Commun. Optim. Theory 2019 (2019), Article ID 13 https://doi.org/10.23952/cot.2019.13

Communications in Optimization Theory

Available online at http://cot.mathres.org

SUZUKI TYPE FIXED POINT RESULTS IN p-METRIC SPACES

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Abstract. In this paper, we introduce the structure of modified b-metric spaces as a generalization of b-metric. Also, we present the notions of p-contractive mappings in the modified b-metric spaces and investigate the existence of fixed points for such mappings under various contractive conditions. We provide examples to illustrate our main results.

Keywords. Fixed point; Complete metric space; p-metric space.

2010 Mathematics Subject Classification. 47H10, 54H25.

1. Introduction and preliminaries

There is a large number of generalizations of the Banach contraction principle via using different forms of contractive conditions in various generalized metric spaces. Some of such generalizations are obtained via contractive conditions expressed by rational terms (see, e.g., [1–4]).

The concept of b-metric spaces, as one of the useful generalizations of standard metric spaces, was first used by Bakhtin in [5] and Czerwik in [6]. Recall (see, e.g., [5, 6]) that a b-metric d on a set X is a generalization of the standard metric, where the triangular inequality is replaced by

$$d(x,z) \leq b[d(x,y)+d(y,z)], x,y,z \in X,$$

for some fixed $b \geq 1$. In 2008 Suzuki introduced an interesting generalization of Banach fixed point principle. This interesting fixed-point result is as follows.

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Received January 28, 2018; Accepted July 15, 2019.

Theorem 1.1. *[7] Let (X,d) be a complete metric space. Let $T : X \longrightarrow X$ be a self-maps and $\theta =: [0,1) \longrightarrow (\frac{1}{2},1]$ be defined by*

$$\theta(r) = \begin{cases} 1 & , \quad 0 \le r \le \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2} & , \quad \frac{\sqrt{5}-1}{2} \le r \le \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & , \quad \frac{1}{\sqrt{2}} \le r < 1. \end{cases}$$

If there exists $r \in [0,1)$ such that, for each $x,y \in X$,

$$\theta(r)d(x,Tx)) \le d(x,y) \Longrightarrow d(Tx,Ty) \le rd(x,y)),$$

then

(i) T has a unique fixed point $a \in X$.

(ii) For each $x \in X$, the sequence $\{T^n x\}$ converges to a.

Remark 1.2. Let $f,g : [0,\infty) \longrightarrow \mathbb{R}$ be two functions such that

$$f(0) = g(0)$$

and

$$\frac{df(x)}{dx} = f'(x) \le g'(x) = \frac{dg(x)}{dx}.$$

Then $f(x) \le g(x)$, $\forall x \in [0,\infty)$.

Let Ψ denote a family of mappings such that, for each $\Omega \in \Psi$, $\Omega : [0,\infty) \longrightarrow [0,\infty)$,

(1) $t \le \Omega(t)$ for every $t \in [0,\infty)$ and Ω is onto.

(2) Ω is differentiable for all $t \in [0,\infty)$ and Ω' is increasing.

Lemma 1.3. *Let $\Omega \in \Psi$. For every $x,y \in [0,\infty)$, $r \in (0,1)$ and for every $n \in \mathbb{N}$, we have*

(1) $\Omega(x+y) \ge \Omega(x) + \Omega(y)$,

(2) Ω *is continuous and is strictly increasing,*

(3) $r\Omega(x) \ge \Omega(rx)$,

(4) $\Omega^{-1}(x+y) \le \Omega^{-1}(x) + \Omega^{-1}(y)$,

(5) $\Omega^{-1}(rx) \ge r\Omega^{-1}(x)$,

(6) $\Omega^n(x+y) \ge \Omega^n(x) + \Omega^n(y)$,

(7) $\Omega^{-n}(x+y) \le \Omega^{-n}(x) + \Omega^{-n}(y)$.

