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ON THE UPPER SECURITY NUMBER OF A GRAPH

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Abstract. In this paper, we investigate the properties of the $S(G)$-set, and the upper security number of standard graphs. We also characterize the graphs with $S(G) \leq 3$ and the graphs with $S(G) \geq n-2$.

Keywords. Secure sets; Graph; Upper security number.

1. Introduction

The concept of *alliance* and some of its variants was introduced by Kristiansen, Hedetniemi, and Hedetniemi [1, 2]. In more realistic settings, alliances are formed so that any attack on the entire alliance or any subset of the alliance can be safeguarded. Considering the model of this situation, Brigham, Dutton, and Hedetniemi [3, 4] introduced the concept of *secure sets* as a generalization of defensive alliances in a graph.

Let $G = (V,E)$ be a graph and $v \in V$. Then the *open neighbourhood* of v is $N(v) = \{u : uv \in E\}$ and the *closed neighbourhood* of v is $N[v] = N(v) \cup \{v\}$. For any set $S \subseteq V$, the *open neighbourhood* of S is $N(S) = \cup_{v \in S} N(v)$ and the closed neighbourhood of S is $N[S] = \cup_{v \in V} N[v]$. The boundary of S is the set $\partial S = N[S] - S$.

Let $S = \{s_1, s_2, \ldots, s_k\} \subseteq V$. An *attack* on S is a collection of k mutually disjoint sets $A = \{A_1, A_2, \ldots, A_k\}$, where $A_i \subseteq N[s_i] - S$, $1 \leq i \leq k$. A *defence* of S is a collection of k mutually disjoint sets $D = \{D_1, D_2, \ldots, D_k\}$, where $D_i \subseteq N[s_i] \cap S$, $1 \leq i \leq k$.

(1) An attack A is *defendable* if there exists a defence D such that $|D_i| \geq |A_i|$ for $1 \leq i \leq k$.
(2) A set S is *secure* if and only if every attack on S is defendable.
(3) The minimum cardinality of a secure set in a graph G is called the *security number* of G, denoted by $s(G)$.

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(4) A secure set of cardinality $s(G)$ is called an $s(G)$-set or a *minimum secure set*.
(5) A secure set S is called *minimal secure set* if none of its proper subsets are secure.
(6) The maximum cardinality of a minimal secure set of G is called the *upper security number* of G, denoted by $S(G)$.
(7) A minimal secure set with cardinality $S(G)$ is called an $S(G)$-*set*.
(8) A subset $X \subset S$ is called $S-secure$ if every attack on S in which $A_i = \emptyset$, whenever $s_i \notin X$, is defendable.

Let $G = (V,E)$ be a graph with k components $G_1, G_2, \ldots, G_k$. Then

$$s(G) = \min\{s(G_i) : 1 \leq i \leq k\}$$

and

$$S(G) = \max\{S(G_i) : 1 \leq i \leq k\}.$$

Therefore, throughout this paper, we consider only connected graphs and follow standard notations as in [5]. For the similar work, we refer to [1, 2, 6–8] and our earlier results [9]. The following observation is straightforward.

Observation 1.1. For a nontrivial graph G of order n, any set of $n-1$ vertices is a secure set and $1 \leq s(G) \leq S(G) \leq n-1$.

We recall the following results for immediate references.

Observation 1.2 (Brigham, Dutton, and Hedetniemi [3])**.** Let $G = (V,E)$ be a connected graph and $S \subseteq V$.

(1) If $X \subseteq S$ is S-secure, then $|N[X] \cap S| \geq |N[X] - S|$. The set S is secure if and only if X is S-secure for every $X \subseteq S$.
(2) Every $s(G)$-set is connected.
(3) If S_1 and S_2 are vertex disjoint secure sets in the graph G, then $S_1 \cup S_2$ is a secure set in G.

Theorem 1.3 (Brigham, Dutton, and Hedetniemi [3])**.** *Let $G = (V,E)$. A set $S \subseteq V$ is secure if and only if $|S| \geq |N[S] - S|$ and every $X \subseteq S$ is S-secure whenever $|X| \leq |N[X] - S| - 1 - \kappa(\langle S \rangle)$, where $\kappa(\langle S \rangle)$ is the vertex connectivity of the subgraph induced by S.*

Theorem 1.4 (Brigham, Dutton, and Hedetniemi [3])**.** *Let $G = (V,E)$ be a graph.*

