Commun. Optim. Theory 2019 (2019), Article ID 9 https://doi.org/10.23952/cot.2019.9

Communications in Optimization Theory
Available online at http://cot.mathres.org

A NEW GENERAL ITERATIVE ALGORITHM FOR SOLVING A VARIATIONAL INEQUALITY PROBLEM WITH A QUASI-NONEXPANSIVE MAPPING IN BANACH SPACES

T.M.M. SOW

Department of Mathematics, Gaston Berger University, Saint Louis, Senegal

Abstract. In this paper, by using a modified Krasnoselskii-Mann algorithm, we introduce and study a new iterative method with a strongly accretive and Lipschitzian operator for solving a variational inequality problem with a quasi-nonexpansive maping in a Banach space. We prove the strong convergence of the proposed iterative scheme without imposing any compactness condition on the mapping or the space. Applications are also considered in Hilbert spaces.

Keywords. Fixed point; Iterative method; Quasi-nonexpansive mapping; Variational inequality.

2010 Mathematics Subject Classification. 47H06, 65K05.

1. INTRODUCTION

Let E be a real Banach space and let K be a nonempty, closed and convex subset of E. We denote by J the normalized duality map from E to 2^{E^*}, where E^* is the dual space of E, defined by

$$J(x) := \{x^* \in E^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\}, \quad \forall x \in E.$$

Recall that a Banach space E is said to be smooth if $\lim_{t\to 0}(\|x+ty\| - \|x\|)/t$ exists for each $x, y \in S_E$, where $S_E := \{x \in E : ||x|| = 1\}$ is the unit sphere of E. E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S_E$, and E is Frechet differentiable if it is smooth and the limit is attained uniformly for $y \in S_E$. It is known that E is smooth if and only if each duality map J is single-valued, E is Frechet differentiable if and only if duality map J is norm-to-norm continuous on E, and E is uniformly smooth if and only if duality map J is norm-to-norm uniformly continuous on bounded subsets of E. Let E be a real normed linear space. The modulus of smoothness of E, ρ_E, is defined by:

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}; \quad \tau > 0.$$

E-mail address: sowthierno89@gmail.com.

Received October 9, 2018; Accepted May 4, 2019.

It is known that a normed linear space E is *uniformly smooth* if $\lim_{\tau\to 0}\rho_E(\tau)/\tau = 0$. If there exists a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \le c\tau^q$, then E is said to be q-uniformly smooth. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$, where

$$L_p\ (or\ l_p)\ or\ W_p^m \text{ is } \begin{cases} 2-\text{uniformly smooth and } p-\text{uniformly convex} & \text{if } \ 2 \le p < \infty; \\ 2-\text{uniformly convex and } p-\text{uniformly smooth} & \text{if } \ 1 < p < 2. \end{cases}$$

Let J_q denote the generalized duality mapping from E to 2^{E^*} defined by

$$J_q(x) := \left\{ f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1} \right\},$$

where $\langle .,.\rangle$ denotes the generalized duality pairing. J_2 is called the normalized duality mapping and is denoted by J as above. Notice that for $x \neq 0$, $J_q(x) = \|x\|^{q-2}J(x)$, $q > 1$.

Following Browder [1], we say that a Banach space has a weakly continuous normalized duality map if J is a single-valued and is weak-to-weak* sequentially continuous, i.e., if $(x_n) \subset E$, $x_n \rightharpoonup x$, then $J(x_n) \rightharpoonup J(x)$ in E^*.

The weak continuity of duality map J plays an important role in the fixed point theory for nonlinear operators. Recall that a Banach space E satisfies the Opial property [2] if

$$\liminf_{n\to+\infty}\|x_n - x\| < \liminf_{n\to+\infty}\|x_n - y\|$$

whenever $x_n \rightharpoonup x$ and $x \neq y$. A Banach space E that has a weakly continuous normalized duality map satisfies the Opial's property.

