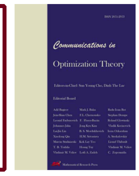
Commun. Optim. Theory 2020 (2020), Article ID 18 https://doi.org/10.23952/cot.2020.18

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org



GENERAL DECAY OF SOLUTIONS IN ONE-DIMENSIONAL POROUS-ELASTIC SYSTEM WITH MEMORY AND DISTRIBUTED DELAY TERM WITH SECOND SOUND

DJAMEL OUCHENANE[1], KHALED ZENNIR[2,3,*]

[1]Laboratory of Pure and Applied Mathematics , Amar Teledji Laghouat University, Algeria

[2]Department of Mathematics, College of Sciences and Arts, Qassim University, Ar-Rass, Saudi Arabia

[3]Laboratoire de Mathématiques Appliquées et de Modélisation,
Université 8 Mai 1945 Guelma. B.P. 401 Guelma 24000 Algérie

Abstract. We investigate a one-dimensional porous-elastic system with the presence of both memory and distributed delay terms in the second equation with second sound. Using the well known energy method combined with Lyapunov functionals approach, we obtain a general decay result.

Keywords. Porous system; General decay; Exponential Decay; Memory term; Distributed delay term.

1. INTRODUCTION

Let $\mathscr{H} = (0,1) \times (\tau_1, \tau_2) \times (0,\infty)$. We are interested in the following problem

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g(t-s)\,\phi_{xx}(x,s)\,ds + \mu_1\phi_t \int_{\tau_1}^{\tau_2} |\mu_2(s)|\phi_t(x,t-s)\,ds + \gamma\theta_x = 0, \\ \rho_3\theta_t + \kappa q_x + \gamma\phi_{tx} = 0, \\ \tau_0 q_t + \delta q + \kappa\theta_x = 0, \end{cases}$$

where $(x,s,t) \in \mathscr{H}$. As in [1], taking the following new variable $z(x,\rho,s,t) = \phi_t(x,t-s\rho)$, we obtain

$$\begin{cases} sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \\ z(x,0,s,t) = \phi_t(x,t). \end{cases}$$

*Corresponding author.

E-mail addresses: ouchenanedjamel@gmail.com (D. Ouchenane), k.Zennir@qu.edu.sa (K. Zennir).

Received March 22, 2020; Accepted August 19, 2020.

Consequently, the problem is equivalent to

$$\begin{cases} \rho u_{tt}-\mu u_{xx}-b\phi_x=0, \\ J\phi_{tt}-\delta\phi_{xx}+bu_x+\xi\phi+\int_0^t g(t-s)\,\phi_{xx}(x,s)\,ds+\mu_1\phi_t+\int_{\tau_1}^{\tau_2}|\mu_2(s)|\phi_t(x,t-s\,)\,ds+\gamma\theta_x=0, \\ \rho_3\theta_t(x,t)+\kappa q_x(x,t)+\gamma\phi_{tx}(x,t)=0, \\ \tau_0 q_t(x,t)+\delta q(x,t)+\kappa\theta_x(x,t)=0, \\ sz_t(x,\rho,s,t)+z_\rho(x,\rho,s,t)=0, \end{cases} \tag{1.1}$$

where $(x,\rho,s,t)\in(0,1)\times\mathscr{H}$. The system with memory and delay term acting only on the porous equation together with the initial data

$$\begin{cases} u(x,0)=u_0(x),u_t(x,0)=u_1(x), \\ \phi(x,0)=\phi_0(x),\phi_t(x,0)=\phi_1(x), \\ \theta(x,0)=\theta_0(x),q(x,0)=q_0(x),\ x\in(0,1), \end{cases}$$

and boundary conditions

$$u_x(0,t)=u_x(1,t)=\phi(0,t)=\phi(1,t)=q(0,t)=q(1,t)=0,\ t\geq 0, \tag{1.2}$$

where u is the longitudinal displacement, ϕ is the volume fraction of the solid elastic material and $\rho,\mu,b,J,\delta,\xi,\rho_3,\ \gamma,\ \tau_0,\ \delta,\ \kappa$ are positive constants with μ,ξ,b satisfying $\mu\xi>b^2$. The integral represents the memory and delay term with $\tau_1,\tau_2>0$ are a time delay, μ_1 is positive constant, μ_2 is an L^∞function and g is the relaxation function satisfying

(H1) $g\in C^1(\mathbb{R}_+,\mathbb{R}_+)$ is a non-increasing function satisfying

$$g(0)>0,\ b-\int_0^\infty g(s)\,ds=l>0. \tag{1.3}$$

(H2) There exists a positive non-increasing differentiable function $\vartheta\in(\mathbb{R}_+,\mathbb{R}_+)$ satisfying

$$g'(t)\leq-\vartheta(t)\,g(t),t\geq 0. \tag{1.4}$$

(H3) $\mu_2:[\tau_1,\tau_2]\to\mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\leq\mu_1. \tag{1.5}$$