Proof. (1) If $g(x) = \Omega(x+b)$ and $f(x) = \Omega(x) + \Omega(b)$, then $f(0) = g(0)$ and

$$f'(x) = \Omega'(x) \le \Omega'(x+b) = g'(x).$$

Therefore, for every $x \in [0,\infty)$, we have $f(x) \le g(x)$, that is,

$$\Omega(x+y) \ge \Omega(x) + \Omega(y).$$

(2) It is clear that Ω is continuous. Since

$$\Omega'(x) = \lim_{h\to 0} \frac{\Omega(x+h) - \Omega(x)}{h} \ge \lim_{h\to 0} \frac{\Omega(h)}{h} \ge \lim_{h\to 0} \frac{h}{h} = 1,$$

Ω is strictly increasing.

(3) If $f(x) = \Omega(rx)$ and $g(x) = r\Omega(x)$ for every $r \in (0,1)$, then

$$f(0) = g(0) = 0$$

and

$$f'(x) = r\Omega'(rx) \leq r\Omega'(x) = g'(x).$$

For every $x \in [0,\infty)$, $f(x) \leq g(x)$, that is, $\Omega(rx) \leq r\Omega(x)$.

(4) By part (2), we know that the inverse of Ω, that is, Ω^{-1} exists and it is strictly increasing. Hence, if $x = \Omega^{-1}(x)$ and $y = \Omega^{-1}(y)$ in (1), then

$$\Omega(\Omega^{-1}(x) + \Omega^{-1}(y)) \geq \Omega(\Omega^{-1}(x)) + \Omega(\Omega^{-1}(y)) = x + y.$$

That is,

$$\Omega^{-1}(x+y) \leq \Omega^{-1}(x) + \Omega^{-1}(y).$$

(5) If $x = \Omega^{-1}(x)$ in (3), then

$$\Omega(r\Omega^{-1}(x)) \leq r\Omega(\Omega^{-1}(x)).$$

That is, $r\Omega^{-1}(x) \leq \Omega^{-1}(rx))$.

(6) For $n = 1$, it is obvious. Suppose that (5) holds for some $n \geq 2$. Since

$$\begin{aligned}\Omega^{n+1}(x+y) &= \Omega(\Omega^{n}(x+y)) \\ &\geq \Omega(\Omega^{n}(x) + \Omega^{n}(y)) \\ &\geq \Omega(\Omega^{n}(x)) + \Omega(\Omega^{n}(y)) = \Omega^{n+1}(x) + \Omega^{n+1}(y),\end{aligned}$$

we can obtain inequality (5) by induction.

(7) Similarly, this part is obtain from (4) and (5) obviously.

□

Remark 1.4. For every $\Omega \in \Psi$ and for every $t \in [0,\infty)$ we have $\Omega^{-1}(t) \leq t \leq \Omega(t)$ and $\Omega^{-1}(0) = 0 = \Omega(0)$.

For example, if $\Omega : [0,\infty) \longrightarrow [0,\infty)$ is defined by $\Omega(t) = e^t - 1$, $\Omega(t) = te^t$ and $\Omega(t) = t^2 + 2t$ for every $t \in [0,\infty)$, then it is easy to see that $\Omega \in \Psi$.

Now, we introduce the concept of the extended b-metric spaces as follows.

Definition 1.5. Let X be a (nonempty) set. A function $\widetilde{d} : X \times X \to \mathbb{R}^+$ is a p-metric if there exists a $\Omega \in \Psi$ such that, for all $x,y,z \in X$, the following conditions hold:

($\widetilde{d}_1$) $\widetilde{d}(x,y) = 0$ iff $x = y$,

($\widetilde{d}_2$) $\widetilde{d}(x,y) = \widetilde{d}(y,x)$,

($\widetilde{d}_3$) $\widetilde{d}(x,z) \leq \Omega(\widetilde{d}(x,y)) + \Omega(\widetilde{d}(y,z))$.

In this case, the pair $(X,\widetilde{d})$ is called a p-metric space, or an extended b-metric space.

A b-metric [6] is a p-metric, with $\Omega(t) = bt$. For some fixed $b \geq 1$, every metric is a p-metric, for every $\Omega \in \Psi$.