Let E be a real Banach space and let K be a nonempty subset of E. A map $T : K \to E$ is said to be Lipschitz if there exists an $L \ge 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|, \quad \forall x, y \in K.$$

If $L < 1$, T is called a contraction and if $L = 1$, T is called a nonexpansive mapping. We denote by $F(T)$ the set of fixed points of mapping T, that is, $F(T) := \{x \in K : x = Tx\}$. A map T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - p\| \le \|x - p\|$, $\forall x \in K, p \in F(T)$. It is easy to see that the class of quasi-nonexpansive mappings properly includes that of nonexpansive maps with fixed points.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$. An operator $A : H \to H$ is said to be monotone if

$$\langle Ax - Ay, x - y\rangle_H \ge 0, \quad \forall\, x, y \in H.$$

A is said to be k-strongly monotone if there exists $k \in (0,1)$ such that

$$\langle Ax - Ay, x - y\rangle_H \ge k\|x - y\|^2, \quad \forall\, x, y \in H.$$

$A : H \to H$ is said to be strongly positive if there exists a constant $k > 0$ such that

$$\langle Ax, x\rangle_H \ge k\|x\|^2, \quad \forall\, x \in H.$$

Let E be a smooth Banach space. A map $A : E \to E$ is said to be accretive if

$$\langle Ax - Ay, J_q(x - y)\rangle \ge 0, \quad \forall x, y \in E.$$

Let $q > 1$ be a fixed real number. A is said to be k-strongly accretive if there exists $k \in (0,1)$ such that

$$\langle Ax - Ay, J_q(x - y)\rangle \ge k\|x - y\|^q, \quad \forall x, y \in E.$$

In Hilbert spaces, the normalized duality map is the identity map. Hence, strongly monotonicity and strongly accretivity coincide in Hilbert spaces.

Let C be a nonempty closed convex subset of a smooth Banach space E and let $A : C \to E$ be a nonlinear mapping. Let T be a mapping on C such that $F(T) \neq \emptyset$. A point $u \in F(T)$ is said to be a solution of the variational inequality problem $VI(A,T)$ provided that

$$\langle Au, J(u-v)\rangle \leq 0, \quad \forall v \in F(T).$$

The variational inequality problem, which finds a number of applications in diverse disciplines, such as, differential equations, time-optimal control, optimization, mathematical programming, mechanics, and finance has been studied by many authors; see, for example, [3, 4, 5, 6] and the references therein.

Recently, Yao, Zhou and Liou [7] proved the following convergence theorem.

Theorem 1.1. [7] *Let H be a real Hilbert space. Let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For given $x_0 \in H$, let $\{x_n\}$ and $\{y_n\}$ be the sequences generated iteratively by $x_0 \in H$ by:*

$$\begin{cases} y_n = (1-\alpha_n)x_n \\ x_{n+1} = \lambda y_n + (1-\lambda)Ty_n, \end{cases} \tag{1.1}$$

where λ is a constant in $(0,1)$ and $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying: (i) $\lim_{n\to\infty}\alpha_n = 0$ and (ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.1) converge strongly to fixed point of T.

Iterative methods for nonexpansive mappings have been applied to solve convex minimization problems; see, e.g., [8, 9, 10] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle \tag{1.2}$$

In [8], Xu proved that the sequence $\{x_n\}$ defined by iterative method below with initial guess $x_0 \in H$ chosen arbitrarily:

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n \; n \geq 0,$$

where $\{\alpha\}$ is a real number sequence in $(0,1)$, T is a nonexpansive mapping and A a strongly positive bounded linear operator, converges strongly to the unique solution of minimization problem (1.2).

Recently, Marino and Xu [11] considered a general iterative method for a nonexpansive mapping. Let f be a contraction on H and let $A : H \to H$ be a strongly positive bounded linear operator. The sequence $\{x_n\}$ defined by iterative method below with initial guess $x_0 \in H$ chosen arbitrarily:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0$$

converges strongly to the fixed point of T, which is a unique solution of the following variational inequality

$$\langle Ax^* - \gamma f(x^*), x^* - p\rangle \leq 0, \quad \forall p \in F(T).$$

However, there are few results on iterative solutions of variational inequality problems involving a quasi-nonexpansive mapping. Motivated by Marino and Xu [11], Yao, Zhou and Liou [7], we construct a Krasnoselskii-Mann iterative algorithm for finding solutions of a variational inequality problem involving a quasi-nonexpansive mapping in Banach spaces. Applications are also investigated in the framework of Hilbert spaces.

2. Preliminaries

We start with the following demiclosedness principle for nonexpansive mappings (Browder [1]).

