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GENERALIZED α-ψ-GERAGHTY TYPE CONTRACTIONS IN GENERALIZED METRIC SPACES

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Abstract. In this paper, we obtain fixed point theorems for generalized α-ψ-Geraghty type contraction mappings in the context of generalized metric spaces introduced recently by Jleli and Samet (Fixed Point Theory Appl. 2015 (2015), Article ID 61). A fixed point result in the framework of generalized metric spaces with partial order is obtained. Some useful examples are given to illustrate our main results.

Keywords. Fixed point; Generalized metric spaces; α-ψ-Geraghty type contraction; Partial order.

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1. Introduction

Banach fixed point theorem, which asserts the existence of a unique fixed point for a contraction mapping in a complete metric space, is one of the most useful and fundamental result in nonlinear analysis. Several generalizations of this concept has been done and many notions of metric-type spaces were introduced (b-metric, dislocated spaces, generalized metric spaces, quasi-metric spaces, symmetric spaces etc.). In 2015, Jleli and Samet [1] introduced a very interesting concept of a generalized metric space and extended some well known fixed point results. The generalized metric spaces recovers different well-known metric structures including metric spaces, b-metric spaces, dislocated metric spaces and modular spaces. Since then, many researchers have focused on this concept of generalized metric spaces and extended some well known fixed point results for nonlinear contractions to this space. For more details, see [2–5] and the references therein. Very recently, Sastry et al. [4] proved the existence and uniqueness of fixed points for α-Geraghty contraction type mappings in generalized metric spaces and extended it to the case of generalized metric spaces with partial order. Motivated by Sastry et al. [4], we establish and generalize some fixed point results for Geraghty type contractions in this new class of generalized metric spaces in this paper.

The aim of this paper is to obtain conditions for the existence of fixed points for a generalized α-ψ-Geraghty type contractions in a generalized metric space with coefficient k. We use a relation between

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the function β and the coefficient k of the generalized metric space so that the proof is valid even when $k \neq 1$. We also assume that the auxiliary function ψ is scalar multiplicative. Further, conditions for the existence of fixed points for a generalized metric space with partial order is obtained.

2. PRELIMINARIES

For the sake of completeness, we recall some basic definitions and results.

Notation: [1] Let X be a non-empty set and $D : X \times X \to [0,\infty)$ be a given mapping. For every $x \in X$, let us define the set

$$C(D,X,x) = \left\{ (x_n) \subset X : \lim_{n\to\infty} D(x_n,x) = 0 \right\}.$$

Definition 2.1. Let X be a non-empty set and $D : X \times X \to [0,\infty)$ be a function which satisfies the following conditions:

(D1) $D(x,y) = 0$ implies $x = y$;
(D2) $D(x,y) = D(y,x)$ for all $x,y \in X$;
(D3) there exists $k > 0$ such that
if $x,y \in X$ and $(x_n) \in C(D,X,x)$, then $D(x,y) \leq k \limsup_{n\to\infty} D(x_n,y)$.

Then D is called a generalized metric and the pair (X,D) is called a generalized metric space with coefficient k.

Remark 2.2. If the set $C(D,X,x)$ is empty for every $x \in X$, then (X,D) is a generalized metric space if and only if $(D1)$ and $(D2)$ are satisfied.

Definition 2.3. [5] Let (X,D) be a generalized metric space. Then a sequence (x_n) in X is said to be:

(i) D-convergent to $x \Leftrightarrow (x_n) \in C(D,X,x)$;
(ii) D- Cauchy $\Leftrightarrow \lim_{n,m\to\infty} D(x_n,x_{n+m}) = 0$.

Definition 2.4. [1] Let (X,D) be a generalized metric space. It is said to be D-complete if every D-Cauchy sequence in X is convergent to some element in X.

Remark 2.5. Senapati et al. [5] established with an example that a convergent sequence may not be Cauchy in a Generalized metric space.

Proposition 2.6. *[1] Let (X,D) be a generalized metric space. Let (x_n) be a sequence in X and $x,y \in X$. If (x_n) D-converges to x and (x_n) D-converges to y, then $x = y$.*

Proposition 2.7. *[2] $C(D,X,x)$ is a nonempty set if and only if $D(x,x) = 0$.*

Definition 2.8. [3] Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}^+$ be a function. Then T is said to be triangular α-admissible if for all $x,y,z \in X$,

(i): $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$;
(ii): $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$ implies $\alpha(x,y) \geq 1$.