Example 1.6. Let (X,d) be a b-metric space with coefficient $b \geq 1$ and let $\widetilde{d}(x,y) = \sinh(d(x,y))$. We show that $\widetilde{d}$ is a p -metric with $\Omega(t) = \sinh(2bt)$ for all $t \geq 0$ (and $\Omega^{-1}(u) = \frac{1}{2b}\sinh^{-1}(2bu)$ for $u \geq 0$). Obviously, conditions $(\widetilde{d}_1)$ and $(\widetilde{d}_2)$ of Definition 1.5 are satisfied. Since $\sinh(x)$ is an increasing function. Hence, for every $x, y \geq 0$,

$$\sinh(x+y) \leq \sinh(2\max\{x,y\}) \leq \sinh(2x) + \sinh(2y).$$

Therefore, for each $x, y, z \in X$, we have

$$\begin{aligned}
\widetilde{d}(x,z) &= \sinh(d(x,z)) \\
&\leq \sinh(bd(x,y) + bd(y,z)) \leq \sinh(b\sinh(d(x,y)) + b\sinh(d(y,z))) \\
&= \sinh(b\widetilde{d}(x,y) + b\widetilde{d}(y,z)) \\
&\leq \sinh(2b\widetilde{d}(x,y)) + \sinh(2b\widetilde{d}(y,z)) \\
&= \Omega(\widetilde{d}(x,y)) + \Omega(\widetilde{d}(y,z)).
\end{aligned}$$

So, condition $(\widetilde{p}_3)$ of Definition 1.5 is also satisfied and $\widetilde{d}$ is a p-metric. Note that $\sinh|x-y|$ is not a metric on $\mathbb{R}$, e.g.,

$$\sinh 5 \approx 74.203 \nleq 3.627 + 10.0179 \approx \sinh 2 + \sinh 3.$$

Similarly, although $d(x,y) = (x-y)^2$ is a b-metric on $\mathbb{R}$ with $b = 2$, there is no $b \neq 1$ such that $\widehat{d}(x,y) = \sinh(x-y)^2$ is a b-metric with parameter b. Indeed, putting $z = 0$ and $y = 1$, we have

$$\sinh x^2 \leq b(\sinh(x-1)^2 + \sinh 1),$$

which does not hold for any fixed b and x sufficiently large.

Definition 1.7. Let $(X,\widetilde{d})$ be a p-metric space. A sequence $\{x_n\}$ in X is said to be:

(1) p-Cauchy if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m,n \geq n_0$, $\widetilde{d}(x_n,x_m) < \varepsilon$.

(2) p-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $n \geq n_0$, $\widetilde{d}(x,x_n) < \varepsilon$.

(3) An p-metric space X is said to be complete if every p-Cauchy sequence is p-convergent in X.

Lemma 1.8. *Let $(X,\widetilde{d})$ be a p-metric space. If sequence $\{x_n\}$ in X p-converges to x, then x is unique.*

Proof. Let $\{x_n\}$ p-converge to x and y. Using the rectangle inequality in the p-metric space, it is easy to see that

$$\widetilde{d}(x,y) \leq \Omega(\widetilde{d}(x,x_n)) + \Omega(\widetilde{d}(y,x_n)).$$

Taking the limit as $n \to \infty$ in the above inequality, we obtain $\widetilde{d}(x,y) = 0$ so $x = y$. This completes the proof. □

Lemma 1.9. *Let $(X,\widetilde{d})$ be a p-metric space. If sequence $\{x_n\}$ in X is p-convergent to x, then $\{x_n\}$ is a p-Cauchy sequence.*

Proof. Since $\lim_{n\to\infty} x_n = x$ and using the rectangle inequality in the p-metric space, it is easy to see that

$$\widetilde{d}(x_n,x_m) \leq \Omega(\widetilde{d}(x_n,x)) + \Omega(\widetilde{d}(x,x_m)).$$

Taking the limit as $n,m \to \infty$ in the above inequality, we obtain

$$\lim_{n,m\to\infty} \widetilde{d}(x_n,x_m) = 0.$$

Hence $\{x_n\}$ is a p-Cauchy sequence. The proof is complete. □

We need the following simple lemma for the p-convergent sequences.