Time delays arise in many applications because most phenomena naturally depend not only on the present state but also on some past occurrences In recent years, the control of PDEs with time delay effects has become an active area of research; see, e.g., [2] and the references therein. In many cases, it was shown that the delay is a source of instability unless additional conditions or control terms are used; see [3]. Therefore, the stability issue of systems with delay is of theoretical and practical great importance. It is well know that, in the single wave equation, if $\mu_2=0$, that is, in absence of a decay, the energy of the system exponentially decays (see [4]). On the contrary, if $\mu_1=0$, that is, there exists only the delay part in the interior, the system becomes unstable (see [3]). It is shown that a small delay in a boundary control can turn such a well-behaved hyperbolic system into a wild one, therefore, delay becomes a source of instability. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see [1], [5]). Researchers from various fields were interested in elasticity problems, and they have been mainely attracted by the qualitative studies of different type of this problems. In the one-dimensional case, for instance, the combination of the elastic equations with thermal consequences causes a negative exponential to control the decay of solutions. The one-dimensional porous-elastic model is given by

$$\begin{aligned} \rho_0 u_{tt} &= \mu u_{xx} + \beta \varphi_x, \text{ in } (0,l) \times (0,L), \\ \rho_0 k \varphi_{tt} &= \alpha \varphi_{tt} - \beta u_x - \tau \varphi_t - \xi \varphi, \text{ in } (0,l) \times (0,L), \end{aligned}$$

and it was studied by many authors. The first contribution in this direction was obtained by Quintanilla [6]. To be more precise, it was developed in [7], and the authors showed that the classical elasticity theory to porous media by introducing the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition to the usual elastic effects, the materials with voids possess a microstructure with the property that the mass at each point is obtained as the product of the mass density of the material matrix by the volume fraction. This concept was introduced in the pioneered work in [8] with advanced nonlinear theory of elastic materials with voids (see [9], [10]). The basic evolution equations for one-dimensional theories of porous materials with memory effect are given by

$$\rho u_{tt} = T_x, J\phi_{tt} = H_x + G, \tag{1.6}$$

where T is the stress tensor, H is the equilibrated stress vector and G is the equilibrated body force. The variables u and ϕ are the displacement of the solid elastic material and the volume fraction, respectively. The constitutive equations are

$$T = \mu u_x + b\phi, H = \delta \phi_x - \int_0^t g(t-s)\,\phi_x(s)\,ds, G = -bu_x - \xi \phi. \tag{1.7}$$

In [6], substituting (1.7) into (1.6), Quintanilla considered

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \text{ in } (0,1) \times (0,\infty), \\ J\phi_{tt} - \delta \phi_{xx} + bu_x + \xi \phi + \int_0^t g(t-s)\,\phi_{xx}(x,s)\,ds = 0, \text{ in } (0,1) \times (0,\infty). \end{cases} \tag{1.8}$$

A porous-elastic system with memory term and Neumann-Dirichlet boundary conditions where g is the relaxation function was proved; see, for more detail, [11]. Quintanilla investigated

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x, x \in (0,L), t > 0, \\ J\phi_{tt} = \delta \phi_{xx} - bu_x - \xi \phi - \tau \phi_t, x \in (0,L), t > 0, \end{cases} \tag{1.9}$$

with initial and mixed boundary conditions and supposed that the damping in the porous equation $(-\tau\phi_t)$ is not enough to obtain an exponential decay but only a slow decay can be obtained. To improve this decay, several other damping mechanisms were considered. In [12], Casas and Quintanilla considered

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x - \beta \theta_x, x \in (0,L), t > 0, \\ J\phi_{tt} = \delta \phi_{xx} - bu_x - \xi \phi + m\theta - \tau \phi_t, x \in (0,L), t > 0, \\ c\theta_t = k\theta_{xx} - \beta u_{xt} - m\phi_t, x \in (0,L), t > 0, \end{cases} \tag{1.10}$$

where θ is the temperature difference with initial and Dirichlet-Neumann boundary conditions. The authors applied the semigroup theory and the method proposed and developed in [13] to establish the exponential decay of the solutions. Later, with $\tau = 0$, in [14], the same authors proposed that the heat effect alone is not strong sufficient to bring an exponential decay but only a slow decay could be established. However, the heat effect together with micro-temperature created an exponential decay result. Similarly, when $\tau = 0$ and γu_{xxt} is added to the first equation in (1.6), Rivera, Pamplona and Quintanilla [15], proved the lack of exponential stability. For $\tau = 0$, problem (1.6) was considered in [16] with the following boundary conditions

$$\begin{cases} u(0,t) = \phi(0,t) = \theta(0,t) = \theta(L,t) = 0, t \geq 0, \\ u(L,t) = -\int_0^t g_1(t-s)[\mu u_x(L,s) + b\phi(L,s)]\, ds, t \geq 0, \\ \phi(L,t) = -\delta \int_0^t g_2(t-s)\, \phi_x(L,s)\, ds, t \geq 0, \end{cases}$$

where g_1 and g_2 are positive decreasing functions. A general decay result in which the usual exponential and polynomial decay rates are just special cases was obtained; see, [16], [17], [18], [19], [20] and the references therein. The viscoelastic damping is represented by a memory term in the form of a convolution which arises in the constitutive equation between the stress $\sigma(x,t)$ and the strain $\varepsilon(x,t)$ (see [11], [21]) $\sigma(x,t) = \varepsilon(x,t) + \int_0^t g(t-s)\, \varepsilon(x,s)\, ds$. This type of the viscoelastic dissipation could be said to coincide to viscosity with null initial history because it is assumed that the strains have been zero for $-\infty < t < 0$ or, equivalently, if any past strains have occurred sufficiently long ago that the effect is trivial. In other words, there will be a time prior to which all the strains which have previously occurred will have a trivial contribution. Thus, an experiment generally starts at some time $(t = 0)$ when the material is free of stresses. We must mention the pioneer works recently published in [4], the author considered a one-dimensional porous thermo-elastic system with memory effects and proved a general decay result, for which exponential and polynomial decay results are special cases, depending only on the kernel of the memory effects. The results were established irrespective of the wave speeds of the system (see [22], [23]). In [24], Fareh and Messaoud investigated a porous thermo-elastic system where the heat conduction is given by Cattaneo's law and the energy associated with the solution is not necessary positive. They introduced a stability number and proved an exponential and polynomial decay results.