Lemma 2.9. *[6] Let T be a triangular α-admissible map on a non empty set X. Assume that there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$. Define $x_{n+1} = Tx_n$ for $n = 0,1,2,\cdots$. Then $\alpha(x_n,x_m) \geq 1$ for all $m,n \in \mathbb{N} \cup \{0\}$ with $n < m$.*

In 1973, Geraghty [7] introduced an interesting class of auxiliary functions to refine the Banach contraction mapping principle as follows.

Theorem 2.10. [7]. *Let (X,d) be a complete metric space and let $T : X \to X$ be a self-mapping such that, for all $x,y \in X$,*

$$d(Tx,Ty) \leq \beta(d(x,y))d(x,y),$$

where $\beta : [0,\infty) \to [0,1)$ is a function satisfying $\beta(t_n) \to 1$ implies $t_n \to 0$ as $n \to \infty$. Then T has a unique fixed point $z \in X$.

In 2013, Cho et al. [8] introduced the notion of α-Geraghty contraction type mappings and assured the unique fixed point theorems for such mappings in complete metric spaces. On the other hand, Karapinar [9] defined the concept of α-ψ-Geraghty contraction as well as generalized $\alpha-\psi$-Geraghty contraction mapping in metric spaces where the auxiliary function ψ is assumed to be subadditive. Later, in [10], Karapinar showed that the subadditivity property is not required in many cases and therefore, can be removed. Let $\mathbb{F}$ be the family of all functions $\beta : [0,\infty) \to [0,1)$ satisfying the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \Rightarrow \lim_{n\to\infty} t_n = 0.$$

Example 2.11. Consider $\beta : [0,\infty) \to [0,1)$ defined by

$$\begin{aligned} \beta(t) &= e^{-t} \text{ for } t > 0 \\ &= 0 \text{ for } t = 0. \end{aligned}$$

Then $\beta \in \mathbb{F}$.

Now, we define the following class of auxiliary functions, which will be used densely in the sequel. Let Ψ denote the class of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(a) ψ is nondecreasing;
(b) ψ is continuous;
(c) $\psi(t) = 0 \Rightarrow t = 0$;
(d) $\psi(ct) = c\psi(t)$ for any constant c.

Example 2.12. Consider $\psi : [0,\infty) \to [0,\infty)$ defined by $\psi(t) = \ln 2^t$, $t \geq 0$. We see that $\psi \in \Psi$. Also, $\psi(t) = \lambda t$, where λ is a constant and $t \geq 0$, is a trivial example.

We now define the generalized α-ψ-Geraghty contraction mapping in a generalized metric space.

Definition 2.13. Let (X,D) be a Generalized metric space and $\alpha : X \times X \to [0,\infty)$ be a function. A mapping $T : X \to X$ is called a generalized α-ψ-Geraghty contraction mapping if there exists $\beta \in \mathbb{F}$ and $\psi \in \Psi$ such that for all $x,y \in X$

$$\alpha(x,y)\psi(D(Tx,Ty)) \leq \beta(\psi(M(x,y))\psi(M(x,y)) \tag{2.1}$$

where $M(x,y) = \max\{D(x,y), D(x,Tx), D(y,Ty)\}$ whenever $D(x,y) < \infty$.

3. Main Results

Now onwards, $\mathbb{N}$ denotes the set of natural numbers and X is a generalized metric space (X,D). We first prove the following proposition which will be used to prove our main theorem.

Proposition 3.1. *Let (X,D) be a generalized metric space, $\beta \in \mathbb{F}$ and $\alpha : X \times X \to [0,\infty)$ be a function. Let T be a generalized $\alpha - \psi$-Geraghty contraction map. Suppose that z is a fixed point of T with $\alpha(z,z) \geq 1$, and $D(z,z) < \infty$. Then $D(z,z) = 0$.*

Proof. Given $Tz = z$ and $D(z,z) < \infty$, since T is a generalized α-ψ-Geraghty contraction map and $\alpha(z,z) \geq 1$, we have

$$\psi(D(z,z)) \leq \alpha(z,z)\psi(D(Tz,Tz)) \leq \beta(\psi(M(z,z))\psi(M(z,z)),$$

where $M(z,z) = \max\{D(z,z), D(z,Tz), D(z,Tz)\} = D(z,z) < \infty$. Hence,

$$\begin{aligned}\psi(D(z,z)) &\leq \alpha(z,z)\psi(D(z,z)) \\ &\leq \beta(\psi(D(z,z))\psi(D(z,z)) \\ &< \psi(D(z,z)).\end{aligned}$$

This is a contradiction. Hence $\psi(D(z,z)) = 0$, which implies that $D(z,z) = 0$. □

Now we state and prove our main result.

