $n\in\mathbb{N}$.

(b) T is a $\preceq$-closed mapping.

Then T has a fixed point.

Proof. Define a function $\alpha\colon X\times X\to[0,\infty)$ by

$$\alpha(x,y)=\begin{cases}1, & \text{if } x\preceq y;\\ 0, & \text{otherwise.}\end{cases}$$

Then, if $\alpha(x,y)\geq 1$, i.e., $x\preceq y$, then by the order perverseness of T , $Tx\sqsubseteq Ty$. So, $\alpha_\star(Tx,Ty)=\inf\{\alpha(a,b)\colon a\in Tx, b\in Ty\}=1$. This shows that T is $\alpha_\star$-admissible. Again, condition (II) shows that there exists $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\geq 1$. Also, if (a) of (III) is satisfied, then for any trajectory $\{x_n\}\in\mathfrak{A}_\alpha$ converging to $x\in X$ we must have $\alpha(x_n,x)\geq 1$ for all $n\in\mathbb{N}$. $\preceq$-closedness of T implies the α-closedness of T and $\preceq$-completeness of (X,d) implies its α-completeness. Finally, since T is a set-valued generalized ordered (ψ,L)-weak contraction, by definition of α the mapping T is a set-valued generalized $\alpha_\star$-(ψ,L)-weak contraction. Hence, the result follows from Theorem 3.7 and Theorem 3.11. □

We next consider another version of our main results for the metric spaces with which a graphical structure is associated.

Let X be a nonempty set. Let $\mathscr{G}=(V,E)$ be a graph defined by the set of vertex $V=X$, the set of edges $E\subseteq X\times X$ with the diagonal $\Delta=\{(x,x)\colon x\in X\}\subseteq E$ and E has no parallel edges. In this case, we say that X is endowed with the graph $\mathscr{G}$. Suppose that $A,B\subseteq X$ are nonempty and $T\colon X\to 2^X$. Then, we write ${}_A\mathscr{G}_B$ if and only if $(a,b)\in E$ for all $a\in A$ and $b\in B$. We say that the mapping T is edge preserving if $(x,y)\in E$ implies ${}_{Tx}\mathscr{G}_{Ty}$ for all $x,y\in X$.

Definition 3.14. Let X be a nonempty set endowed with a graph G such that d be a metric on X and $T\colon X\to 2^X$ a mapping. We say that T is a set-valued generalized $\mathscr{G}$-(ψ,L)-weak contraction if there exist $\psi\in\Psi$ and $L\geq 0$ such that

$$H(Tx,Ty)\leq\psi(M(x,y,T))+Ld(y,Tx) \tag{3.5}$$

for all $x,y\in X$ with ${}_{Tx}\mathscr{G}_{Ty}$.

Let $\{x_n\}$ be a sequence in X. Then the sequence $\{x_n\}$ is called a $\mathscr{G}$-sequence if $(x_n,x_{n+1})\in E(\mathscr{G})$ for all $n\in\mathbb{N}$. By $\mathfrak{A}_\mathscr{G}$, we denote the class of all $\mathscr{G}$-sequences in X. For the mapping T, we denote by $G_\mathscr{G}(T)$ the $\mathscr{G}$-graph of T and

$$G_\mathscr{G}(T)=\{(x_n,x_{n+1})\in X\times X\colon x_1\in X \text{ and } \{x_n\}\in\mathfrak{A}_\mathscr{G}\cap\mathscr{T}(x_1,T)\}.$$

The mapping T is said to be $\mathscr{G}$-closed if $G_\mathscr{G}(T)$ is a closed subset of $X\times X$. The space (X,d) is called $\mathscr{G}$-complete if every Cauchy sequence in $\mathfrak{A}_\mathscr{G}$ is convergent in $X\times X$ (see, Shukla et al. [13]).

Corollary 3.15. Let X be a nonempty set endowed with a graph G such that d be a metric on X such that (X,d) is a $\mathscr{G}$-complete metric space. Suppose that $T\colon X\to CB(X)$ is a set-valued generalized $\mathscr{G}$-(ψ,L)-weak contraction and the following conditions are satisfied:

(I) T is edge preserving.