Our purpose in this paper is to give a general decay result of solutions in one dimensional porous-elastic system with memory and distributed delay term. In what follows, we consider (u, ϕ, θ, q) to be a solution of system (1.1)-(1.2) with the regularity needed to justify the calculations in this paper. We specify Section 2 to the statements and prove of our stability result. We use c throughout this paper to denote a generic positive constant. Meanwhile, from (1.1) and (1.2), it follows that

$$\frac{d^2}{dt^2} \int_0^1 u(x,t)\, dx = 0. \tag{1.11}$$

So, by solving (1.11) and using the initial data of u, θ, we get

$$\int_0^1 u(x,t)\, dx = t \int_0^1 u_1(x)\, dx + \int_0^1 u_0(x)\, dx.$$

Consequently, if

$$\begin{aligned} \bar{u}(x,t) &= u(x,t) - t \int_0^1 u_1(x)\, dx - \int_0^1 u_0(x)\, dx, \\ \overline{\theta}(x,t) &= \theta(x,t) - \int_0^1 \theta_0(x) dx, \end{aligned} \tag{1.12}$$

then $\int_0^1 \bar{u}(x,t)\,dx = 0, \forall t \geq 0$. From the third equation in (1.1), we easily verify that $\int_0^1 \overline{\theta}(x,t)\,dx = 0$. Therefore, the use of Poincare's inequality for $\bar{u}, \overline{\theta}$ is justified. In addition, simple substitution shows that $\left(\bar{u}, \phi, \overline{\theta}, q\right)$ satisfies system (1.1) with initial data for $\bar{u}$ given as

$$\bar{u_0}(x) = u_0(x) - \int_0^1 u_0(x)\,dx \text{ and } \bar{u_1}(x) = u_1(x) - \int_0^1 u_1(x)\,dx.$$

Henceforth, we work with $\bar{u}, \overline{\theta}$ instead of u, θ but write u, θ for simplicity of the notations.

2. Main Results

In this section, we state and prove our decay result for the energy of system (1.1)-(1.2) using the multiplier technique.

We need the following Lemmas.

Lemma 2.1. *The energy functional E, defined by*

$$\begin{aligned} & E(t) \\ = \; & \frac{1}{2}\int_0^1 \left[\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + \left(\delta - \int_0^t g(s)\,ds\right)\phi_x^2 + \xi\phi^2 + 2bu_x\phi + \rho_3\theta^2\right]dx \quad (2.1) \\ & + \frac{1}{2}\int_0^1 \tau_0 q^2 dx + \frac{1}{2} g \circ \phi_x + \frac{1}{2}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t)\,ds\,d\rho\,dx, \end{aligned}$$

satisfies

$$\frac{dE(t)}{dt} = -\delta\int_0^1 q^2 dx + \frac{1}{2} g' \circ \phi_x - \frac{1}{2} g(t)\int_0^1 \phi_x^2 dx - \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds\right)\int_0^1 \phi_t^2 dx,$$

and

$$\frac{dE(t)}{dt} \leq -\delta\int_0^1 q^2 dx + \frac{1}{2} g' \circ \phi_x - \eta_0 \int_0^1 \phi_t^2 dx \leq 0, \tag{2.2}$$

where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds \geq 0$ and $g \circ v = \int_0^1 \int_0^t g(t-s)\left(v_x(t) - v_x(s)\right)^2 ds\,dx$.

Proof. Multiplying the first equation of (1.1) by u_t and the second equation by ϕ_t, the third equation in (1.1) by θ, and the fourth equation in (1.1) by q we integrate by parts, using Young's inéquality we get (2.2) □

Lemma 2.2. *The functional*

$$D_1(t) := J\int_0^1 \phi_t \phi\,dx + \frac{b\rho}{\mu}\int_0^1 \phi \int_0^x u_t(y)\,dy\,dx - \frac{\gamma\tau_0}{\kappa}\int_0^1 \phi q\,dx, \tag{2.3}$$

satisfies

$$\begin{aligned} \frac{dD_1(t)}{dt} \leq \; & -\frac{l}{2}\int_0^1 \phi_x^2 dx - \mu_3\int_0^1 \phi^2 dx + \varepsilon_1\int_0^1 u_t^2 dx + c\left(1 + \frac{1}{\varepsilon_1} + \frac{\gamma\tau_0\varepsilon_1}{2\kappa}\right)\int_0^1 \phi_t^2 dx \\ & \left(\frac{\delta\gamma}{2\kappa\varepsilon_1} + \frac{\gamma\tau_0}{2\kappa\varepsilon_1}\right)\int_0^1 q^2 dx + cg \circ \phi_x + c\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\,ds\,dx, \quad (2.4) \end{aligned}$$

where $\mu_3 = \xi - \frac{b^2}{\mu} > 0$.

Proof. Direct computation using integration by parts and Young's inequality, for $\varepsilon_1 > 0$, yields

$$
\begin{aligned}
& \frac{dD_1(t)}{dt} \\
= \; & -\delta \int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right) \int_0^1 \phi^2 dx + J \int_0^1 \phi_t^2 dx + \frac{b\rho}{\mu} \int_0^1 \phi_t \int_0^x u_t(y)\, dy dx \\
& + \int_0^1 \phi_x \int_0^t g(t-s)\, \phi_x(s)\, ds dx - \mu_1 \int_0^1 \phi_t \phi dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) ds dx \\
& - \frac{\gamma \tau_0}{\kappa} \int_0^1 \phi_t q dx + \frac{\delta\gamma}{\kappa} \int_0^1 \phi q dx \qquad (2.5) \\
\leq \; & -\delta \int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right) \int_0^1 \phi^2 dx + c\left(1 + \frac{1}{\varepsilon_1} + \frac{\gamma\tau_0\varepsilon_1}{2\kappa}\right) \int_0^1 \phi_t^2 dx \\
& + \varepsilon_1 \int_0^1 \left(\int_0^x u_t(y)\, dy\right)^2 dx \\
& \left(\frac{\delta\gamma}{2\kappa\varepsilon_1} + \frac{\gamma\tau_0}{2\kappa\varepsilon_1}\right) \int_0^1 q^2 dx + \int_0^1 \phi_x \int_0^t g(t-s)\, \phi_x(s)\, ds dx \\
& - \mu_1 \int_0^1 \phi_t \phi dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) ds dx. \qquad (2.6)
\end{aligned}
$$