(1) $s(G) = 1$ if and only if $\delta(G) \leq 1$.
(2) $s(G) = 2$ if and only if $\delta(G) \geq 2$ and V has a subset $S = \{u, v\}$, where u and v are adjacent and $|\partial S| \leq 2$.
(3) $s(G) = 3$ if and only if $s(G) > 2$ and V has a subset $S = \{u, v, w\}$, where $|\partial S| \leq 3$ and $\langle S \rangle$ is either K_3 or $P_3 = \langle u, v, w \rangle$ with $|N(u) \cap \partial S|$, $|N(w) \cap \partial S| \leq 2$.
(4) $s(K_n) = \left\lceil \frac{n}{2} \right\rceil$, $s(C_n) = 2$, and $s(K_{m,n}) = \left\lceil \frac{m+n}{2} \right\rceil$.

Theorem 1.5 (Brigham, Dutton, and Hedetniemi [3])**.** *Let $G = (V,E)$ and $S \subseteq V$. Then S is secure if and only if $|N[X] \cap S| \geq |N[X] - S|$ for all $X \subseteq S$.*

Observation 1.6 (Dutton, Lee, and Brigham [4])**.** Let G be a graph of order n with minimum degree $\delta(G)$. Then any minimal secure set has cardinality at most $n - \left\lceil \frac{\delta(G)}{2} \right\rceil$.

Remark 1.7. For a connected graph G of order 4, any pair of adjacent vertices is a secure set and hence $S(G) \leq 2$. Further, $S(G) = 1$ only if $G \equiv K_{1,3}$.

2. PROPERTIES OF $S(G)$-SETS

We now prove a theorem which extends the result of Brigham, Dutton, and Hedetniemi [3].

Theorem 2.1. *For every minimal secure set S of a graph G, the graph $\langle S \rangle$ is connected.*

Proof. Let $S = \{s_1, s_2, \dots s_m\}$ be a minimal secure set of G. Suppose that $\langle S \rangle$ has k components and $k \geq 2$. Then there is a partition of S into k subsets such that $\langle S_1 \rangle, \langle S_2 \rangle, \dots, \langle S_k \rangle$ are the components of $\langle S \rangle$ with $S_1 \cup S_2 \cup \dots \cup S_k = S$ and $S_i \cap S_j = \emptyset$ for every i, j with $i \neq j$. Without loss of generality, we may assume $S_1 = \{s_1, s_2, \dots, s_{m_1}\}$. Let $A = \{A_1, A_2, \dots, A_{m_1}\}$ be any attack on the set S_1 in G. Then $A' = \{A_1, A_2, \dots, A_{m_1}, A_{m_1+1}, \dots, A_m\}$, where $A_i = \emptyset$ for $m_1 + 1 \leq i \leq m$ is a corresponding attack on S in G. Since S is secure, there exists a defence $D' = \{D_1, D_2, \dots, D_m\}$ such that $|D_i| \geq |A_i|$ for $1 \leq i \leq m$. By the definition of defence, $D_i \subseteq N[s_i] \cap S$ for each i, $1 \leq i \leq m$. Since $S_1 \cap S_j = \emptyset$ for all j, $2 \leq j \leq k$, $D_i \subseteq N[s_i] \cap S_1$ for each i, $1 \leq i \leq m_1$. Thus $D = \{D_1, D_2, \dots, D_{m_1}\}$ is a defence of S_1 such that $|D_i| \geq |A_i|$ for $1 \leq i \leq m_1$ and the attack A on S_1 is defendable. Since an attack A on S_1 is arbitrary, it follows that every attack on S_1 is defendable. Hence S_1 is a secure set and $S_1 \subset S$, a contradiction to the minimality of S. Hence $\langle S \rangle$ must be a connected subgraph of G. □

Observation 2.2. Let $G = (V, E)$ be any graph and S be a secure set of G. Then for any vertex $v \in S$, at least $\left\lfloor \frac{\deg(v)}{2} \right\rfloor$ vertices, which are adjacent to v must belong to S. Hence $|S| \geq \left\lfloor \frac{\deg(v)}{2} \right\rfloor + 1 = \left\lceil \frac{\deg(v)+1}{2} \right\rceil$.

From the definition, $S(G) \geq s(G)$ and hence all the lower bounds for $s(G)$ holds good for $S(G)$ also. In the following theorem, we obtain an upper bound for $S(G)$ in terms of order of G.