Theorem 3.2. *Let (X,D) be a D-complete generalized metric space and $T : X \to X$ be a mapping. Assume that the following conditions are satisfied:*

(1) T is triangular α-admissible.
(2) T is a generalized $\alpha - \psi$- Geraghty contraction type mapping with respect to β.
(3) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and there exists δ in $\mathbb{R}$ such that $\sup_n \psi(D(x_0, T^n x_0)) < \delta < \infty$ and $D(T^n x_0, T^{n+1}x_0) < \infty$ for every n.

Then $(T^n x_0)$ is D-Cauchy and hence D-converges, say to $z \in X$. Further, assume that

(4) $\beta(\psi(D(x,Tx))) < \frac{1}{k}$ for all $x \in X$ for which $D(x,Tx) < \infty$.
(5) if $\{y_n\}$ is a sequence in X such that $\alpha(y_n, y_{n+1}) \geq 1$ for all n and $y_n \to y$, then $\alpha(y_n, y) \geq 1$ for all n and
(6) $\lim_{n\to\infty} \sup D(T^n x_0, Tz) < \infty$.

Then z is a fixed point of T. Moreover,

(7) assume that z' is another fixed point of T such that $D(z,z) < \infty$, $D(z,z') < \infty$, and $D(z',z') < \infty$. Then either $\alpha(z,z') < 1$ or $\alpha(z',z) < 1$ or $z = z'$.

Proof. Let $x_0 \in X$ satisfying assumption (3) of the theorem. We write $x_{n+1} = Tx_n$, for $n = 0,1,2,\cdots$. By lemma 2.9, we have $\alpha(x_n, x_m) \geq 1$ for all $m,n \in \mathbb{N} \cup \{0\}$ with $n < m$. Assume further that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Since T is a generalized $\alpha - \psi$- Geraghty contraction type mapping, it follows that

$$\begin{aligned}\psi(D(x_{n+1}, x_{n+2})) &= \psi(D(Tx_n, Tx_{n+1})) \\ &\leq \alpha(x_n, x_{n+1})\psi(D(Tx_n, Tx_{n+1})) \\ &\leq \beta(\psi(M(x_n, x_{n+1})))\,\psi(M(x_n, x_{n+1})), \qquad (3.1)\end{aligned}$$

where $M(x_n,x_{n+1}) = \max\{D(x_n,x_{n+1}), D(x_n,x_{n+1}), D(x_{n+1},x_{n+2})\}$. Suppose $M(x_n,x_{n+1}) = D(x_{n+1},x_{n+2})$. Then $\psi(D(x_{n+1},x_{n+2})) < \psi(D(x_{n+1},x_{n+2}))$. This is a contradiction. Therefore $M(x_n,x_{n+1}) = D(x_n,x_{n+1})$. Then, from (3.1),

$$\begin{aligned}\psi(D(x_{n+1},x_{n+2})) &\leq \beta(\psi(D(x_n,x_{n+1})))\,\psi(D(x_n,x_{n+1}))\\ &< \psi(D(x_n,x_{n+1})).\end{aligned} \tag{3.2}$$

Since ψ is nondecreasing, we obtain from (3.2) that $D(x_{n+1},x_{n+2}) < D(x_n,x_{n+1})$ for each $n \in \mathbb{N}\cup\{0\}$. Thus we observe that $\{D(x_n,x_{n+1})\}$ is nonnegative and non-increasing. As a result, there exists $r \geq 0$ such that $\lim_{n\to\infty} D(x_n,x_{n+1}) = r$. Further from (3.1), we get

$$\frac{\psi(D(x_{n+1},x_{n+2}))}{\psi(D(x_n,x_{n+1}))} \leq \beta(\psi(D(x_n,x_{n+1}))) < 1,$$

which implies that $\lim_{n\to\infty} \beta(\psi(D(x_n,x_{n+1}))) = 1$. That is, $\lim_{n\to\infty} \psi(D(x_n,x_{n+1})) = 0$ as $\beta \in \mathbb{F}$. Continuity of ψ implies that $\psi\left(\lim_{n\to\infty} D(x_n,x_{n+1})\right) = 0$ and hence $r = \lim_{n\to\infty} D(x_n,x_{n+1}) = 0$. By the assumption (3) of the theorem, we observe that $D(x_n,x_m) < \infty$ for all $n < m$. Now, for $m > n$, we have