Lemma 1.10. *Let $(X,\widetilde{d})$ be a p-metric space with function Ω,*

1. Suppose that $\{x_n\}$ and $\{y_n\}$ are p-convergent to x and y, respectively. Then

$$\Omega^{-2}(\widetilde{d}(x,y)) \le \liminf_{n\longrightarrow\infty} \widetilde{d}(x_n,y_n) \le \limsup_{n\longrightarrow\infty} \widetilde{d}(x_n,y_n) \le \Omega^{2}(\widetilde{d}(x,y)).$$

In particular, if $x = y$, then $\lim_{n\to\infty} \widetilde{d}(x_n,y_n) = 0$.

2. Suppose that $\{x_n\}$ is p-convergent to x and $z \in X$ is arbitrary. Then

$$\Omega^{-1}(\widetilde{d}(x,z)) \le \liminf_{n\longrightarrow\infty} \widetilde{d}(x_n,z) \le \limsup_{n\longrightarrow\infty} \widetilde{d}(x_n,z) \le \Omega(\widetilde{d}(x,z)).$$

Proof. 1. Using the rectangle inequality in the p-metric space, it is easy to see that

$$\begin{aligned}\widetilde{d}(x,y) &\le \Omega(\widetilde{d}(x,x_n)) + \Omega(\widetilde{d}(y,x_n))\\ &\le \Omega(\widetilde{d}(x,x_n)) + \Omega\big[\Omega(\widetilde{d}(y,y_n)) + \Omega(\widetilde{d}(x_n,y_n))\big]\end{aligned}$$

and

$$\begin{aligned}\widetilde{d}(x_n,y_n) &\le \Omega(\widetilde{d}(x_n,x)) + \Omega(\widetilde{d}(y_n,x))\\ &\le \Omega(\widetilde{d}(x_n,x)) + \Omega\big[\Omega(\widetilde{d}(y_n,y)) + \Omega(\widetilde{d}(x,y))\big].\end{aligned}$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality, we obtain the desired result.

2. Using the rectangle inequality, we see that

$$\widetilde{d}(x,z) \le \Omega(\widetilde{d}(x,x_n)) + \Omega(\widetilde{d}(x_n,z)).$$

Taking the lower limit as $n \to \infty$ in the above inequality, we have

$$\widetilde{d}(x,z) \le \Omega(\liminf_{n\longrightarrow\infty} \widetilde{d}(x_n,z)).$$

Hence

$$\Omega^{-1}(\widetilde{d}(x,z)) \le \liminf_{n\longrightarrow\infty} \widetilde{d}(x_n,z).$$

Also

$$\widetilde{d}(x_n,z) \le \Omega(\widetilde{d}(x_n,x)) + \Omega(\widetilde{d}(z,x)).$$

Taking the upper limit as $n \to \infty$ in the above inequality, we obtain the desired result. □

2. MAIN RESULTS

In this section, let $(X,\widetilde{d})$ be a p-metric space with function Ω satisfying the following condition: For every $0<q<1$, $\alpha\geq 0$ and every $m>n$ sequence

$$\Omega^{m-n}[\sum_{i=n}^{m-1}\Omega^{-(m+i-1)}(q^i\alpha)],$$

is convergent to 0 when $n,m\to\infty$, that is,

$$\lim_{n,m\longrightarrow\infty}\Omega^{m-n}[\sum_{i=n}^{m-1}\Omega^{-(m+i-1)}(q^i\alpha)]=0.$$

For example if $\Omega(t)=bt$ for every $b\geq 1$, then it is easy to see that

$$\lim_{n,m\longrightarrow\infty}b^{m-n}[\sum_{i=n}^{m-1}b^{-(m+i-1)}(q^i\alpha)]=0.$$

With this condition, we start our work by proving the following crucial Theorem. Henceforth, we assume that Ω is with above condition.