Theorem 2.3. *For a connected graph $G = (V, E)$ of order $n \geq 4$, $S(G) \leq n - 2$.*

Proof. From Remark 1.7, the result holds for $n = 4$. Let $n \geq 5$ and v be any vertex of G. Then by Observation 1.2, $s(G) \leq n - 1$ and any set of $n - 1$ vertices is secure. That is, for any vertex $v \in V$, the set $S = V - \{v\}$ is a secure set. We now show that S is not a minimal secure set.

Assume that S is a minimal secure set and $u \in S$. Then for some $u \in S$, $S - \{u\}$ is not a secure set. Therefore there exists an attack A on $S - \{u\}$, which is not defendable. Since $V - (S - \{u\}) = \{u, v\}$ and A is not defendable, there exists at least one vertex w in $S - \{u\}$ such that $\deg(w) = 2$ and w is adjacent to both v and u (as shown in the Figure 1). But as G is a connected graph of order $n \geq 5$, there exists a vertex y in $S - \{u, v, w\}$ such that y is adjacent to u or v. If y is adjacent to u, then $S - \{w\}$ is a secure set (shown in the graph G_1 of Figure 1), which is a contradiction to the assumption that S is a minimal secure set. Suppose that there is no vertex in $S - \{u, v, w\}$ is adjacent to u. Then y is adjacent to only v (the graph G_2 of the Figure 1). The set $S_1 = S - \{u, w\} \neq \emptyset$ and $N[S_1] - S_1 = \{v\}$. Therefore for any subset X of S, $|N[X] - S_1| \leq 1$ and $|N[X] \cap S_1| \geq 1$ and S_1 is a secure set in G. This is a contradiction to the minimality of S. Therefore if S is a minimal secure set, then $|S| \leq n - 2$ and $S(G) \leq n - 2$. □

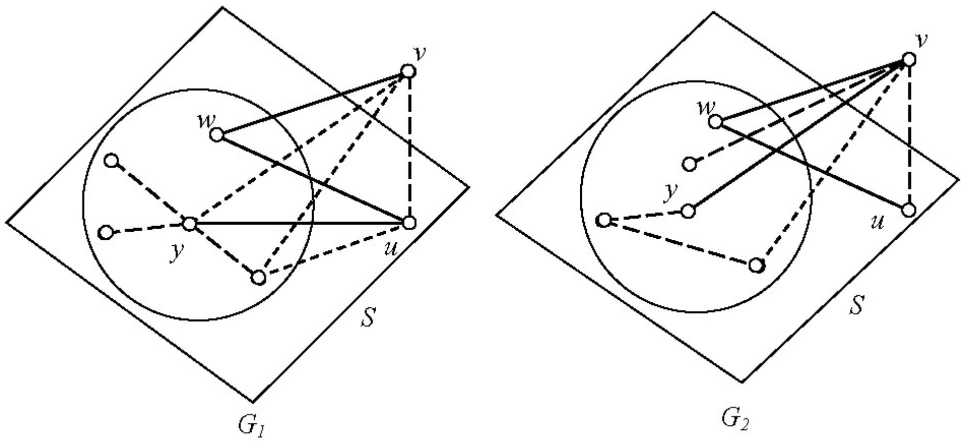

FIGURE 1. The graphs G_1 and G_2 for which $V-\{u,v\}$ is not a secure set.

From Theorem 2.3, a general upper bound for security number of a graph in terms of its order is as follows.

Corollary 2.4. *For a graph G of order $n \geq 4$, $s(G) \leq n-2$ and every set of $n-2$ vertices of G contains a secure set. Further, each set of $n-2$ vertices, not containing a vertex of degree 2, is secure.*

The join of two graphs $G=(V_1,E_1)$ and $H=(V_2,E_2)$ is denoted by $G+H$ and is graph, whose vertex set is $V_1 \cup V_2$, and consists of all the edges of $E_1 \cup E_2$ and the edges joining every vertex of G with every vertex of H.

The upper bound obtained in the above Theorem 2.3 is tight bound for $n=4$ because $S(C_4)=2$. For an integer $n \geq 5$, we consider the graph $G=C_{n-2}+\overline{K}_2$. Let $\{u_1,u_2\}$ be the vertex set of $\overline{K}_2$ and $\{v_1,v_2,\dots,v_{n-2}\}$ be the vertex set of C_{n-2}, with v_i adjacent to v_{i+1} for $1 \leq i \leq n-3$ and v_{n-2} adjacent to v_1. Then the set $S=\{v_1,v_2,\dots,v_{n-2}\}$ is a minimal secure set in G and $S(G)=n-2$. Thus we state the following;

Proposition 2.5. *For every integer $n \geq 4$, there is a graph G of order n with $S(G)=n-2$.*