$$\begin{aligned}\psi(D(x_n,x_m)) &= \psi(D(Tx_{n-1},Tx_{m-1}))\\ &\leq \alpha(x_{n-1},x_{m-1})\,\psi(D(Tx_{n-1},Tx_{m-1}))\\ &\leq \beta(\psi(M(x_{n-1},x_{m-1})))\,\psi(M(x_{n-1},x_{m-1})),\end{aligned}$$

where

$$\begin{aligned}M(x_{n-1},x_{m-1}) &= \max\{D(x_{n-1},x_{m-1}), D(x_{n-1},x_n), D(x_{m-1},x_m)\}\\ &= D(x_{n-1},x_{m-1}) \text{ for large } n \text{ since } \lim_{n\to\infty} D(x_{n-1},x_n) = 0.\end{aligned}$$

Therefore,

$$\psi(D(x_n,x_m)) \leq \beta(\psi(D(x_{n-1},x_{m-1})))\,\psi(D(x_{n-1},x_{m-1})). \tag{3.3}$$

Let $a_{n,m} = D(x_n,x_m)$. Since $(\beta(t))^n \to 0$ uniformly as $n \to \infty$, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$(\beta(t))^n < \frac{\varepsilon}{\delta} \text{ for all } n \geq N. \tag{3.4}$$

From (3.3), for $m > n > N+1$, we have

$$\begin{aligned}\psi(D(x_n,x_m)) &\leq \beta(\psi(a_{n-1,m-1}))\,\psi(a_{n-1,m-1})\\ &\leq \beta(\psi(a_{n-1,m-1}))\,\beta(\psi(a_{n-2,m-2}))\cdots\beta(\psi(a_{N+1,m-n+N+1}))\,\psi(a_{N+1,m-n+N+1})\\ &\leq \{\max\{\beta(\psi(a_{n-1,m-1}))\,\beta(\psi(a_{n-2,m-2}))\cdots\beta(\psi(a_{N+1,m-n+N+1}))\}\}^{n-(N+1)}\\ &\qquad \psi(a_{N+1,m-n+N+1})\\ &= (\beta(t))^{n-(N+1)}\psi(a_{N+1,m-n+N+1})\\ &\qquad \text{for some } t \in \{a_{n-1,m-1}, a_{n-2,m-2}, \cdots, a_{N+1,m-n+N+1}\}\\ &\leq (\beta(t))^n\psi(a_{0,m-n})\\ &< (\beta(t))^n\delta(\text{ by condition (3)of the theorem})\\ &< \frac{\varepsilon}{\delta}\delta = \varepsilon\ (by(3.4)).\end{aligned}$$

Therefore $\{x_n\}$ is a D-Cauchy sequence. Since (X,D) is D-complete, there exists some $z \in X$ such that $\lim_{n\to\infty} D(T^n x_0, z) = 0$.

Now

$$\begin{aligned}\psi(D(x_{n+1},Tz)) &= \psi(D(Tx_n,Tz))\\ &\leq \alpha(x_n,z)\psi(D(Tx_n,Tz)) \text{ by condition(5)}\\ &\leq \beta(\psi(M(x_n,z)))\,\psi(M(x_n,z)),\end{aligned}$$

where $M(x_n,z) = \max\{D(x_n,z), D(x_n,x_{n+1}), D(z,Tz)\} = D(z,Tz)$ for large n. Therefore

$$\begin{aligned}\psi(D(x_{n+1},Tz)) &\leq \beta(\psi(D(z,Tz)))\,\psi(D(z,Tz))\\ \limsup_{n\to\infty}\psi(D(x_{n+1},Tz)) &\leq \beta(\psi(D(z,Tz)))\,\psi(D(z,Tz))\\ &< \frac{1}{k}\psi(D(z,Tz)).\end{aligned} \tag{3.5}$$

$D(z,Tz) \leq k \limsup_{n\to\infty} D(x_n,Tz)$ (by (D3)) implies

$$\begin{aligned}\psi(D(z,Tz)) &\leq \psi\left(k\limsup_{n\to\infty} D(x_n,Tz)\right)\\ &= k\psi\left(\limsup_{n\to\infty} D(x_n,Tz)\right).\end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.5), we get

$$\limsup_{n\to\infty}\psi(D(x_{n+1},Tz)) < \limsup_{n\to\infty}\psi(D(x_n,Tz)).$$