Theorem 2.1. *Let $(X,\widetilde{d})$ be a complete p-metric space with $\Omega\in\Psi$. Let $T:X\longrightarrow X$ be a self-maps and $\theta =: [0,1)\longrightarrow(\frac{1}{2},1]$ be defined by*

$$\theta(r)=\begin{cases}1 & , \quad 0\leq r\leq\frac{\sqrt{5}-1}{2}\\[2ex] \frac{1-r}{r^2} & , \quad \frac{\sqrt{5}-1}{2}\leq r\leq\frac{1}{\sqrt{2}}\\[2ex] \frac{1}{1+r} & , \quad \frac{1}{\sqrt{2}}\leq r<1.\end{cases} \tag{2.1}$$

If there exists $r\in[0,1)$ such that, for each $x,y\in X$,

$$\theta(r)\Omega^{-1}(\widetilde{d}(x,Tx))\leq\widetilde{d}(x,y)\Longrightarrow\widetilde{d}(Tx,Ty)\leq r\Omega^{-2}(\widetilde{d}(x,y)). \tag{2.2}$$

Then

(i) T has a unique fixed point $a\in X$.

(ii) For each $x\in X$, the sequence $\{T^nx\}$ converges to a.

Proof. Put $y=Tx$ in (2.2). From

$$\theta(r)\Omega^{-1}(\widetilde{d}(x,Tx))\leq\widetilde{d}(x,Tx),$$

we have

$$\widetilde{d}(Tx,T^2x)\leq r\Omega^{-2}(\widetilde{d}(x,Tx)), \tag{2.3}$$

for every $x \in X$. Let $x_0 \in X$ be arbitrarily fixed. For the sequence $\{x_n\}$, letting $x_1 = Tx_0$ and $x_{n+1} = Tx_n = T^{n+1}x_0$ for $n \in \mathbb{N} \cup \{0\}$. We show that $x_{n+1} = Tx_n$ is a p-Cauchy sequence. By (2.3), we have

$$\begin{aligned}
\widetilde{d}(x_n, x_{n+1}) &= \widetilde{d}(Tx_{n-1}, T^2x_{n-1}) \\
&\leq r\Omega^{-2}(\widetilde{d}(x_{n-1}, Tx_{n-1})) = r\Omega^{-2}(\widetilde{d}(x_{n-1}, x_n)) \\
&\leq r\Omega^{-2}(r\Omega^{-2}(\widetilde{d}(x_{n-2}, x_{n-1}))) \\
&\leq \Omega^{-4}(r^2\widetilde{d}(x_{n-2}, x_{n-1})) \\
&\vdots \\
&\leq \Omega^{-2n}(r^n\widetilde{d}(x_0, x_1)).
\end{aligned}$$

By Lemma 1.3 and the triangle inequality in p-metric spaces, for $m > n$, we have

$$\begin{aligned}
&\Omega^{-(m-n)}(\widetilde{d}(x_n, x_m)) \\
\leq\ &\Omega^{-(m-n)}[\Omega(\widetilde{d}(x_n, x_{n+1})) + \Omega(\widetilde{d}(x_{n+1}, x_m))] \\
\leq\ &\Omega^{-(m-n-1)}(\widetilde{d}(x_n, x_{n+1})) + \Omega^{-(m-n-1)}(\widetilde{d}(x_{n+1}, x_m)) \\
\leq\ &\Omega^{-(m-n-1)}\widetilde{d}(x_n, x_{n+1}) + \Omega^{-(m-n-2)}(\widetilde{d}(x_{n+1}, x_{n+2})) + \Omega^{-(m-n-2)}(\widetilde{d}(x_{n+2}, x_m)) \\
&\vdots \\
\leq\ &\Omega^{-(m-n-1)}(\widetilde{d}(x_n, x_{n+1})) + \Omega^{-(m-n-2)}(\widetilde{d}(x_{n+1}, x_{n+2})) \\
+\ &\cdots + \Omega^{-1}(\widetilde{d}(x_{m-2}, x_{m-1})) + \widetilde{d}(x_{m-1}, x_m).
\end{aligned}$$