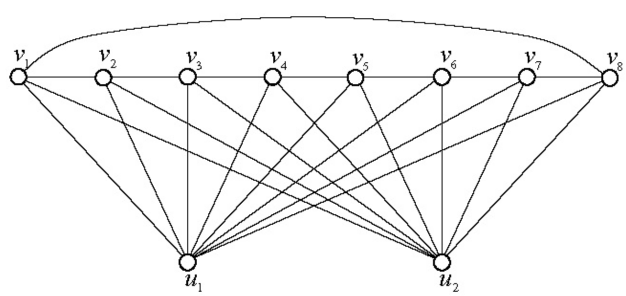

FIGURE 2. The graph $C_8+\overline{K}_2$.

2.1. **Upper Security Number of Standard Graphs.** We recall that $S(K_n)=s(K_n)=1$, for $n=1,2$. In this section, the upper security number of some other standard graphs of order at least 3 are computed.

Proposition 2.6. *For an integer $n \geq 3$, $S(K_n) = s(K_n) = \lceil \frac{n}{2} \rceil$, $S(C_n) = s(C_n) = 2$,*

$$S(P_n) = \begin{cases} 1, & for\ n = 3, \\ 2, & for\ n \geq 4. \end{cases}$$

The proof of the above proposition is a direct consequence of the definition of the upper security number.

Theorem 2.7. *For positive integers m,n with $m \leq n$, we have*

$$S(K_{m,n}) = \begin{cases} 1, & for\ m = 1\ (or\ n = 1), \\ \lceil \frac{m+n}{2} \rceil, & for\ m,n \geq 2. \end{cases}$$

Proof. Every vertex of $K_{1,n}$ is a pendant vertex except the central vertex and each singleton set containing a pendant vertex is a secure set. Therefore, every subset of vertices containing more than one vertex is not a minimal secure set. So $S(K_{1,n}) = s(K_{1,n}) = 1$. Let $m,n \geq 2$. By Observation 1.2 and the statement (4) of Theorem 1.4, we get

$$S(K_{m,n}) \geq \left\lceil \frac{m+n}{2} \right\rceil.$$

Let $V_1 = \{u_1, u_2, \ldots u_m\}$, $V_2 = \{v_1, v_2, \ldots v_n\}$ be the bipartition of vertex set and $S \subseteq V$ be a minimal secure set of $K_{m,n}$. Since $\langle V_2 \rangle$ is totally disconnected, by Theorem 2.1, we get $S \cap V_1 \neq \emptyset$. Let $v \in S \cap V_1$. Then by Theorem 1.5 with $X = \{v\}$, we get $|N[v] \cap S| \geq |N[v] - S|$. This implies that

$$|S \cap V_2| + 1 \geq |V_2| - |S \cap V_2| \Rightarrow |S \cap V_2| \geq \frac{n-1}{2}.$$

Since $|S \cap V_2|$ is an integer, we have

$$|S \cap V_2| \geq \left\lceil \frac{n-1}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor.$$

Similarly, we get $|S \cap V_1| \geq \lfloor \frac{m}{2} \rfloor$. Let S' be any secure set with $|S'| > \lceil \frac{m+n}{2} \rceil$. Then S' contains a minimal secure set. Therefore by the above arguments, $|S' \cap V_1| \geq \lfloor \frac{m}{2} \rfloor$ and $|S' \cap V_2| \geq \lfloor \frac{n}{2} \rfloor$. Further, we observe $|S' \cap V_1| \geq \lfloor \frac{m}{2} \rfloor + 1$ or $|S' \cap V_2| \geq \lfloor \frac{n}{2} \rfloor + 1$ (otherwise, $|S'| = |S' \cap V_1| + |S' \cap V_2| \leq \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor \leq \lceil \frac{m+n}{2} \rceil$, a contradiction). Without loss of generality, we assume that

$$|S' \cap V_1| \geq \left\lfloor \frac{m}{2} \right\rfloor + 1.$$

Let $v \in S' \cap V_1$, $S_1 = S' - \{v\}$ and $X \subseteq S_1$. Then,

$$|N[X] \cap S_1| \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 \geq n - \left\lfloor \frac{n}{2} \right\rfloor \geq |N[X] - S|, \text{ if } X \subseteq S_1 \cap V_1;$$

$$|N[X] \cap S_1| \geq \left\lfloor \frac{m}{2} \right\rfloor + 1 \geq m - \left\lfloor \frac{m}{2} \right\rfloor \geq |N[X] - S|, \text{if } X \subseteq S_1 \cap V_2;$$

and

$$|N[X] \cap S_1| \geq \left\lceil \frac{m+n}{2} \right\rceil \geq (m+n) - \left\lceil \frac{m+n}{2} \right\rceil \geq |N[X] - S|, \text{ otherwise.}$$