Lemma 2.1 (Demiclosedness Principle)**.** *Let E be a Banach space satisfying Opial's property. Let K be a closed convex subset of E, and let $T: K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Then $I-T$ is demiclosed, that is,*

$$\{x_n\} \subset K,\; x_n \rightharpoonup x \in K \text{ and } (I-T)x_n \to y \text{ implies that } (I-T)x = y.$$

Lemma 2.2 ([12])**.** *Let E be a Banach space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x,y \in E.$$

Lemma 2.3. [12] *Let $q>1$ be a fixed real number and E be a smooth Banach space. Then the following statements are equivalent:*
(i) E is q-uniformly smooth.
(ii) There is a constant $d_q > 0$ such that

$$\|x+y\|^q \leq \|x\|^q + q\langle y\,, J_q(x)\rangle + d_q\|y\|^q, \quad \forall x,y \in E.$$

(iii) There is a constant $c_1 > 0$ such that

$$\langle x-y\,, J_q(x) - J_q(y)\rangle \leq c_1\|x-y\|^q, \quad \forall x,y \in E.$$

Lemma 2.4 (Zalinescu [13])**.** *Let E be a uniformly convex real Banach space. For arbitrary $r>0$, let $B_r := \{x \in E : ||x|| \leq r\}$ and $\lambda \in [0,1]$. Then there exists a continuous, strictly increasing and convex function*

$$g : [0,2r] \to \mathbb{R}^+,\; g(0) = 0$$

such that

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - (1-\lambda)\lambda g(\|x-y\|), \quad \forall x,y \in B_r.$$

Lemma 2.5 (Xu, [14])**.** *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n$$

for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that (a) $\sum_{n=0}^{\infty}\alpha_n = \infty$, (b) $\limsup_{n\to\infty}\sigma_n \leq 0$ or $\sum_{n=0}^{\infty}|\sigma_n\alpha_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. [15] *Let H be a real Hilbert space. Let K be a nonempty, closed convex subset of H and $A : K \to H$ be a k-strongly monotone and L-Lipschitzian operator. Assume that $0 < \eta < \dfrac{2k}{L^2}$ and $\tau = \eta\Big(k - \dfrac{L^2\eta}{2}\Big)$. Then, for each $t \in \Big(0, min\{1, \dfrac{1}{\tau}\}\Big)$,*

$$\|(I-t\eta A)x - (I-t\eta A)y\| \leq (1-t\tau)\|x-y\|,\; \forall x,y \in K.$$

Lemma 2.7. *Let $q > 1$ be a fixed real number and let E be a q-uniformly smooth real Banach space with constant d_q. Let K be a nonempty, closed convex subset of E and let $A : K \to E$ be a k-strongly accretive and L-Lipschitzian operator. Assume that $0 < \eta < \left(\frac{kq}{d_q L^q}\right)^{\frac{1}{q-1}}$ and $\tau = \eta\left(k - \frac{d_q L^q \eta^{q-1}}{q}\right)$. Then, for each $t \in \left(0, \min\{1, \frac{1}{\tau}\}\right)$,*

$$\|(I - t\eta A)x - (I - t\eta A)y\| \leq (1 - t\tau)\|x - y\|, \ \forall x, y \in K.$$

Proof. Using (ii) of Lemma 2.3 and properties of A, we have

$$\begin{aligned}
& \|(I - t\eta A)x - (I - t\eta A)y\|^q \\
\leq \ & \|x - y\|^q + q\langle t\eta Ay - t\eta Ax, J_q(x - y)\rangle + d_q\|t\eta Ax - t\eta Ay\|^q \\
\leq \ & \|x - y\|^q - qt\eta\langle Ax - Ay, J_q(x - y)\rangle + d_q(t\eta)^q\|Ax - Ay\|^q \\
\leq \ & \|x - y\|^q - qtk\eta\|x - y\|^q + d_q(Lt\eta)^q\|x - y\|^q \\
\leq \ & \left(1 - qtk\eta + d_q L^q t^q \eta^q\right)\|x - y\|^q.
\end{aligned}$$

Therefore

$$\|(I - t\eta A)x - (I - t\eta A)y\| \leq \left(1 - qtk\eta + d_q L^q t^q \eta^q\right)^{\frac{1}{q}}\|x - y\|. \tag{2.1}$$