Using

$$\widetilde{d}(x_n, x_{n+1}) \leq \Omega^{-2n}(r^n\widetilde{d}(x_0, x_1)),$$

we have

$$\begin{aligned}
&\Omega^{-(m-n)}(\widetilde{d}(x_n, x_m)) \\
\leq\ &\Omega^{-(m-n-1)}(\Omega^{-2n}(r^n\widetilde{d}(x_0, x_1))) + \Omega^{-(m-n-2)}(\Omega^{-2(n+1)}(r^{n+1}\widetilde{d}(x_0, x_1))) \\
+\ &\cdots + \Omega^{-1}(\Omega^{-2(m-2)}(r^{m-2}\widetilde{d}(x_0, x_1))) + \Omega^{-2(m-1)}(r^{m-1}\widetilde{d}(x_0, x_1)) \\
=\ &\Omega^{-(m+n-1)}(r^n\widetilde{d}(x_0, x_1)) + \Omega^{-(m+n)}(r^{n+1}\widetilde{d}(x_0, x_1)) \\
+\ &\cdots + \Omega^{-(2m-3)}(r^{m-2}\widetilde{d}(x_0, x_1)) + \Omega^{-2(m-1)}(r^{m-1}\widetilde{d}(x_0, x_1)) \\
=\ &\sum_{i=n}^{m-1} \Omega^{-(m+i-1)}(r^i\widetilde{d}(x_0, x_1)).
\end{aligned}$$

It follows that

$$\widetilde{d}(x_n, x_m) \leq \Omega^{m-n}(\sum_{i=n}^{m-1} \Omega^{-(m+i-1)}(r^i\widetilde{d}(x_0, x_1))).$$

Since, $\lim_{n,m\longrightarrow\infty} \Omega^{m-n}[\sum_{i=n}^{m-1} \Omega^{-(m+i-1)} r^i \alpha] = 0$, we have

$$\lim_{n,m\longrightarrow\infty} \widetilde{d}(x_n, x_m) = 0.$$

It follows that $\{x_n\}$ is a p-Cauchy sequence. Since X is complete, we conclude that x_n p-converges to a for some $a \in X$. Putting $x = T^{n-1}a$ in (2.3), we get that

$$\widetilde{d}(T^n a, T^{n+1}a) \leq r\Omega^{-2}(\widetilde{d}(T^{n-1}a, T^n a)), \tag{2.4}$$

holds for each $n \in \mathbb{N}$ (where $T^0 a = a$). It follows by induction that

$$\widetilde{d}(T^n a, T^{n+1} a) \leq \Omega^{-2n}(r^n \widetilde{d}(a, Ta)). \tag{2.5}$$

Let us now prove that

$$\widetilde{d}(a, Tx) \leq r\widetilde{d}(a, x), \tag{2.6}$$

holds for each $x \neq a$. By Lemma 1.10, we have

$$0 < \Omega^{-1}(\widetilde{d}(a,x)) \leq \limsup_{n \longrightarrow \infty} \widetilde{d}(x_n, x).$$

Thus $\limsup_{n \longrightarrow \infty} \widetilde{d}(x_n, x) > 0$. Since $\widetilde{d}(x_n, Tx_n) \longrightarrow 0$, it follows that there exists a $n_0 \in \mathbb{N}$ such that

$$\theta(r)\Omega^{-1}(\widetilde{d}(x_n, Tx_n)) \leq \widetilde{d}(x_n, x),$$

holds for every $n \geq n_0$. Assumption (2.2) implies that for such n

$$\widetilde{d}(Tx_n, Tx) \leq r\Omega^{-2}(\widetilde{d}(x_n, x)),$$

and by Lemma 1.10, we have

$$\begin{aligned} \Omega^{-1}(\widetilde{d}(a, Tx)) &\leq \limsup_{n \longrightarrow \infty} \widetilde{d}(Tx_n, Tx) \\ &\leq \limsup_{n \longrightarrow \infty} r\Omega^{-2}(\widetilde{d}(x_n, x)) \\ &\leq r\Omega^{-2}(\Omega(\widetilde{d}(a,x))) = r\Omega^{-1}(\widetilde{d}(a,x)). \end{aligned}$$