By assumption (6) of the theorem and considering the fact that ψ is continuous and non decreasing, we get

$$\limsup_{n\to\infty} D(x_n,Tz) = 0.$$

Hence $x_n \to Tz$, consequently $z = Tz$. Let z' be another fixed point of T. Suppose $\alpha(z,z') \not< 1$ and $\alpha(z',z) \not< 1$. Then $\alpha(z,z') \geq 1$ and $\alpha(z',z) \geq 1$ so that $\alpha(z,z) \geq 1$ and $\alpha(z',z') \geq 1$. Now

$$\begin{aligned}\psi(D(z,z')) &= \psi(D(Tz,Tz'))\\ &\leq \alpha(z,z')\psi(D(Tz,Tz'))\\ &\leq \beta(M(z,z'))\psi(M(z,z')),\end{aligned}$$

where $M(z,z') = \max\{D(z,z'), D(z,z), D(z',z')\} = D(z,z')$ by proposition (3.1). Then we have $\psi(D(z,z')) \leq \beta(D(z,z'))\psi(D(z,z')) < \psi(D(z,z'))$ if $z \neq z'$, which is a contradiction. T has a unique fixed point. □

If $\psi(t) = t$ in Theorem3.2, then we get the following result by Sastry et al. [4] as a corollary.

Corollary 3.3. *Let (X,D) be a D-complete generalized metric space with coefficient λ, $\alpha : X \times X \to \mathbb{R}^+$ be a function and $f : X \to X$ be a mapping. Assume that the following conditions are satisfied:*

(1) f is triangular α-admissible;

(2) f is a generalized α-Geraghty contraction type map with respect to β;

(3) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$, and there exists $\gamma \in \mathbb{R}$ satisfying

$$\sup_n D(x_0, f^n(x_0)) < \gamma < \infty \text{ and } D(f^n(x_0), f^{n+1}(x_0)) < \infty \text{ for every } n.$$

(4) $\beta(D(x,fx)) < \dfrac{1}{\lambda}$ *for all* $x \in X$ *for which* $D(x,fx) < \infty$.

Then $\{f^n(x_0)\}$ *is D-Cauchy and hence D-converges, say to* $w \in X$.
Further assume that

(5) if $\{z_n\}$ *is a sequence in* X *such that* $\alpha(z_n,z_{n+1}) \geq 1$ *for all* n *and* $z_n \to z$ *then* $\alpha(z_n,z) \geq 1$ *for all* n *and*

(6) $\limsup_n D(f^n(x_0,fw) < \infty$.

Then f *has a fixed point w(say) in* X.
Moreover

(7) assume that w' *is another fixed point of* f *such that* $D(w,w) < \infty$, $D(w,w') < \infty$ *and* $D(w',w') < \infty$.

Then either $\alpha(w,w') < 1$ *or* $\alpha(w',w) < 1$ *or* $w = w'$. *It is in this sense* f *has a unique fixed point.*

We now prove one fixed point theorem for $\alpha-\psi$-Geraghty type mappings in a generalized metric space with partial order.

Definition 3.4. Let (X,D) be a generalized metric space and $\preceq$ be a partial order on X. Then we say that $(X,D,\preceq)$ is a generalized metric space with partial order $\preceq$. Two elements $x,y \in X$ are comparable if either $x \preceq y$ or $y \preceq x$.

A sequence (x_n) in X is said to be non decreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. If $x_{n+1} \preceq x_n$ for all n, we say that (x_n) is a decreasing sequence. Suppose that $(X,D,\preceq)$ is a generalized metric space with partial order $\preceq$. Let $T : X \to X$ be a selfmap of X. We say that T is an increasing function if $x \preceq y$ implies $Tx \preceq Ty$ and we say T is decreasing if $x \succeq y$ implies $Tx \preceq Ty$.

Definition 3.5. Let $(X,D,\preceq)$ be a generalized metric space with partial order $\preceq$, $\beta \in \mathbb{F}$, $\alpha : X \times X \to \mathbb{R}^+$ be a function and $T : X \to X$ be a mapping. We say that T is a generalized $\alpha-\psi$-Geraghty contraction type in $(X,D,\preceq)$ if

$$\alpha(x,y)\psi(D(Tx,Ty)) \leq \beta(\psi(M(x,y))M(x,y), \tag{3.7}$$

where $M(x,y) = \max\{D(x,y),D(x,Tx),D(y,Ty)\}$ and x and y are comparable with $M(x,y) < \infty$.