Using definition of η, inequality (2.1) and inequality $(1 + x)^s \leq 1 + sx$, for $x > -1$ and $0 < s < 1$, we have

$$\begin{aligned}
\|(I - t\eta A)x - (I - t\eta A)y\| & \leq \left(1 - tk\eta + \frac{d_q L^q t^q \eta^q}{q}\right)\|x - y\| \\
& \leq \left(1 - t\eta(k - \frac{d_q L^q \eta^{q-1}}{q})\right)\|x - y\| \\
& \leq (1 - t\tau)\|x - y\|,
\end{aligned}$$

which completes this lemma. □

Let C be a nonempty subsets of real Banach space E. A mapping $Q_C : E \to C$ is said to be sunny if

$$Q_C(Q_C x + t(x - Q_C x)) = Q_C x$$

for each $x \in E$ and $t \geq 0$. A mapping $Q_C : E \to C$ is said to be a retraction if $Q_C x = x$ for each $x \in C$.

Lemma 2.8. [16] *Let C and D be nonempty subsets of a real Banach space E with $D \subset C$ and $Q_D : C \to D$ a retraction from C into D. Then Q_D is sunny and nonexpansive if and only if*

$$\langle z - Q_D z, j(y - Q_D z)\rangle \leq 0$$

for all $z \in C$ and $y \in D$.

Remark 2.9. If K is a nonempty closed convex subset of a Hilbert space H, then the nearest point projection P_K from H to K is the sunny nonexpansive retraction.

Lemma 2.10. *Let H be a real Hilbert space. Let K be a nonempty, closed convex subset of H and let $A : K \to H$ be a k-strongly monotone and L-Lipschitzian operator. Let T be a quasi-nonexpansive mapping on K. Then, $VI(A, T)$ is nonempty.*

Proof. Set η and τ two real numbers such that $0 < \eta < \dfrac{2k}{L^2}$ and $\tau = \eta\Big(k - \dfrac{L^2\eta}{2}\Big)$. Let t_0 be a fixed real number such that $t_0 \in \Big(0, \min\{1, \dfrac{1}{\tau}\}\Big)$. We observe that $P_{F(T)}(I - t_0\eta A)$ is a contraction. Indeed, for all $x, y \in K$, we have from Lemma 2.6 that

$$\begin{aligned}\|P_{F(T)}(I -_0 \eta A)x - P_{F(T)}(I - t_0\eta A)y\| &\leq \|(I - t_0\eta A)x - (I - t_0\eta A)y\| \\ &\leq (1 - t_0\tau)\|x - y\|.\end{aligned}$$

Banachs Contraction Mapping Principle guarantees that $P_{F(T)}(I - t_0\eta A)$ has a unique fixed point, say $x_1 \in H$. That is, $x_1 = P_{F(T)}(I - t_0\eta A)x_1$. Thus, in view of Lemma 2.8 and Remark 2.9, it is equivalent to the following variational inequality problem

$$\langle Ax_1, x_1 - p\rangle \leq 0 \ \forall p \in F(T).$$

Hence, $x_1 \in VI(A, T)$. This completes this proof. □

3. Iterative Algorithms

In this section, we present and analyze our general iterative method in Banch spaces. In what follows, we use the following explicit iteration scheme. Let $q > 1$ be a fixed real number and let E be a q-uniformly smooth and uniformly convex real Banach space having a weakly continuous normalized duality map $J : E \to E^*$. Let K be a nonempty, closed and convex subset of E which is a nonexpansive retract of E with Q_K as the nonexpansive retraction. Let $A : K \to E$ be a k-strongly accretive and L-Lipschitzian operator, $T : K \to K$ be a quasi-nonexpansive mapping. Let $\{x_n\}$ and $\{y_n\}$ be a sequences defined iteratively by:

$$\begin{cases} y_n = Q_K(I - \alpha_n\eta A)x_n \\ x_{n+1} = \lambda y_n + (1-\lambda)Ty_n, \end{cases} \tag{3.1}$$

where λ is a constant in $(0,1)$ and $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying: (i) $\lim_{n\to\infty}\alpha_n = 0$ and (ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Assumption 3.1. (a) $0 < \eta < \Big(\dfrac{kq}{d_qL^q}\Big)^{\frac{1}{q-1}}$.
(b) $I - T$ is demiclosed at the origine.
(c) $VI(A, T)$ is nonempty.