It follows for each $x \neq a$ that

$$\widetilde{d}(a, Tx) \leq r\widetilde{d}(a, x).$$

We next prove that

$$\widetilde{d}(T^n a, a) \leq \widetilde{d}(Ta, a), \tag{2.7}$$

for each $n \in \mathbb{N}$. For $n = 1$, this relation is obvious. Suppose that it holds for some $m \in \mathbb{N}$. If $T^m a = a$, then $T^{m+1} a = Ta$. Therefore,

$$\widetilde{d}(T^{m+1} a, a) = \widetilde{d}(Ta, a) \leq \widetilde{d}(Ta, a).$$

If $T^m a \neq a$, we can apply (2.6) and the induction hypothesis to get

$$\begin{aligned} \widetilde{d}(a, T^{m+1} a) = \widetilde{d}(T^{m+1} a, a) &\leq r\widetilde{d}(T^m a, a) \\ &\leq r\widetilde{d}(Ta, a) \leq \widetilde{d}(Ta, a). \end{aligned}$$

(2.7) is proved by induction.

In order to prove that $Ta = a$. Let $Ta \neq a$. We consider two possible cases.

Case I. $0 \leq r < \frac{1}{\sqrt{2}}$ (and hence $\theta(r) \leq \frac{1-r}{r^2}$). We prove first that

$$\widetilde{d}(T^n a, Ta) \leq r\Omega^{-1}(\widetilde{d}(Ta, a)) \tag{2.8}$$

for each $n \in \mathbb{N}$. For $n = 1$, it is obvious. For $n = 2$, it follows from (2.4) due to

$$\widetilde{d}(T^2 a, Ta) \leq r\Omega^{-2}(\widetilde{d}(Ta, a)) \leq r\Omega^{-1}(\widetilde{d}(Ta, a)).$$

Suppose that (2.8) holds for some $n > 2$. Since

$$\begin{aligned}\Omega^{-1}(\widetilde{d}(Ta,a)) &\leq \widetilde{d}(a,T^n a)+\widetilde{d}(T^n a,Ta)\\ &\leq \widetilde{d}(a,T^n a)+r\Omega^{-1}(\widetilde{d}(a,Ta)),\end{aligned}$$

we have $(1-r)\Omega^{-1}(\widetilde{d}(a,Ta)) \leq \widetilde{d}(a,T^n a)$. Using (2.4), we have

$$\begin{aligned}\widetilde{d}(T^n a,T^{n+1}a) &\leq r\Omega^{-2}(\widetilde{d}(T^{n-1}a,T^n a))\\ &\leq \Omega^{-1}(r\Omega^{-1}(\widetilde{d}(T^{n-1}a,T^n a)))\\ &\leq \Omega^{-1}(r\Omega^{-1}(r\Omega^{-2}(\widetilde{d}(T^{n-2}a,T^{n-1}a))))\\ &\leq \Omega^{-3}(r^2\Omega^{-1}(\widetilde{d}(T^{n-2}a,T^{n-1}a)))\\ &\vdots\\ &\leq \Omega^{-2n+1}(r^n\Omega^{-1}(\widetilde{d}(a,Ta))).\end{aligned}$$

It follows from (2.5) that

$$\begin{aligned}\theta(r)\Omega^{-1}(\widetilde{d}(T^n a,T^{n+1}a)) &\leq \frac{1-r}{r^2}\Omega^{-1}(\widetilde{d}(T^n a,T^{n+1}a))\\ &\leq \frac{1-r}{r^n}\Omega^{-1}(\widetilde{d}(T^n a,T^{n+1}a))\\ &\leq \frac{1-r}{r^n}\widetilde{d}(T^n a,T^{n+1}a)\\ &\leq \frac{1-r}{r^n}\Omega^{-2n+1}(r^n\Omega^{-1}(\widetilde{d}(a,Ta)))\\ &\leq \frac{(1-r)r^n}{r^n}\Omega^{-1}(\widetilde{d}(a,Ta)) = (1-r)\Omega^{-1}(\widetilde{d}(a,Ta))\\ &\leq \widetilde{d}(a,T^n a).\end{aligned}$$