(II) there exist $x_0\in X$ and $x_1\in Tx_0$ with $(x_0,x_1)\in E$;

(III) at least one of the following conditions is satisfied:

(a) for a trajectory $\{x_n\}\in\mathfrak{A}_\mathscr{G}$, starting at x_1 and converging to $x\in X$, we have $(x_n,x)\in E$ for all $n\in\mathbb{N}$.

(b) T is a $\mathscr{G}$-closed mapping.

Then T has a fixed point.

Proof. Define a function $\alpha\colon X\times X\to[0,\infty)$ by

$$\alpha(x,y)=\begin{cases}1, & \text{if } (x,y)\in E;\\ 0, & \text{otherwise.}\end{cases}$$

Then, if $\alpha(x,y)\geq 1$, i.e., (x,y), then by edge preserveness of T, we have ${}_{Tx}\mathscr{G}_{Ty}$. So, $\alpha_\star(Tx,Ty)=\inf\{\alpha(a,b)\colon a\in Tx, b\in Ty\}=1$. This shows that T is $\alpha_\star$-admissible. Again, condition (II) shows that there exists $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\geq 1$. Also, if (a) of (III) is satisfied, then, for any sequence $x_n\in X$ converging to $x\in X$ and $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}$, we must have $\alpha(x_n,x)\geq 1$ for all $n\in\mathbb{N}$. $\mathscr{G}$-closedness of T implies the α-closedness of T and $\mathscr{G}$-completeness of (X,d) implies its α-completeness. Finally, since T is a set-valued generalized $\mathscr{G}$-(ψ,L)-weak contraction, by definition of α the mapping, T is a set-valued generalized $\alpha_\star$-(ψ,L)-weak contraction. Hence, the result follows from Theorem 3.7 and Theorem 3.11. □

In view of Remark 3.3, if we take $L=0$ in Theorem 3.7, then we obtain the following improvement of one of the main results of Hussain et al. [11].

Corollary 3.16. Let (X,d) be a α-complete metric space and $T\colon X\to CB(X)$ be an $\alpha_\star$-admissible multifunction on X. Assume that for $\psi\in\Psi$, we have

$$\alpha_\star(Tx,Ty)H(Tx,Ty)\quad\leq\quad\psi(M(x,y,T))$$

for all $x,y\in X$. Also suppose that the following assertions hold:

(i) there exist $x_0\in X$ and $x_1\in Tx_0$ with $\alpha(x_0,x_1)\geq 1$;
(ii) at least one of the following conditions is satisfied:
  (a) for a sequence $x_n\in X$ converging to $x\in X$ and $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}$, we have $\alpha(x_n,x)\geq 1$ for all $n\in\mathbb{N}$.
  (b) T is a α-closed mapping.

Then T has a fixed point.

Definition 3.17. Let (X,d) be a metric space, $T\colon X\to 2^X$ a mapping, and $\alpha: X\times X\to[0,\infty)$ be a function. We say that T is a set-valued generalized α-(ψ,L)-weak contraction if there exist $\psi\in\Psi$ and $L\geq 0$ such that

$$H(Tx,Ty)\leq\psi(M(x,y,T))+Ld(y,Tx)\text{ for all }x,y\in X\text{ with }\alpha(x,y)\geq 1.$$

In the next theorem, we omit the completeness of space, closedness and the $\alpha_\star$-admissibility of mapping T, and obtain a fixed point result for T when it a set-valued generalized α-(ψ,L)-weak contraction.

Theorem 3.18. *Let (X,d) be a metric space and $T\colon X\to CB(X)$ be a set-valued generalized α-(ψ,L)-weak contraction, where ψ is continuous. Suppose that there exists $u\in X$ such that:*

(i) $\alpha(u,v)\geq 1$ for all $v\in Tu$;
(ii) $d(u,Tu)\leq d(v,Tv)$ for all $v\in Tu$.

Then T has a fixed point.