Theorem 3.2. *Let Assumptions 3.1 hold. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (3.1) converge strongly to $x^* \in F(T)$, which is a unique solution of variational inequality $VI(A, T)$.*

Proof. We first show that the uniqueness of a solution of the variational inequality $VI(A,T)$. Using Assumption (c), we have that $F(T)$ is nonempty $((VI(A,T) \subset F(T))$. If both $x^* \in F(T)$ and $x^{**} \in F(T)$ are solutions to $VI(A, T)$, then

$$\langle Ax^*, J(x^* - x^{**})\rangle \leq 0 \tag{3.2}$$

and

$$\langle Ax^{**}, J(x^{**} - x^*)\rangle \leq 0 \tag{3.3}$$

Adding up (3.2) and (3.3) yields

$$\langle Ax^{**} - Ax^*, J(x^{**} - x^*)\rangle \leq 0.$$

which implies that $x^* = x^{**}$.

Below we use x^* to denote the unique solution of $VI(A,T)$. We prove that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Let $p \in F(T)$. Without loss of generality, we can assume $\alpha_n \in \Big(0, \min\{1, \dfrac{1}{\tau}\}\Big)$, where

$$\tau = \eta\Big(k - \frac{d_q L^q \eta^{q-1}}{q}\Big).$$

Using (3.1) and the condition that T is quasi-nonexpansive, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\lambda y_n + (1-\lambda)Ty_n - p\| \\ &\leq \lambda\|y_n - p\| + (1-\lambda)\|Ty_n - p\| \\ &\leq \lambda\|y_n - p\| + (1-\lambda)\|y_n - p\| \\ &\leq \|y_n - p\|.\end{aligned}$$

By Lemma 2.7, we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq \|Q_K(I - \alpha_n\eta A)x_n - p\| \\ &\leq (1 - \tau\alpha_n)\|x_n - p\| + \alpha_n\|\eta Ap\| \\ &\leq \max\{\|x_n - p\|, \frac{\|\eta Ap\|}{\tau}\}.\end{aligned}$$

By induction, it is easy to see that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|\eta Ap\|}{\tau}\},\ n \geq 1.$$

Hence $\{x_n\}$ is bounded, So are $\{y_n\}$ and $\{Ax_n\}$.

Using Lemma 2.4 and (3.1), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\lambda y_n + (1-\lambda)Ty_n - p\|^2 \\ &\leq (1-\lambda)\|Ty_n - p\|^2 + \lambda\|y_n - p\|^2 - \lambda(1-\lambda)g(\|y_n - Ty_n\|) \\ &\leq (1-\lambda)\|y_n - p\|^2 + \lambda\|y_n - p\|^2 - \lambda(1-\lambda)g(\|y_n - Ty_n\|) \\ &\leq \|y_n - p\|^2 - \lambda(1-\lambda)g(\|y_n - Ty_n\|).\end{aligned}$$

It follows from Lemma 2.2 that

$$\begin{aligned}\lambda(1-\lambda)g(\|y_n - Ty_n\|) &\leq \|y_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n\eta\langle Ax_n, J(p - y_n)\rangle.\end{aligned}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, n there exists a constant $B > 0$ such that

$$\langle x_n, J(p - y_n)\rangle \leq B,\ \text{for all},\ n \geq 0.$$

Hence,

$$\lambda(1-\lambda)g(\|y_n - Ty_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n B. \tag{3.4}$$

Now we prove that $\{x_n\}$ converges strongly to x^*. We divide the proof into two cases.

Case 1. Assume that the sequence $\{\|x_n - p\|\}$ is monotonically decreasing sequence. Then $\{\|x_n - p\|\}$ is convergent. Clearly, we have

$$\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0.$$

It then implies from (3.4) that

$$\lim_{n\to\infty} \lambda(1-\lambda)g(\|y_n - Ty_n\|) = 0. \tag{3.5}$$

Using the fact that λ is a constant in $(0,1)$ and property of g, we have

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0. \tag{3.6}$$

From (3.1), we have

$$\begin{aligned} \|y_n - x_n\| &= \|Q_K(I-\alpha_n\eta A)x_n - x_n\| \\ &\leq \alpha_n\eta\|Ax_n\|. \end{aligned}$$

Hence,

$$\|y_n - x_n\| \to 0, \text{ as } n\to\infty. \tag{3.7}$$

Next, we prove that $\limsup_{n\to+\infty}\langle Ax^*, J(x^*-y_n)\rangle \leq 0$. Since E is reflexive and $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that y_{n_k} converges weakly to a in K and

$$\limsup_{n\to+\infty}\langle Ax^*, J(x^*-y_n)\rangle = \lim_{k\to+\infty}\langle Ax^*, J(x^*-y_{n_k})\rangle.$$