Therefore, by Theorem 1.5, S_1 is a secure set. Thus S' is not minimal. Since S' is arbitrary, it follows that, $S(K_{m,n}) \leq \lceil \frac{m+n}{2} \rceil$. Hence $S(K_{m,n}) = \lceil \frac{m+n}{2} \rceil$. ☐

Remark 2.8. Let V_1 and V_2 be vertex partition of $K_{m,n}$. If S is any subset of vertices of $K_{m,n}$ with $|S \cap V_1| \geq \lceil \frac{m}{2} \rceil$ and $|S \cap V_2| \geq \lceil \frac{n}{2} \rceil$, then S is secure in $K_{m,n}$.

Remark 2.9. If S is a secure set of $K_{m,n}$ and S' is any set containing S, then S' is also a secure set in $K_{m,n}$.

The graph $C_n + K_1$ is said to be *wheel* and is denoted by $W_{1,n}$. The graph $W_{1,n}$ is of order $n+1$, consists one vertex of degree n, called the *central vertex* and all other vertices are of degree 3, called *rim vertices*. The edges between a rim vertex and the central vertex are called *spokes* and the other edges are considered as *rim edges*.

Theorem 2.10. *For an integer $n \geq 3$, we have*

$$s(W_{1,n}) = \begin{cases} 2, & for\ n = 3, \\ 3, & for\ n \geq 4. \end{cases}$$

Proof. Observe the graph $W_{1,3} \equiv K_4$. Hence, by Proposition 1.4, we have $s(W_{1,3}) = 2$. Let $n \geq 4$, and $V(W_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$ with v as its central vertex; $v_1, v_2, \ldots v_n$ as rim vertices such that v_i is adjacent to v_{i+1} for each i, $1 \leq i \leq n-1$; and v_n adjacent to v_1. We note that if S is any set of vertices of $W_{1,n}$ with $|S| = 2$ and $|N[S] - S| \geq 3$, by Theorem 1.3, we have that S is not a secure set and $s(W_{1,n}) \geq 3$. In $W_{1,4}$, any set of three vertices is a secure set and $s(W_{1,4}) = 3$. Let $n \geq 5$. Consider the set $S = \{v_1, v_2, v_3\}$. Then $N[S] - S = \{v_n, v_4, v\}$. For any set $X \subseteq S$, we observe that $|N[X] \cap S| \geq |N[X] - S|$. Hence by Theorem 1.5, S is a secure set and $s(W_{1,n}) = 3$ for all $n \geq 4$. □

Remark 2.11. For an integer $n \geq 4$, let $V(W_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$ with v as its central vertex and $v_1, v_2, \ldots v_n$ as rim vertices such that v_i is adjacent to v_{i+1} for each i, $1 \leq i \leq n-1$ and v_n adjacent to v_1. Then, for each $i, 1 \leq i \leq n$, the set $S_i = \{v_i, v_{i+1}, v_{i+2}\}$ is a secure set, where the suffix additions are under modulo n.

Theorem 2.12. *For an integer $n \geq 3$, we have*

$$S(W_{1,n}) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. Observe that the graph $W_{1,3} \equiv K_4$. Hence by Theorem 2.6, $S(W_{1,3}) = 2$. Let $n \geq 4$, and $V(W_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$ with v as its central vertex and $v_1, v_2, \ldots v_n$ are rim vertices such that v_i is adjacent to v_{i+1} for each i, $1 \leq i \leq n-1$ and v_n adjacent to v_1. We observe the following:

(1) The set $\{v\}$ is not a secure set.
(2) For any i, $1 \leq i \leq n$, $\deg(v_i) = 3$, implies that $\{v_i\}$ is not a secure set.
(3) For any i, j with $1 \leq i, j \leq n$ we have $|N[\{v_i, v_j\}] - \{v_i, v_j\}| \geq 3$ and by Theorem 1.3, the set $\{v_i, v_j\}$ is not a secure set. Also for any i, $1 \leq i \leq n$, $|N[\{v_i, v\}] - \{v_i, v\}| \geq 3$ and the set $\{v_i, v\}$ is not a secure set.
(4) For any i, $1 \leq i \leq n$, the set $\{v_i, v_{i+1}, v_{i+2}\}$ (where addition of suffixes is under modulo n) is a minimal secure set.