(2.7)

By Cauchy-Schwartz inequality, it is clear that

$$
\int_0^1 \left(\int_0^x u_t(y)\, dy\right)^2 dx \leq \int_0^1 \left(\int_0^1 u_t dx\right)^2 dx \leq \int_0^1 {u_t}^2 dx.
$$

So, estimate (2.5) becomes

$$
\begin{aligned}
& \frac{dD_1(t)}{dt} \\
\leq \; & -\delta \int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right) \int_0^1 \phi^2 dx + c\left(1 + \frac{1}{\varepsilon_1} + \frac{\gamma\tau_0\varepsilon_1}{2\kappa}\right) \int_0^1 \phi_t^2 dx + \varepsilon_1 \int_0^1 {u_t}^2 dx \\
& - \mu_1 \int_0^1 \phi_t \phi dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) ds dx \\
& + \left(\frac{\delta\gamma}{2\kappa\varepsilon_1} + \frac{\gamma\tau_0}{2\kappa\varepsilon_1}\right) \int_0^1 q^2 dx + \int_0^1 \phi_x \int_0^t g(t-s)\, \phi_x(s)\, ds dx. \qquad (2.8)
\end{aligned}
$$

The last term in the RHS of (2.8) is estimated as follows:

$$
\begin{aligned}
& \int_0^1 \phi_x \int_0^t g(t-s)\, \phi_x(s)\, ds dx \\
= \; & \int_0^t g(s)\, ds \int_0^1 \phi_x^2 dx - \int_0^1 \phi_x \int_0^t g(t-s) \left(\phi_x(t) - \phi_x(s)\right) ds dx \\
\leq \; & \left(\delta_1 + \int_0^t g(s)\, ds\right) \int_0^1 \phi_x^2 dx + \frac{1}{4\delta_1} \left(\int_0^t g(s)\, ds\right) g \circ \phi_x, \qquad (2.9)
\end{aligned}
$$

where Cauchy-Schwartz, Young's and poincare's inequalities are used, for $\delta_1, \varepsilon_2, \varepsilon_3 > 0$. By substituting (2.9) into(2.5), we obtain

$$\begin{aligned}
&\frac{dD_1(t)}{dt} \\
\leq\ & -\left(\delta - \int_0^t g(s)\,ds - \delta_1 - \mu_1 c\delta_2 - \mu_1 c\delta_3\right)\int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx \\
& + \varepsilon_1 \int_0^1 u_t^2 dx + \left(c(1+\frac{1}{\varepsilon_1}) + \frac{\mu_1}{4\delta_2} + \frac{\gamma\tau_0\varepsilon_1}{2\kappa}\right)\int_0^1 \phi_t^2 dx + \frac{1}{4\delta_1}\left(\int_0^t g(s)\,ds\right) g\circ\phi_x \\
& + \left(\frac{\delta\gamma}{2\kappa\varepsilon_1} + \frac{\gamma\tau_0}{2\kappa\varepsilon_1}\right)\int_0^1 q^2 dx + \frac{1}{4\delta_3}\int_0^t \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\,ds\,dx. \qquad (2.10)
\end{aligned}$$

Bearing in mind that $\mu\xi > b^2$, using the fact that $\delta - \int_0^t g(s)\,ds \geq l$, and letting $\delta_1 = \frac{l}{6}$, $\delta_2 = \delta_3 = \frac{l}{6c\mu_1}$, we obtain estimate (2.4). □

In the following Lemma, we use the essential hypothesis that the wave speeds of the system are equal

$$\frac{\mu}{\rho} = \frac{\delta}{J}. \qquad (2.11)$$

Lemma 2.3. *Assume that (H1) and (2.11) hold. Then the functional*

$$D_2(t) := \int_0^1 \phi_x u_t dx + \int_0^1 \phi_t u_x dx - \frac{\rho}{\mu J}\int_0^1 u_t \int_0^t g(t-s)\,\phi_x(s)\,ds\,dx,$$

satisfies, for any $\varepsilon_2 > 0$

$$\begin{aligned}
\frac{dD_2(t)}{dt} \leq\ & \left(-\frac{b}{2J} + \frac{\gamma}{2J}\right)\int_0^1 u_x^2 dx + \frac{\gamma}{2J}\int_0^1 \theta^2 dx \\
& + c\left(1 + \frac{1}{\varepsilon_2}\right)\int_0^1 \phi_x^2 dx + c\varepsilon_2 \int_0^1 u_t^2 dx \qquad (2.12) \\
& + c\int_0^1 \phi_t^2 + cg\circ\phi_x - \frac{c}{\varepsilon_2} g'\circ\phi_x + c\mu_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\,dx.
\end{aligned}$$