Since (3.6) and $I-T$ is demiclosed, we obtain $a\in F(T)$. On the other hand, the assumption that the duality mapping J is weakly continuous and the fact that x^* solves $VI(A,T)$, we then have

$$\begin{aligned} \limsup_{n\to+\infty}\langle Ax^*, J(x^*-y_n)\rangle &= \lim_{k\to+\infty}\langle Ax^*, J(x^*-y_{n_k})\rangle \\ &= \langle Ax^*, J(x^*-a)\rangle \leq 0. \end{aligned}$$

Finally, we have from recursion formula (3.1) and Lemma 2.2 that

$$\begin{aligned} \|x_{n+1}-x^*\|^2 &= \|\lambda y_n + (1-\lambda)Ty_n - x^*\|^2 \\ &\leq (\lambda\|y_n - x^*\| + (1-\lambda)\|Ty_n - x^*\|)^2 \\ &\leq \|y_n - x^*\|^2 \\ &\leq \|(I-\alpha_n\eta A)x_n - x^*\|^2 \\ &\leq (1-\alpha_n\tau)\|x_n - x^*\|^2 + 2\alpha_n\eta\langle Ax^*, J(x^*-y_n)\rangle. \end{aligned}$$

From Lemma 2.5, its follows that $x_n \to x^*$.

Case 2. Assume that the sequence $\{\|x_n - x^*\|\}$ is not monotonically. Set $B_n = \|x_n - x^*\|$ and $\tau:\mathbb{N}\to\mathbb{N}$ is a mapping for all $n\geq n_0$ (for some n_0 large enough) by $\tau(n) = \max\{k\in\mathbb{N}: k\leq n,\ B_k\leq B_{k+1}\}$. We have that τ is a non-decreasing sequence such that $\tau(n)\to\infty$ as $n\to\infty$ and $B_{\tau(n)}\leq B_{\tau(n)+1}$ for $n\geq n_0$. From (3.4), we have

$$\lambda(1-\lambda)g(\|y_{\tau(n)} - Ty_{\tau(n)}\|) \leq 2\alpha_{\tau(n)}B \to 0 \text{ as } n\to\infty.$$

Furthermore, we have

$$\|y_{\tau(n)} - Ty_{\tau(n)}\| \to 0 \text{ as } n\to\infty.$$

Hence,

$$\lim_{n\to\infty}\|y_{\tau(n)} - Ty_{\tau(n)}\| = 0.$$

By same argument as in Case 1, we can show that $x_{\tau(n)}$ and $y_{\tau(n)}$ converge weakly in K and

$$\limsup_{n\to+\infty}\langle Ax^*, J(x^*-y_{\tau(n)})\rangle \leq 0.$$

We have, for all $n \geq n_0$,

$$0 \leq \|x_{\tau(n)+1}-x^*\|^2 - \|x_{\tau(n)}-x^*\|^2 \leq \alpha_{\tau(n)}[-\tau\|x_{\tau(n)}-x^*\|^2 + 2\eta\langle Ax^*, J(x^*-y_{\tau(n)})\rangle],$$

which implies that

$$\|x_{\tau(n)}-x^*\|^2 \leq \frac{2\eta}{\tau}\langle Ax^*, J(x^*-y_{\tau(n)})\rangle.$$

Then, we have

$$\lim_{n\to\infty}\|x_{\tau(n)}-x^*\|^2 = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Furthermore, for all $n \geq n_0$, we have $B_{\tau(n)} \leq B_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $n > \tau(n)$); because $B_j > B_{j+1}$ for $\tau(n)+1 \leq j \leq n$. As consequence, we have, for all $n \geq n_0$,

$$0 \leq B_n \leq \max\{B_{\tau(n)},\, B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} B_n = 0$, that is, $\{x_n\}$ converges strongly to x^*. Using (3.7), we have that $\{y_n\}$ converges strongly to x^*. This completes the proof. □

We now apply Theorem 3.2 for finding a common element of the set of fixed points of nonexpansive mappings and set of solution of the variational inequality problem without assumption (b).