Now consider the set $S = \{v_1,\ v_2,\ v_4,\ v_5,\ v_7,\ v_8,\ \ldots,\ v_k,\ v\}$, where $k = n - \lceil \frac{n}{4} \rceil$ for $n \equiv 2 (\mathrm{mod}\ 4\ )$, otherwise $k = n - \lceil \frac{n}{4} \rceil - 1$. For any vertex $v_j \in S$, $|N[v_j] \cap S| = 3$, and $|N[v_j] - S| = 1$

implies that $|N[X] \cap S| \geq |N[X] - S|$ for all $X \subseteq S$. So, S is a secure set. Moreover, any non-empty subset of S contains a subset S_1 such that $N[S_1] = \{v_i\}$ or $\{v, v_i\}$ for some i, $1 \leq i \leq n$; or $N[S_1] = \{v_i, v_j\}$ for some i, j, with $1 \leq i, j \leq n$. Hence by the above observations, no proper subset of S is secure and S is a minimal secure set. Thus, $S(W_{1,n}) \geq |S| = \lceil \frac{n+1}{2} \rceil$.

Further, if S is any secure set with $|S| > \lceil \frac{n+1}{2} \rceil$, S is not minimal. This fact is proved in two cases.

Case 1. $v \notin S$.

Since S is secure, for each $v_i, v_j \in S$, we must have $|N[v_i] \cap S| \geq 2$ and $|N[\{v_i, v_j\}]| \geq 3$. By the pigeonhole principle, we can find some j, $1 \leq j \leq n$, such that $S' = \{v_j, v_{j+1}, v_{j+2}\} \subset S$, where the addition of suffixes is under modulo n. Now the set S' is secure set of $W_{1,n}$ and S is not minimal.

Case 2. $v \in S$.

In this case, if there exist two vertices $v_i, v_j \in S$, such that $|i - j| \geq 3$ and

$$|N[v_i] \cap S| = |N[v_j] \cap S| = 2,$$

then S is not a secure set. Therefore, there can be at most one i, where $1 \leq i \leq n$ is such that $|N[v_i] \cap S| = |N[v_{i+2}] \cap S| = 2$. Since $|S| > \lceil \frac{n+1}{2} \rceil$ and some j, we can find a secure set $1 \leq j \leq n$ such that $\{v_j, v_{j+1} v_{j+2}\} \subset S$. Otherwise, we can find a proper subset S_1 of S with at most one vertex $v_i \in S_1$ with $|N[v_i] \cap S_1| = 2$. For any other vertex $v_j \in S_1$, $|N[v_j] \cap S_1| = 3$. Therefore S_1 is a secure set of $W_{1,n}$. Thus $S(W_{1,n}) = \lceil \frac{n+1}{2} \rceil$. $\square$

3. Characterization of Graphs with $S(G) \leq 3$

By Proposition 1.4, $s(G) = 1$ if and only if $\delta(G) \leq 1$. We seek similar results related to upper security number of a graph. It is very clear, for all the graphs with $\delta(G) \geq 2$, $s(G) \geq 2$ and $S(G) \geq 2$. But we have several graphs with $\delta(G) = 1$ and $S(G) \geq 2$. We now characterize the graphs with $S(G) = 1$ in terms of number of pendant vertices in the graph.

Let $G = (V, E)$ be a connected graph of order at least 3 and P be the set of pendant vertices of G. Suppose that $P \neq \emptyset$ and no subset of $V - P$ is secure. Then every secure set of G must contain at least one pendant vertex. A singleton set containing a pendant vertex is a minimal secure set. Therefore, if $P \neq \emptyset$ and no subset of $V - P$ is secure, then $S(G) = 1$. On the other hand if $S(G) = 1$, then $\delta(G) \leq 1$. Thus we have the following proposition.

Proposition 3.1. *Let $G = (V, E)$ be a connected graph and P be the set of all pendant vertices of G. Then $S(G) = 1$ if and only if $P \neq \emptyset$ and no subset of $V - P$ is secure.*

Proposition 3.2. *If $G = (V, E)$ be a connected graph of order at least 3 with $S(G) = 1$ and P be the set of pendant vertices of G, then $|P| > |V - P|$.*

Proof. Let $G = (V, E)$ be a connected graph of order at least 3 with $S(G) = 1$ and P be the set of all pendant vertices of G. Suppose that $|P| \leq |V - P|$. Then the set $V - P$ is a secure set and hence we can find a minimal secure set S such that $S \subseteq V - P$. Since S has no pendant vertex and is secure, we get $S(G) \geq 2$, which is a contradiction. Therefore, $|P| > |V - P|$. $\square$

The converse of the above Proposition 3.2 is not true. In fact, there are several graphs with $|P| > |V-P|$ and $S(G) \geq 2$. For the graphs of the Figure 3, we have $S(G) = 1$. The contrapositive of Proposition 3.2 is quite useful.