Definition 3.6. Suppose that $(X,D,\preceq)$ is a generalized metric space with partial order $\preceq$. We say that

(i) X is D- regular (increasing) if (x_n) is an increasing sequence in X and (x_n) is D- convergent to x implies $x_n \preceq x$ for all n.

(ii) X is D-regular(decreasing) if (x_n) is a decreasing sequence in X and (x_n) is D- convergent to x implies $x_n \succeq x$ for all n.

Now we prove our theorem.

Theorem 3.7. *Let* $(X,D,\preceq)$ *be a D-complete generalized metric space with partial order* $\preceq$. (X,D) *is D-regular.* $\alpha : X \times X \to \mathbb{R}^+$ *be a function and* $T : X \to X$ *be a mapping. Assume that the following conditions hold:*

(i) T *is triangular* α*-admissible;*

(ii) T *is a generalized* $\alpha-\psi$*- Geraghty contraction type in* $(X,D,\preceq)$*;*

(iii) T *is an increasing function;*

(iv) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$, $\alpha(x_0, Tx_0) \geq 1$ *and there exists* δ *in* $\mathbb{R}$ *such that* $\sup_n \psi(D(x_0, T^n x_0)) < \delta < \infty$ *and* $D(T^n x_0, T^{n+1} x_0) < \infty$ *for every* n.

Then $(T^n x_0)$ *is Cauchy and hence converges, say to* $z \in X$. *Further, assume that*

(v) $\beta(\psi(D(x, Tx))) < \frac{1}{k}$ *for all* $x \in X$ *for which* $D(x, Tx) < \infty$.

(vi) *if* $\{y_n\}$ *is a sequence in* X *such that* $\alpha(y_n, y_{n+1}) \geq 1$ *for all* n *and* $y_n \to y$, *then* $\alpha(y_n, y) \geq 1$ *for all* n *and*

(vii) $\lim_{n\to\infty} \sup D(T^n x_0, Tz) < \infty$.

Then T *has a fixed point z(say) in* X. *Moreover,*

(viii) *assume that* z' *is another fixed point of* T *such that* $D(z, z) < \infty$, $D(z, z') < \infty$, *and* $D(z', z') < \infty$. *Then either* $\alpha(z, z') < 1$ *or* $\alpha(z', z) < 1$ *or* $z = z'$.

Proof. Let $x_0 \in X$ be as in (iv) and assume that $x_0 \neq Tx_0$. Since T is increasing and $x_0 \preceq Tx_0$, we have $Tx_0 \preceq TTx_0 = Tx_1$. By induction, we get $x_n \preceq Tx_n = x_{n+1}$ for all n. Therefore (x_n) is an increasing sequence. By Lemma (2.9), we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$. Without loss of generality we may assume that $x_n \neq x_{n+1}$ for all n. Then $\alpha(x_n, x_{n+1}) \geq 1$. Since x_n and x_{n+1} are comparable, we have

$$\begin{aligned}
\psi(D(x_{n+1}, x_{n+2})) &= \psi(D(Tx_n, Tx_{n+1})) \\
&\leq \alpha(x_n, x_{n+1})\psi(D(Tx_n, Tx_{n+1})) \\
&\leq \beta(\psi(M(x_n, x_{n+1})))\psi(M(x_n, x_{n+1})),
\end{aligned}$$

where $M(x_n, x_{n+1}) = \max\{D(x_n, x_{n+1}), D(x_n, x_{n+1}), D(x_{n+1}, x_{n+2})\}$. Suppose $M(x_n, x_{n+1}) = D(x_{n+1}, x_{n+2})$. Then $\psi(D(x_{n+1}, x_{n+2})) < \psi(D(x_{n+1}, x_{n+2}))$, a contradiction. Therefore $M(x_n, x_{n+1}) = D(x_n, x_{n+1})$. Then $\psi(D(x_{n+1}, x_{n+2})) \leq \beta(\psi(D(x_n, x_{n+1})))\psi(D(x_n, x_{n+1})) < \psi(D(x_n, x_{n+1}))$ or $D(x_{n+1}, x_{n+2}) < D(x_n, x_{n+1})$. Hence $\{D(x_n, x_{n+1})\}$ is a decreasing sequence and hence converges to say $r \geq 0$. Then proceeding as in Theorem3.2, we get from (3.1) that

$$\frac{\psi(D(x_{n+1}, x_{n+2}))}{\psi(D(x_n, x_{n+1}))} \leq \beta(\psi(D(x_n, x_{n+1}))) < 1,$$