Theorem 3.3. *Let $q > 1$ be a fixed real number and let E be a q-uniformly smooth and uniformly convex real Banach space having a weakly continuous normalized duality map $J : E \to E^*$. Let K be a nonempty, closed and convex subset of E which is a nonexpansive retract of E with Q_K as the nonexpansive retraction. Let $A : K \to E$ be a k-strongly accretive and L-Lipschitzian operator and let $T : K \to K$ be a nonexpansive mapping. Let $\{x_n\}$ and $\{y_n\}$ be a sequences defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} y_n = Q_K(I-\alpha_n\eta A)x_n \\ x_{n+1} = \lambda y_n + (1-\lambda)Ty_n, \end{cases} \tag{3.8}$$

where λ is a constant in $(0,1)$ and $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying (i) $\lim_{n\to\infty}\alpha_n = 0$ and (ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$. Let assumptions (a) and (c) hold. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (3.8) converge strongly to $x^* \in F(T)$, which is a unique solution of variational inequality $VI(A,T)$.

Proof. Since every nonexpansive mappings is quasi-nonexpansive mappings. It suffices to prove assumption (b) is satisfied. Using the fact that E satisfies Opials property and Lemma 2.1, we have assumption (b) is satisfied. This completes the proof of Theorem 3.3. □

We apply our main results for approximating fixed points of quasi-nonexpansive mapping in Hilbert spaces without assumption (c).

Corollary 3.4. *Let H be a real Hilbert space. Let K be a nonempty, closed and convex subset of H. Let $A : K \to H$ be a k-strongly monotone and L-Lipschitzian operator, and let $T : K \to K$ be a quasi-nonexpansive mapping. Let $\{x_n\}$ and $\{y_n\}$ be a sequences defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} y_n = P_K(I - \alpha_n \eta A)x_n \\ x_{n+1} = \lambda y_n + (1-\lambda)Ty_n, \end{cases} \tag{3.9}$$

where λ is a constant in $(0,1)$ and $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying (i) $\lim_{n\to\infty} \alpha_n = 0$ and (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$. Assume that $0 < \eta < \dfrac{2k}{L^2}$ and $I - T$ is demiclosed. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (3.9) converge strongly to $x^* \in F(T)$, which is a unique solution of variational inequality

$$\langle Ax^*, x^* - p\rangle \leq 0 \ \forall p \in F(T).$$

Proof. Since Hilbert spaces are q-uniformly smooth and uniformly convex. It suffices to prove assumption (c) is satisfied. By Lemma 2.10, we get that $VI(A,T)$ is nonempty. This completes this proof. □

4. Applications

4.1. **The split feasibility problem.** In this subsection, we apply our main results for solving split feasiblity problem. The split feasibility problem (SFP) was first introdued by Censor and Elfving [17] in 1994. The SFP is to find

$$x \in K, \text{ such that } Ax \in Q, \tag{4.1}$$

where K is a nonempty, closed convex subset of a Hilbert space H_1, Q is a nonempty closed convex subset of a Hilbert space H_2, and $A : H_1 \to H_2$ is a bounded linear operator.

The problem (4.1) arises in signal processing and image reconstruction with particular progress in intensity modulated therapy, and many iterative algorithms has been established for it (see, e.g., [1, 17, 18]) and the reference therein. Let Ω be the solution set of the split feasibility problem. From an optimization point of view, $x^* \in \Omega$ if and only if x^* is a solution of the following minimization problem with zero optimal value:

$$\min_{x\in K} f(x), \text{ where } f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2.$$

The following lemma appears in [12].

Lemma 4.1. *Given $x^* \in H$, x^* solves SFP (4.1) if and only if x^* is the solution of the fixed point equation $x = P_K(I - \gamma A^*(I - P_Q)A)x$, where $\gamma > 0$ is a suitable constant.*

The following proposition was also given in [19].

Proposition 4.2. [19] *Let K be a nonempty, closed and convex subset of a Hilbert space H_1, and let Q be a a nonempty, closed and convex subset of a Hilbert space H_2. Let $A : H_1 \to H_2$ is a bounded linear operator. Let P_K and P_Q denote the orthogonal projection onto set K and Q, respectively. Let $0 < \gamma < \frac{2}{\rho}$, where ρ is the spectral raduis of A^*A, and A^* is the adjoint of A. Then, the operator $T := P_K(I - \gamma A^*(I - P_Q)A)$ is nonexpansive on K.*