Remark 3.3. Let $G = (V,E)$ be a connected graph of order at least 3 and P be the set of pendant vertices of G. If $|P| \leq |V-P|$, then $S(G) \geq 2$.

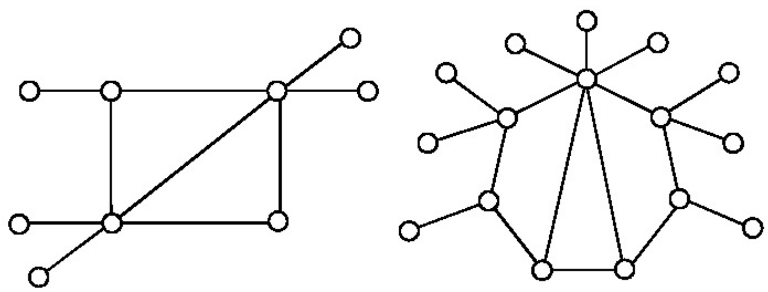

FIGURE 3. The graphs for which $S(G) = 1$.

The following result is a consequence of Theorem 2.1 and Proposition 1.4.

Proposition 3.4. *Let $G = (V,E)$ be a connected graph of order $n \geq 4$. Then $S(G) = 2$ if and only if every set S of G with $|S| \geq 2$ and not containing a pendant vertex contains a subset $S' = \{u,v\}$ such that u,v are adjacent and $|\partial S| \leq 2$.*

Now we determine the properties of graphs with upper security number 3. Let S be a secure set of graph G with $|S| \geq 3$. If S contains a pendant vertex or two adjacent vertices u,v with $\deg(u) = \deg(v) = 2$, then by Proposition 3.1 and Proposition 3.4, S is not a minimal secure set. Therefore, if a secure set with $|S| \geq 3$ is minimal, then for any $u \in S$, $\deg u \geq 2$ and for any two adjacent vertices u,v, $|\partial\{u,v\}| \geq 3$. By Proposition 1.4, it follows that $S(G) = 3$ if and only if every secure set S of G with $|S| \geq 4$ is not minimal and there exists at least one minimal secure set of cardinality 3. Thus we conclude:

Proposition 3.5. *Let $G = (V,E)$ be a connected graph of order $n \geq 6$. Then $S(G) = 3$ if and only if every set S of vertices of G with $|S| \geq 3$ containing no pendant vertex or no two adjacent vertices u,v each of degree 2, contains a subset $S' = \{u,v,w\}$, with $|\partial S'| \leq 3$ and $\langle S' \rangle \equiv K_3$ or P_3.*

4. GRAPHS WITH $s(G) \geq n-2$

In Section 2, we proved that $s(G) \leq n-2$ for a connected graph G of order $n \geq 4$ and any set of $n-2$ vertices of G is secure. In this section, we characterize the graphs of order n with $s(G) = n-1$ and $s(G) = n-2$.

Proposition 4.1. *For a nontrivial connected graph G of order n, $s(G) = n-1$ if and only if $G \equiv K_2$ or $G \equiv K_3$.*

Proof. Let G be a nontrivial connected graph of order n and $s(G) = n-1$. From Corollary 2.4, we have $n \leq 3$. If $n = 3$, then connected graphs of order 3 are P_3 and K_3. By using Proposition 1.4, we have $s(P_3) = 1$ and $s(K_3) = 2$. For $n = 2$ the only connected graph of order 2 is K_2 and

$s(K_2) = 1$. Thus $G \equiv K_2$ or $G \equiv K_3$ for a nontrivial graph G of order n with $s(G) = n-1$. The converse follows directly from Proposition 1.4. $\square$

In [4], Dutton et al. proved that if $d_1 \le d_2 \le \cdots \le d_n$ degree sequence of a graph G, then $s(G) \le n - \left\lceil \frac{d_{k+1}}{2} \right\rceil$, where $k = \max\left\{ i : i \le \left\lceil \frac{d_{i+1}}{2} \right\rceil \right\}$. If $d_4 \ge 5$, then $k \ge 3$ and $d_i \le d_{i+1}$ for all i, $1 \le i \le n$ implies $\left\lceil \frac{d_{k+1}}{2} \right\rceil \ge \left\lceil \frac{d_4}{2} \right\rceil \ge 3$. Then $s(G) \le n-3$.