which implies that $\lim_{n\to\infty} \beta(\psi(D(x_n, x_{n+1}))) = 1$. That is, $\lim_{n\to\infty} \psi(D(x_n, x_{n+1})) = 0$ as $\beta \in \mathbb{F}$. Continuity of ψ implies that $\psi\left(\lim_{n\to\infty} D(x_n, x_{n+1})\right) = 0$ and hence $\lim_{n\to\infty} D(x_n, x_{n+1}) = 0$. ie, $r = 0$. By the assumptions (ii)and (vi) of the theorem, we have

$$\begin{aligned}
\psi(D(x_n, x_m)) = \psi(D(Tx_{n-1}, Tx_{m-1})) &\leq \alpha(x_{n-1}, x_{m-1})\psi(D(Tx_{n-1}, Tx_{m-1})) \\
&\leq \beta(\psi(M(x_{n-1}, x_{m-1})))\psi(M(x_{n-1}, x_{m-1})),
\end{aligned}$$

where $M(x_{n-1}, x_{m-1}) = \max\{D(x_{n-1}, x_{m-1}), D(x_{n-1}, x_n), D(x_{m-1}, x_m)\} = D(x_{n-1}, x_{m-1})$ for large n and m. Therefore $\psi(D(x_n, x_m)) \leq \beta(\psi(D(x_{n-1}, x_{m-1})))\psi(D(x_{n-1}, x_{m-1}))$, which is the same as (3.3) in Theorem 3.2. Now, proceeding as in the proof of Theorem 3.2, we get that (x_n) is a cauchy sequence and hence converges, say to $z \in X$. Since (x_n) is an increasing sequence and converges to z, it follows that $x_n \preceq z$ by the D-regularity of (X, D) and $\alpha(x_n, z) \geq 1$ for every n, by (vi). Now,

$$\begin{aligned}
\psi(D(x_{n+1}, Tz)) = \psi(D(Tx_n, Tz)) &\leq \alpha(x_n, z)\psi(D(Tx_n, Tz)) \\
&\leq \beta(\psi((M(x_n, z)))\psi(M(x_n, z)),
\end{aligned}$$

where $M(x_n,z)=\max\{D(x_n,z),D(x_n,x_{n+1}),D(z,Tz)\}=D(z,Tz)$ for large n. Therefore $\psi(D(x_{n+1},Tz))\leq \beta(\psi(D(z,Tz)))\,\psi(D(z,Tz))$ for large n. Now on taking limit supremum, we get

$$\begin{aligned}\limsup_{n\to\infty}\psi(D(x_{n+1},Tz)) &\leq \beta(\psi(D(z,Tz)))\,\psi(D(z,Tz))\\ &< \frac{1}{k}\psi(k\limsup_{n\to\infty}D(x_n,Tz)\\ &= \frac{1}{k}k\psi(\limsup_{n\to\infty}\psi(D(x_n,Tz)))\\ &= \psi(\limsup_{n\to\infty}\psi(D(x_n,Tz))).\end{aligned}$$

Since ψ is non decreasing, $\limsup_{n\to\infty}D(x_{n+1},Tz)<\limsup_{n\to\infty}D(x_n,Tz)$, which implies that $\limsup_{n\to\infty}D(x_n,Tz)=0$, by (vii). Hence $x_n\to Tz$, consequently $z=Tz$. Suppose that z' is a fixed point of T comparable with z such that $D(z,z)<\infty$, $D(z,z')<\infty$ and $D(z',z')<\infty$. Suppose $\alpha(z,z')\nless 1$ and $\alpha(z',z)\nless 1$. Then $\alpha(z,z')\geq 1$ and $\alpha(z',z)\geq 1$ so that $\alpha(z,z)\geq 1$ and $\alpha(z',z')\geq 1$. Now

$$\begin{aligned}\psi(D(z,z')) &= \psi(D(Tz,Tz'))\\ &\leq \alpha(z,z')\psi(D(Tz,Tz'))\\ &\leq \beta(M(z,z'))\psi(M(z,z'))\end{aligned}$$

where $M(z,z')=\max\{D(z,z'),D(z,z),D(z',z')\}=D(z,z')$ by proposition(3.1). Then we have $\psi(D(z,z'))\leq \beta(D(z,z'))\psi(D(z,z'))<\psi(D(z,z'))$ and hence $D(z,z')<D(z,z')$ if $z\neq z'$, a contradiction. Therefore T has a unique fixed point. □

Example 3.8. Let $X=\{1,\frac{1}{3},\frac{1}{3^2},\cdots,\frac{1}{3^n},\cdots\}\cup\{0\}$. Define $D:X\times X\to[0,\infty)$ by

$$\begin{aligned}D(1,x)=D(x,1) &= \infty \text{ if } x\in X\\ D(x,y) &= |x-y|,\ \text{otherwise}.\end{aligned}$$

Let us first check the axioms of a D-generalized metric space.