Proof. We claim that u is a fixed point of T. On contrary, suppose that $u \notin Tu$, i.e., $D(u) = d(u,Tu) > 0$. Set $D(x) = d(x,Tx)$ for $x \in X$. Then, by (ii) we have

$$0 < D(u) \leq D(v) \text{ for all } v \in Tu. \tag{3.6}$$

The above condition shows that $D(v) > 0$ for all $v \in Tu$. Letting $v \in Tu$, we have $\alpha(u,v) \geq 1$. Since T is a set-valued generalized α-(ψ,L)-weak contraction, we have $H(Tu,Tv) \leq \psi(M(u,v,T)) + Ld(v,Tu) = \psi(M(u,v,T))$. Therefore,

$$0 < D(u) \leq D(v) = d(v,Tv) \leq H(Tu,Tv) \leq \psi(M(u,v,T)), \tag{3.7}$$

where

$$\begin{aligned} M(u,v,T) &= \max\left\{d(u,v), d(u,Tu), d(v,Tv), d(u,Tv)d(v,Tu), \frac{d(u,Tu)d(v,Tv)}{1+d(u,v)}\right\} \\ &= \max\left\{d(u,v), D(u), D(v), \frac{D(u)D(v)}{1+d(u,v)}\right\}. \end{aligned}$$

We consider the following cases:

Case I. If $\max\left\{d(u,v), D(u), D(v), \frac{D(u)D(v)}{1+d(u,v)}\right\} = d(u,v)$, then since $d(u,v) > 0$ it follows from (3.7) that

$$0 < D(u) \leq \psi(d(u,v)) < d(u,v).$$

This inequality is true for all $v \in Tu$, hence taking infimum over $v \in Tu$ and using the continuity of ψ we obtain $0 < D(u) \leq \psi(d(u,Tu)) = \psi(D(u)) < D(u)$. This yields a contradiction.

Case II. If $\max\left\{d(u,v), D(u), D(v), \frac{D(u)D(v)}{1+d(u,v)}\right\} = D(u)$, then it follows from (3.7) that $0 < D(u) \leq \psi(D(u)) < D(u)$. This is a contradiction.

Case III. If $\max\left\{d(u,v), D(u), D(v), \frac{D(u)D(v)}{1+d(u,v)}\right\} = D(v)$, then it follows from (3.7) that $0 < D(u) \leq D(v) \leq \psi(D(v)) < D(v)$. This is a contradiction.

Case IV. If $\max\left\{d(u,v), D(u), D(v), \frac{D(u)D(v)}{1+d(u,v)}\right\} = \frac{D(u)D(v)}{1+d(u,v)}$, then it follows from (3.7) that $0 < D(u) \leq D(v) \leq \psi\left(\frac{D(u)D(v)}{1+d(u,v)}\right) < \frac{D(u)D(v)}{1+d(u,v)}$, i.e.,

$$1 + d(u,v) < D(u)$$

since $D(u), D(v) \neq 0$. By definition, $D(u) = d(u,Tu) \leq d(u,y)$ for all $y \in Tu$, therefore, we obtain $1 + d(u,v) < d(u,v)$. This is again a contradiction.

Thus, in all possible cases, we reach a contradiction. This shows that the quantity $D(u) = d(u,Tu) = 0$, i.e., $u \in Tu$. Therefore, u is a fixed point of T. □

As a corollary of the above result, we give the following fixed point result for a set-valued mapping satisfying contractive condition in an open ball.

Corollary 3.19. *Let (X,d) be a metric space and $T\colon X \to CB(X)$ be a mapping and there exist $u \in X$, $r > 0$ and a continuous $\psi \in \Psi$ such that the following contractive condition holds:*

$$H(Tx,Ty) \leq \psi(M(x,y,T)) + Ld(y,Tx) \text{ for all } x,y \in B(u,r) \tag{3.8}$$

where $B(u,r) = \{x \in X\colon d(x,u) < r\}$. If $Tu \subset B(u,r)$ and $d(u,Tu) \leq d(v,Tv)$ for all $v \in Tu$, then T has a fixed point.

Proof. Define a function $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x, y \in B(u,r); \\ 0, & \text{otherwise.} \end{cases}$$

Then, it is obvious that T is a set-valued generalized α-(ψ, L)-weak contraction. Since $Tu \subset B(u,r)$, we have $\alpha(u,v) = 1$ for all $v \in Tu$. Thus, all the conditions of Theorem 3.18 are satisfied. Hence the conclusion follows. □

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