Remark 4.2. Let $d_1 \le d_2 \le \cdots \le d_n$ be the degree sequence of a graph G and $s(G) = n-2$. Then $d_4 \le 4$ and hence G has at least 4 vertices of degree not exceeding 4.

Lemma 4.3. *Let G be a connected graph of order n. If $s(G) = n-2$, then $n \le 5$.*

Proof. Let $G = (V,E)$ be a connected graph of order n with $s(G) = n-2$ and $S = V - \{u,v\}$ be a $s(G)$-set in G. Then, for any $x \in S$, we consider the set $S_x = S - \{x\}$. Since S is $s(G)$-set, we have that S_x is not a secure set. But $S_x = V - \{u,v,x\}$ implies that $|S_x| = n-3$. By Theorem 1.5, we have that there exists a subset $X \subseteq S_x$ such that $|N[X] \cap S_x| < |N[X] - S_x|$. Since $N[X] - S_x \subseteq V - S_x = \{u,v,x\}$, we get $|N[X] - S_x| \le 3$. Therefore, $|N[X] \cap S_x| < 3$.

If $X = S_x$, then $N[X] \cap S_x = S_x$, which implies $|S_x| < 3$. But we have $|S_x| = n-3$. Therefore $n-3 < 3$ and hence $n \le 5$. Now we consider the case $X \ne S_x$. Since S is secure and $S_x \subset S$, by Observation 1.2, we have that $S_x - y$ is S-secure for every $y \in S_x$. If $|S_x| \ge |N[S_x] - S_x|$, then we find from Theorem 1.5 that S_x is a secure set, which contradicts the fact that S is a $s(G)$-set. Thus $|S_x| < |N[S_x] - S_x|$. But we have $|N[X] - S_x| \le 3$. Therefore $|S_x| = n-3 < 3$. Hence, we get $n \le 5$ in this case. This completes the proof of the lemma. $\square$

Theorem 4.4. *For a connected graph G of order n, $s(G) = n-2$ if and only if $G \in \Im = \{P_3,\ C_4,\ K_4,\ K_4 - e,\ K_5,\ K_5 - e,\ C_4 + K_1,\ C_3 + 2K_1, J_4\}$, where J_4 is the gear graph of order 5 obtained from $W_{1,4}$ by removing the alternating spokes.*

Proof. By Proposition 1.4, we get $s(P_3) = 1$, $s(C_4) = 2$, $s(K_4) = 2$, and $s(K_5) = 3$. Let e be any edge in K_4. Then we observe that $\delta(K_4 - e) \ge 2$ implies $s(K_4 - e) \ge 2$ and the set S containing any two adjacent vertices is a secure set. Therefore $s(K_4 - e) = 2$. Now for each $G \in \Gamma = \{K_5 - e,\ C_4 + K_1,\ C_3 + 2K_1$, and $J_4\}$, the order of G is 5 and every minimal secure set in it is also a dominating set. Hence $s(G) \ge 3$. Also for all the graphs in $\{K_5 - e, C_4 + K_1$, $C_3 + 2K_1$, and $J_4\}$,we observe that any set S containing 3 or more vertices is a secure set. Therefore, $s(G) = 3$ for each $G \in \Gamma$. Thus if G is graph of order n and $G \in \Im$, then $s(G) = n-2$.

Conversely, let G be a connected graph of order n with $s(G) = n-2$. Using by Lemma 4.3, we have $n \le 5$. By Proposition 1.4, for $n = 3$, the only possibility is $G \equiv P_3$, and for $n = 4$, the only possibilities are $G \equiv C_4$, $G \equiv K_4$ or $G \equiv K_4 - e$. Now for $n = 5$, there are only 5 graphs satisfying the property that $\delta(G) \ge 2$ and V has no subset $S = \{u,v\}$ with u and v are adjacent and $|\partial S| \le 2$, and they are K_5, $K_5 - e$, $C_4 + K_1$, $C_3 + 2K_1$ and J_4. Therefore, $s(G) = 3$ and order of G is 5, and G must be one among the graphs K_5, $K_5 - e$, $C_4 + K_1$, $C_3 + 2K_1$, J_4. Thus, if G is a connected graph of order n with $s(G) = n-2$, then $G \in \Im$. $\square$

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