Proof. By differentiating D_2 and integrating by parts, we find from (1.2) and (1.1) that

$$\begin{aligned}
&\frac{dD_2(t)}{dt} \\
=\ & -\frac{b}{J}\int_0^1 u_x^2 dx + \left(\frac{\delta}{J} - \frac{\mu}{\rho}\right)\int_0^1 u_x \phi_{xx} dx + \frac{b}{\rho}\int_0^1 \phi_x^2 dx - \frac{\rho g(0)}{\mu J}\int_0^1 u_t \phi_x dx \\
& + \frac{\gamma}{J}\int_0^1 u_x \theta dx - \frac{\xi}{J}\int_0^1 u_x \phi dx - \frac{b}{\mu J}\int_0^1 \phi_x \int_0^t g(t-s)\,\phi_x(s)\,ds\,dx \\
& - \frac{\rho}{\mu J}\int_0^1 u_t \int_0^t g'(t-s)\,\phi_x(s)\,ds\,dx \\
& - \frac{\mu_1}{J}\int_0^1 \phi_t u_x dx - \frac{1}{J}\int_0^1 u_x \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\,ds\,dx. \qquad (2.13)
\end{aligned}$$

In what follows, we estimate the last five terms in the right hand side of (2.13). Using Young's, Cauchy-Schwartz, and Poincare's inequalities. For $\delta_4, \delta_5, \varepsilon_2 > 0$, we have

$$-\frac{\xi}{J}\int_0^1 u_x\phi dx \le \frac{\xi}{J}\delta_4\int_0^1 u_x^2 dx + \frac{\xi}{4J\delta_4}\int_0^1 \phi^2 dx.$$

By letting $\delta_4 = \dfrac{b}{6\xi}$ and using Poincar's inequality, we get

$$-\frac{\xi}{J}\int_0^1 u_x\phi dx \le \frac{b}{6J}\int_0^1 u_x^2 dx + c\int_0^1 \phi^2 dx, \tag{2.14}$$

$$\begin{aligned} -\frac{b}{\mu J}\int_0^1 \phi_x\int_0^t g(t-s)\phi_x(s)\,dsdx &= \frac{b}{\mu J}\int_0^1 \phi_x\int_0^t g(t-s)(\phi_x(t)-\phi_x(s))\,dsdx \\ &\quad -\frac{b}{\mu J}\int_0^t g(s)\,ds\int_0^1 \phi_x^2 dx \\ &\le \left(\delta_5 - \frac{b}{\mu J}\right)\int_0^t g(s)\,ds\int_0^1 \phi_x^2 dx + \frac{c}{\delta_5} g\circ\phi_x. \end{aligned}$$

Letting $\delta_5 = \dfrac{b}{\mu J}$, we conclude that

$$-\frac{b}{\mu J}\int_0^1 \phi_x\int_0^t g(t-s)\phi_x(s)\,dsdx \le cg\circ\phi_x, \tag{2.15}$$

$$\begin{aligned} &-\frac{\rho}{\mu J}\int_0^1 u_t\int_0^t g'(t-s)\phi_x(s)\,dsdx \\ =\;& \frac{b}{\mu J}\int_0^1 u_t\int_0^t g'(t-s)(\phi_x(t)-\phi_x(s))\,dsdx \\ &-\frac{b}{\mu J}\int_0^t g'(s)\,ds\int_0^1 u_t\phi_x dx \\ \le\;& \frac{\rho\varepsilon_2}{2\mu J}\int_0^1 u_t^2 dx + \frac{\rho g(0)}{\mu J}\int_0^1 u_t\phi_x dx - \frac{\rho g(t)}{\mu J}\int_0^1 u_t\phi_x dx \\ &+\frac{\rho}{2\mu J\varepsilon_2}\int_0^1 g'(s)\,ds\int_0^1\int_0^t g'(t-s)(\phi_x(t)-\phi_x(s))^2\,dsdx \\ \le\;& \frac{\rho\varepsilon_2}{\mu J}\int_0^1 u_t^2 dx + \frac{\rho}{2\mu J\varepsilon_2}\left(\int_0^1 g'(s)\,ds\right) g'\circ\phi_x. \\ &+\frac{\rho g(0)}{\mu J}\int_0^1 u_t\phi_x dx + \frac{\rho g(t)}{2\mu J\varepsilon_2}\int_0^1 u_t\phi_x dx \\ \le\;& \frac{\rho\varepsilon_2}{\mu J}\int_0^1 u_t^2 dx + \frac{\rho}{2\mu J\varepsilon_2}\left(\int_0^1 g'(s)\,ds\right) g'\circ\phi_x. \\ &+\frac{\rho g(0)}{\mu J}\int_0^1 u_t\phi_x dx + \frac{\rho\,(g(t))^2}{2\mu J\varepsilon_2}\int_0^1 \phi_x^2 dx \\ \le\;& c\varepsilon_2\int_0^1 u_t^2 dx + \frac{c}{\varepsilon_2}\int_0^1 \phi_x^2 dx + \frac{\rho g(0)}{\mu J}\int_0^1 u_t\phi_x dx - \frac{c}{\varepsilon_2} g'\circ\phi_x, \end{aligned} \tag{2.16}$$

$$-\frac{\mu_1}{J}\int_0^1 \phi_t u_x dx \leq \frac{\mu_1\delta_6}{2J}\int_0^1 \phi_t^2 dx + \frac{\mu_1}{2J\delta_6}\int_0^1 u_x^2 dx, \tag{2.17}$$

and

$$\begin{aligned}\frac{1}{J}\int_0^1 u_x \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) ds dx \;\; &\leq \;\; \frac{\delta_7\mu_1}{2J}\int_0^1 u_x^2 dx \\ &+\frac{1}{2J\delta_7}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds.\end{aligned} \tag{2.18}$$

Substituting (2.14)-(2.18) into (2.13), and letting $\delta_6 = \delta_7 = \frac{b}{4\mu_1}$, we conclude from (2.11) that (2.12). □