Theorem 4.3. *Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $A^* : H_2 \to H_1$ be the adjoint operator of A. Let K be a nonempty, closed and convex subset of a Hilbert space H_1, and let Q be a a nonempty, closed and convex subset of a Hilbert space H_2. Let*

$B: K \to H_1$ be a k-strongly monotone and L-Lipschitzian operator. Let $0 < \gamma < \frac{2}{\rho}$, ρ is the spectral radius of A^*A. Let $\{x_n\}$ and $\{y_n\}$ be a sequences defined iteratively from arbitrary $x_0 \in K$ by:

$$\begin{cases} y_n = P_K(I - \alpha_n \eta B)x_n \\ x_{n+1} = \lambda y_n + (1-\lambda) P_K(I - \gamma A^*(I - P_Q)A)y_n, \end{cases} \tag{4.2}$$

where λ is a constant in $(0,1)$ and $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying (i) $\lim_{n\to\infty} \alpha_n = 0$ and (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$. Assume that $0 < \eta < \dfrac{2k}{L^2}$ and Ω is nonempty. Then, $\{x_n\}$ and $\{y_n\}$) converge strongly to a solution of (4.1).

Proof. From Lemma 4.1, we know $x \in \Omega$ if and only if $x = P_K(I - \gamma A^*(I - P_Q)A)x$. From Proposition, we have operator $T := P_K(I - \gamma A^*(I - P_Q)A)$ is nonexpansive on K. Using Theorem 3.3, we can obtain the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to a solution of (4.1). □

4.2. **The constrained minimization problem.** In this subsection, we study the problem of finding a minimizer of a convex function f over the set of fixed points of quasi-nonexpansive mappings.

Let H be a real Hilbert space. Let K be a nonempty, closed and convex subset of H and let T be a quasi-nonexpansive mappings on K. We consider the following minimization problem over the set of fixed points of a quasi-nonexpansive mapping:

$$\min_{x \in F(T)} f(x). \tag{4.3}$$

where $f: K \to \mathbb{R}$ is a differentiable, and convex real-valued function.

Finding an optimal point over the set of fixed points of a quasi-nonexpansive mapping is one that occurs frequently in various areas of mathematical sciences and engineering.

Definition 4.4. Let E be normed linear space. A function $f: E \to \mathbb{R}$ is said to be strongly convex if there exists $\alpha > 0$ such that, for every $x, y \in E$ with $x \neq y$ and $\lambda \in (0,1)$, the following inequality holds:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) - \alpha \|x - y\|^2. \tag{4.4}$$

Lemma 4.5. *Let E be normed linear space and $f: E \to \mathbb{R}$ a real-valued differentiable convex function. Assume that f is strongly convex. Then the differential map $\nabla f: E \to E^*$ is strongly monotone, i.e., there exists a positive constant k such that*

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq k\|x - y\|^2 \ \forall x, y \in E. \tag{4.5}$$

Lemma 4.6. *Let K be a nonempty, closed convex subset of E be normed linear space and let $f: K \to \mathbb{R}$ be a real valued differentiable convex function. Then x^* is a minimizer of f over K if and only if x^* solves the following variational inequality $\langle \nabla f(x^*), y - x^* \rangle \geq 0$ for all $y \in K$.*

Remark 4.7. By Lemma 4.6, x^* is a solution of (4.3) if and only if $x^* \in F(T)$ and x^* solves the following variational inequality problem:

$$\langle \nabla f(x^*), x^* - p \rangle \leq 0$$

for all $p \in F(T)$.

Hence, one has the following result.

Theorem 4.8. *Let H be a real Hilbert space. Let K be a nonempty, closed and convex subset of H. Let $f : K \to \mathbb{R}$ be a differentiable, strongly convex real-valued function. Suppose the differential map $\nabla f : K \to H$ is L-Lipschitz. Let $T : K \to K$ be a quasi-nonexpansive mapping. Let $\{x_n\}$ and $\{y_n\}$ be a sequences defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} y_n = P_K(I - \alpha_n \eta \nabla f)x_n \\ x_{n+1} = \lambda y_n + (1-\lambda)Ty_n, \end{cases} \tag{4.6}$$

where λ is a constant in $(0,1)$ and $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying: (i) $\lim_{n\to\infty} \alpha_n = 0$ and (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$. Assume that $0 < \eta < \dfrac{2k}{L^2}$ and $I - T$ is demiclosed. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (4.6) converge strongly to a solution of (4.3).

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