(i) $D(x,y)=0\Rightarrow|x-y|=0\Rightarrow x=y$.
(ii) It is clear that $D(x,y)=D(y,x)$ for all $x,y\in X$.
(iii) We have $C(D,X,x)=\emptyset$ for all $x\neq 0$. If $x=0$, we can find a sequence (x_n) in X converging to 0. So for any $y\in X$, there exists a number $k\geq 1$ such that $D(0,y)=|0-y|=y\leq k\limsup_{n\to\infty}D(x_n,y)$.

The axioms $(D1)-(D3)$ are satisfied and hence (X,D) is a D-generalized metric space. Since every D-cauchy sequence (x_n) in X converges to an element in X, (X,D) is a D-complete generalized metric space with $k=1$.
We define

$$Tx=\begin{cases}1, & \text{if } x=1,\\ \frac{x}{3}, & \text{otherwise.}\end{cases}$$

Also, we define $\alpha(x,y)=1$ for all $x,y\in X$, $\psi\in\Psi$ by $\psi(t)=\frac{t}{2}$, $t\in[0,\infty)$, and $\beta:[0,\infty)\to[0,1)$ by $\beta(t)=\frac{1}{2}$. Then T is a generalized α-ψ-Geraghty contraction type mapping and satisfies all the hypothesis except (7) of Theorem 3.2. T has two fixed points 0 and 1 and we observe that $D(0,1)=\infty$ and condition (7) fails to hold.

Example 3.9. Let $X=[0,\infty]$. We define the distance function $D: X\times X\to[0,\infty)$ as follows:

$$D(x,1)=D(1,x)=\infty \text{ if } x\in X,$$

$$D(x,y)=|x-y|, \text{ otherwise.}$$

Then (X,D) is a generalized metric space with $k=1$. We define the partial order $\preceq$ on X as $x\preceq y$ if $x\geq y$ and x and y are of the form $\frac{1}{2^n}$, $n=0,1,2,\cdots$. Then $(X,D,\preceq)$ is a D-complete generalized metric space with partial order $\preceq$. We define the mapping

$$Tx=\begin{cases}\frac{x^2}{4}, & \text{if } x=\frac{1}{2^n}; n=0,1,\cdots\\ 2x, & \text{otherwise.}\end{cases}$$

Also, we define $\alpha(x,y)=1$ for all $x,y\in X$ and β and ψ as in Example 3.8. Then T satisfies all the hypothesis of Theorem 3.7 and T has a unique fixed point $x=0$. We see that T satisfies condition (3.7) but condition (2.1) is not satisfied. In particular, we choose two elements $x=\frac{1}{4}$ and $y=\frac{1}{3}$, which are not comparable. Then

$$\begin{aligned}\alpha(x,y)\psi(D(Tx,Ty)) &= \alpha(\frac{1}{4},\frac{1}{3})\psi\left(D(\frac{1}{64},\frac{2}{3})\right)\\ &= \psi\left(|\frac{1}{64}-\frac{2}{3}|\right)=\psi\left(\frac{125}{192}\right)=\frac{125}{384}=0.3255\end{aligned}$$

and

$$\begin{aligned}M(x,y) &= \max\{D(x,y),D(x,Tx),D(y,Ty)\}\\ &= \max\{D(\frac{1}{4},\frac{1}{3}),D(\frac{1}{4},\frac{1}{64}),D(\frac{1}{3},\frac{2}{3})\}\\ &= \max\{\frac{1}{12},\frac{15}{64},\frac{1}{3}\}=\frac{1}{3}.\end{aligned}$$

$\beta(\psi(M(x,y)))\psi(M(x,y))=\frac{1}{2}\psi(\frac{1}{3})=\frac{1}{12}=0.0833$. Hence, in this case we see that $\alpha(x,y)\psi(D(Tx,Ty))\not\leq \beta(\psi(M(x,y)))\psi(M(x,y))$ for any $\beta\in\mathbb{F}$.

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