Lemma 2.4. *The functional*

$$D_3(t) := -\rho\int_0^1 u_t u dx,$$

satisfies

$$\frac{dD_3(t)}{dt} \leq -\rho\int_0^1 u_t^2 dx + \frac{3\mu}{2}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx. \tag{2.19}$$

Proof. Direct computations give

$$\frac{dD_3(t)}{dt} = -\rho\int_0^1 u_t^2 dx + \mu\int_0^1 u_x^2 dx + b\int_0^1 u_x\phi dx.$$

By using Young's and Poincaré inequalities, we find from (2.19) that

$$\begin{aligned}\frac{dD_3(t)}{dt} \;\; &\leq \;\; -\rho\int_0^1 u_t^2 dx + \mu\int_0^1 u_x^2 dx + b\varepsilon\int_0^1 u_x^2 dx + \frac{b}{4\varepsilon}\int_0^1 \phi^2 dx \\ &\leq \;\; -\rho\int_0^1 u_t^2 dx + \mu\int_0^1 u_x^2 dx + b\varepsilon\int_0^1 u_x^2 dx + \frac{bc}{4\varepsilon}\int_0^1 \phi_x^2 dx.\end{aligned}$$

Letting $\varepsilon = \frac{\mu}{2b}$, we obtain (2.19). This completes the proof. □

Now, let us introduce the following functional.

Lemma 2.5. *The functional*

$$D_4(t) := \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x,\rho,s,t) ds d\rho dx,$$

satisfies

$$\begin{aligned}\frac{dD_4(t)}{dt} \;\; &\leq \;\; -\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t) ds d\rho dx + \mu_1\int_0^1 \phi_t^2 dx \\ &\quad -\eta_1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx,\end{aligned} \tag{2.20}$$

where η_1 is a positive constant.

Proof. By differentiating D_4, with respect to t and using the last equation in $(H3)$, we have

$$\begin{aligned}\frac{dD_4(t)}{dt} &= -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}|\mu_2(s)|zz_\rho(x,\rho,s,t)\,dsd\rho dx\\ &= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)|z^2(x,\rho,s,t)\,dsd\rho dx\\ &\quad -\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|[e^{-s}z^2(x,1,s,t)-z^2(x,0,s,t)]dsdx.\end{aligned}$$

Using the fact that $z(x,0,s,t)=\phi_t(x,t)$, and $e^{-s}\le e^{-s\rho}\le 1$, for all $0<\rho<1$, we obtain

$$\begin{aligned}\frac{dD_4(t)}{dt} &= -\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)\,dsd\rho dx\\ &\quad -\int_0^1\int_{\tau_1}^{\tau_2} e^{-s}|\mu_2(s)|z^2(x,1,s,t)\,dsdx+\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\int_0^1\phi_t^2dx.\end{aligned}$$

□

Finally, setting $\eta_1=e^{-\tau_2}$ and recalling (H3), we obtain (2.20). We are now ready to prove the main result. Now, in order to obtain a negative term of $\int_0^1\theta^2dx$, we introduce the following functional.

Lemma 2.6. *The functional*

$$D_5(t):=-\tau_0\rho_3\int_0^L q(t,x)\left(\int_0^x\theta(t,y)dy\right)dx. \tag{2.21}$$

satisfies

$$\begin{aligned}\frac{dD_5(t)}{dt} &\le \left(-\rho_3\kappa+\frac{\varepsilon_5\rho_3\delta c}{2}\right)\int_0^1\theta^2dx+\frac{\varepsilon_5'\tau_0\gamma}{2}\int_0^1\phi_t^2dx\\ &\quad+\left(\tau_0\kappa+\frac{\rho_3\delta}{2\varepsilon_5}+\frac{\tau_0\gamma}{2\varepsilon_5'}\right)\int_0^1 q^2dx.\end{aligned} \tag{2.22}$$

The above Lemma was proved in [**?**, Inequality (33)].

Theorem 2.7. *Assume (H1), (H2), (H3) and (2.11) hold. Then, for any $t_0>0$, there exist positive constants α and β such that the energy functional given by (2.1) satisfies*

$$E(t)\le\alpha e^{-\beta\int_{t_0}^t\vartheta(s)ds},\forall t\ge t_0. \tag{2.23}$$

Proof. We define a Lyapunov functional

$$\mathscr{L}(t):=NE(t)+N_1D_1(t)+N_2D_2(t)+D_3(t)+N_4D_4(t)+N_5D_5(t), \tag{2.24}$$

where N, N_1, N_2, N_4 and N_5 are positive constants to be selected later. By differentiating (2.24) and using (2.1), (2.4), (2.12), (2.19), (2.20) and (2.22), we have

$$\begin{aligned}
\mathscr{L}'(t) \leq & -\left[\frac{lN_1}{2} - cN_2(1+\frac{1}{\varepsilon_2}) - c\right]\int_0^1 \phi_x^2 dx - [\rho - N_1\varepsilon_1 - N_2 c\varepsilon_2]\int_0^1 u_t^2 dx \\
& -\left[\frac{bN_2}{2J} + N_2\frac{\gamma}{2J} - \frac{3\mu}{2}\right]\int_0^1 u_x^2 dx \\
& -\left[\eta_0 N - cN_1(1+\frac{1}{\varepsilon_1}) - N_2 c - \mu_1 N_4 + N_5\frac{\varepsilon_5'\tau_0\gamma}{2}\right]\int_0^1 \phi_t^2 dx \\
& -N_1\mu_3\int_0^1 \phi^2 dx - [N_4\eta_1 - cN_1 - cN_2]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds dx \\
& + \left(-N\delta + N_1\left(\frac{\delta\gamma}{2\kappa\varepsilon_1} + \frac{\gamma\tau_0}{2\kappa\varepsilon_1}\right) + N_5\left(\tau_0\kappa + \frac{\rho_3\delta}{2\varepsilon_5} + \frac{\tau_0\gamma}{2\varepsilon_5'}\right)\right)\int_0^1 q^2 dx \\
& + \left(N_2\frac{\gamma}{2J} + N_5\left(-\rho_3\kappa + \frac{\varepsilon_5\rho_3\delta c}{2}\right)\right)\int_0^1 \theta^2 dx + c[N_1+N_2]\, g\circ\phi_x \\
& + \left[\frac{N}{2} - \frac{cN_2}{\varepsilon_2}\right] g'\circ\phi_x - N_4\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t)\, ds d\rho dx.
\end{aligned}$$

By setting $\varepsilon_1 = \frac{\rho}{4N_1}$, $\varepsilon_2 = \frac{\rho}{4cN_2}$, and $\varepsilon_5 \leq \dfrac{\kappa}{\delta c}$, we fix ε_5' small enough such that $\varepsilon_5' \leq \frac{\gamma\rho_2}{4\tau_0\gamma}$. Finally, once all the above constants are fixed, we choose N, N_2 large enough such that

$$\begin{cases}
\dfrac{CN}{2} \geq cN_1(1+\dfrac{1}{\varepsilon_1}) - N_2 c - \mu_1 N_4 + N_5\dfrac{\varepsilon_5'\tau_0\gamma}{2}, \\
\dfrac{N\delta}{2} \geq N_1\left(\dfrac{\delta\gamma}{2\kappa\varepsilon_1} + \dfrac{\gamma\tau_0}{2\kappa\varepsilon_1}\right) + N_5\left(\tau_0\kappa + \dfrac{\rho_3\delta}{2\varepsilon_5} + \dfrac{\tau_0\gamma}{2\varepsilon_5'}\right) \\
\dfrac{bN_2}{2J} + N_2\dfrac{\gamma}{2J} - \dfrac{3\mu}{2} > 0.
\end{cases}$$

Thus,

$$\begin{aligned}
& \mathscr{L}'(t) \\
\leq & -\eta_1\int_0^1\left[\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + \left(\delta - \int_0^t g(s)\,ds\right)\phi_x^2 + \xi\phi^2 + 2bu_x\phi + \rho_3\theta^2\right] dx \\
& -\eta_1\int_0^1 \tau_0 q^2 dx - \eta_1 g\circ\phi_x - \eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t)\, ds d\rho dx,
\end{aligned} \tag{2.25}$$

which implies by (2.1) that there also exists a constant $\eta_2 > 0$ such that$\mathscr{L}'(t) \leq -\eta_2 E(t)$. Moreover, letting $\mathfrak{L}(t) = N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t) + N_5 D_5(t)$, we have

$$\begin{aligned}
|\mathfrak{L}(t)| \leq\ & JN_1 \int_0^1 |\phi\phi_t|\, dx + N_2 \int_0^1 \left|\phi_x u_t + u_x \phi_t - \frac{\rho}{\mu J} u_t \int_0^t g(t-s)\, \phi_x(s)\, ds\right| dx \\
& \frac{b\rho N_1}{\mu} \int_0^1 \left|\phi \int_0^x u_t(y)\, dy\right| dx + \rho \int_0^1 |u_t u|\, dx \\
& + N_4 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t)\, ds d\rho dx \\
& + N_5 \left| -\tau_0 \rho_3 \int_0^L q(t,x) \left(\int_0^x \theta(t,y) dy \right) dx \right|.
\end{aligned}$$

Exploiting Young's, Cauchy-Schwartz, and Poincaré inequalities, we obtain

$$\begin{aligned}
|\mathfrak{L}(t)| &\leq c \int_0^1 \left(u_t^2 + \phi_t^2 + \phi_x^2 + u_x^2 + \phi^2 + \theta^2 + q^2\right) dx + c g \circ \phi_x \\
&+ c \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t)\, ds d\rho \\
&\leq cE(t).
\end{aligned}$$

Consequently, we obtain $|\mathfrak{L}(t)| = |\mathscr{L}(t) - NE(t)| \leq cE(t)$, that is,

$$(N - c) E(t) \leq \mathscr{L}(t) \leq (N + c) E(t). \tag{2.26}$$

Now, by choosing N large enough such that $\frac{N}{2} - c > 0$ and exploiting (2.1), estimates (2.25) and (2.26), respectively, give

$$\mathscr{L}'(t) \leq -k_1 E(t) + k_2 g \circ \phi_x, \forall t \geq t_0, \tag{2.27}$$

and

$$c_2 E(t) \leq \mathscr{L}(t) \leq c_3 E(t), \forall t \geq 0, \tag{2.28}$$

for some $k_1, k_2, c_2, c_3 > 0$. Multiplying (2.27) by $\vartheta(t)$, we obtain

$$\vartheta(t) \mathscr{L}'(t) \leq -k_1 \vartheta(t) E(t) + k_2 \vartheta(t) g \circ \phi_x, \forall t \geq t_0. \tag{2.29}$$

The final term in (2.29) is estimated as following. Using (1.4), we have

$$\begin{aligned}
\vartheta(t) g \circ \phi_x &= \vartheta(t) \int_0^1 \int_0^t g(t-s) \left(\phi_x(t) - \phi_x(s)\right)^2 ds dx \\
&\leq \int_0^1 \int_0^t \vartheta(t-s) g(t-s) \left(\phi_x(t) - \phi_x(s)\right)^2 ds dx \\
&\leq -\int_0^1 \int_0^t g'(t-s) \left(\phi_x(t) - \phi_x(s)\right)^2 ds dx = -g' \circ \phi_x \\
&\leq -2E'(t).
\end{aligned}$$

Thus, (2.29) becomes $\vartheta(t) \mathscr{L}'(t) \leq -k_1 \vartheta(t) E(t) - 2k_2 E'(t), \forall t \geq t_0$, which can be rewritten as

$$\left(\vartheta(t) \mathscr{L}(t) + 2k_2 E(t)\right)' - \vartheta'(t) \mathscr{L}(t) \leq -k_1 \vartheta(t) E(t), \forall t \geq t_0.$$

Using the fact that $\vartheta'(t) \leq 0, \forall t \geq 0$, we have

$$\left(\vartheta(t)\mathscr{L}(t) + 2k_2E(t)\right)' \leq -k_1\vartheta(t)E(t), \forall t \geq t_0.$$

By exploiting (2.28), we have

$$\mathscr{R}(t) = \vartheta(t)\mathscr{L}(t) + 2k_2E(t) \sim E(t). \tag{2.30}$$

Consequently, for some positive constant λ, we obtain

$$\mathscr{R}'(t) \leq -\lambda\mathscr{R}(t)\vartheta(t), \forall t \geq t_0. \tag{2.31}$$

A simple integration of (2.31) over (t_0, t) leads to

$$\mathscr{R}(t) \leq \mathscr{R}(t_0)e^{-\lambda\int_{t_0}^{t}\vartheta(s)ds}, \forall t \geq t_0. \tag{2.32}$$

Consequently, (2.23) is established by virtue of (2.28) and (2.32). □

Acknowledgements

The authors would like to thank the anonymous referees and the handling editor for their careful reading and for relevant remarks/suggestions to improve the paper.

References

[1] A.S. Nicaise, C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, Differential and Integral Equations, 21 (2008), 935-958.

[2] I.H. Suh, Z. Bien, Use of time delay action in the controller design, IEEE Trans. Automat. Control 25 (1980), 600-603.

[3] R. Datko, Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks, SIAM J. Control Optim. 26 (1988), 697-713.

[4] T.A. Apalara, On the stabilization of a memory-type porous thermoelastic system, Bull. Malays. Math. Sci. Soc. 43 (2020), 1433-1448.

[5] D. Ouchenane, A stability result of a timoshenko system in thermoelasticity of second sound with a delay term in the internal feedback, Georgian Math. J. 21 (2014), 475-489.

[6] R. Quintanilla, Slow decay in one-dimensional porous dissipation elasticity, Appl. Math. Lett. 16 (2003), 487-491.

[7] M.A. Goodman, S. C. Cowin, A continuum theory for granular materials, Arch. Ration. Mech. Anal. 44 (1972), 249-266.

[8] J.W. Nunziato, S. C. Cowin, A nonlinear theory of elastic materials with voids, Arch. Ration. Mech. Anal. 72 (1979), 175-201.

[9] S.C. Cowin, J. W. Nunziato, Linear elastic materials with voids, J. Elast. 13 (1983), 125-147.

[10] S.C. Cowin, J. W. Nunziato, The viscoelastic behavior of linear elastic materials with voids, J. Elast. 15 (1985), 185-191.

[11] T.A. Apalara, General decay of solution in one-dimensional porous-elastic system with memory, J. Math. Anal. Appl. 469 (2017), 457-471.

[12] P.S. Casas, R. Quintanilla, Exponential decay in one-dimensional porous-thermoelasticity, Mech. Res. Commun. 32 (2005), 652-658.

[13] S. Zheng, Z. Liu, Semigroups Associated with Dissipaive Systems, Chapman Hall/CRC: Boca, Raton, 1999.

[14] P.S. Casas, R. Quintanilla, Exponential stability in thermoelasticity with microtemperatures, Int. J. Eng. Sci. 43 (2005), 33-47.

[15] J.E.M. Rivera, P.X. Pamplona, R. Quintanilla, Stabilization in elastic solids with voids, J. Math. Anal. Appl. 350 (2009), 37-49.

[16] A. Soufyane, Energy decay for porous-thermo-elasticity systems of memory type, Appl. Anal. 87 (2008), 451-464.

[17] A. Magana, R. Quintanilla, On the time decay of solutions in one-dimensional theories of porous materials, Int. J. Solids Struct. 43 (2006), 3414-3427.

[18] J. E. M. Rivera, R. Quintanilla, On the time polynomial decay in elastic solids with voids, J. Math. Anal. Appl. 338 (2008), 1296-1309.

[19] J.E. M. Rivera, P.X. Pamplona, R. Quintanilla, On the decay of solutions for porous-elastic system with history, J. Math. Anal. Appl. 379 (2011), 682-705.

[20] M.L. Santos, D.A. Junior, On porous-elastic system with localized damping, Z. Angew. Math. Phys. 67 (2016), 1-18.

[21] E. H. Dill, Continum Mechanics: Elasticity, Plasticity, Viscoelasticity, CRC Press, Taylor-Francis Group, New York, 2006.

[22] T.A. Apalara, A general decay for a weakly nonlinearly damped porous system, J. Dyn. Control Syst. 25 (2019), 311-322.

[23] T.A. Apalara, C. A.Raposo, J.O. Ribeiro, Analyticity to transmission problem with delay in porous-elasticity, J. Math. Anal. Appl. 466 (2018), 819-834.

[24] A. Fareh, S.A. Messaoud, Energy decay for a porous thermoelastic system with thermoelasticity of second sound and with a non-necessary positive de nite energy, Appl. Math. Comput. 293 (2017), 493-507.