Assumption (2.2) and relation (2.7) imply that

$$\begin{aligned}\widetilde{d}(Ta,T^{n+1}a) &\leq r\Omega^{-2}(\widetilde{d}(a,T^n a)) \leq r\Omega^{-2}(\widetilde{d}(a,Ta))\\ &\leq r\Omega^{-1}(\widetilde{d}(a,Ta)).\end{aligned}$$

So relation (2.8) is proved by induction. Now $Ta \neq a$ and (2.8) imply that $T^n a \neq a$ for each $n \in \mathbb{N}$. Hence, (2.6) implies that

$$\widetilde{d}(a,T^{n+1}a) \leq r\widetilde{d}(a,T^n a) \leq r^2\widetilde{d}(a,T^{n-1}a) \leq \cdots \leq r^n\widetilde{d}(a,Ta).$$

Hence $\lim_{n\to\infty}\widetilde{d}(a,T^{n+1}a) = 0$. Thus $T^n a \longrightarrow a$ and, using Lemma 1.10 in (2.8), we have

$$\Omega^{-1}(\widetilde{d}(a,Ta)) \leq \limsup_{n\longrightarrow\infty}\widetilde{d}(T^n a,Ta) \leq r\Omega^{-1}(\widetilde{d}(a,Ta)),$$

which implies that $\widetilde{d}(a,Ta) = 0$, a contradiction. This implies that $Ta = a$.

Case II. $\frac{1}{\sqrt{2}} \leq r < 1$ (and so $\theta(r) = \frac{1}{1+r}$). We prove that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\theta(r)\Omega^{-1}(\widetilde{d}(x_{n_k},Tx_{n_k})) \leq \widetilde{d}(x_{n_k},a) \tag{2.9}$$

holds for each $k \in \mathbb{N}$. Suppose that

$$\frac{1}{1+r}\Omega^{-1}(\widetilde{d}(x_{n-1},x_n)) > \widetilde{d}(x_{n-1},a),$$

and

$$\frac{1}{(1+r)}\Omega^{-1}(\widetilde{d}(x_n,x_{n+1})) > \widetilde{d}(x_n,a),$$

holds for some $n\in\mathbb{N}$. From (2.3), we know that $\widetilde{d}(x_n,x_{n+1}) \leq r\Omega^{-2}(\widetilde{d}(x_{n-1},x_n))$ holds for each $n\in\mathbb{N}$. Then

$$\begin{aligned}
\Omega^{-1}(\widetilde{d}(x_{n-1},x_n)) &\leq \Omega^{-1}[\Omega(\widetilde{d}(x_{n-1},a))+\Omega(\widetilde{d}(a,x_n))]\\
&\leq \widetilde{d}(x_{n-1},a)+\widetilde{d}(a,x_n)\\
&< \frac{1}{1+r}\Omega^{-1}(\widetilde{d}(x_{n-1},x_n))+\frac{1}{1+r}\Omega^{-1}(\widetilde{d}(x_n,x_{n+1}))\\
&\leq \frac{1}{1+r}\Omega^{-1}(\widetilde{d}(x_{n-1},x_n))+\frac{r}{1+r}\Omega^{-2}(\widetilde{d}(x_{n-1},x_n))\\
&\leq \frac{1}{1+r}\Omega^{-1}(\widetilde{d}(x_{n-1},x_n))+\frac{r}{1+r}\Omega^{-1}(\widetilde{d}(x_{n-1},x_n))\\
&\leq \Omega^{-1}(\widetilde{d}(x_{n-1},x_n)),
\end{aligned}$$

which is impossible. Hence one of the following holds for each n:

$$\theta(r)\Omega^{-1}(\widetilde{d}(x_{n-1},x_n)) \leq \widetilde{d}(x_{n-1},a),$$

or

$$\theta(r)\Omega^{-1}(\widetilde{d}(x_n,x_{n+1})) \leq \widetilde{d}(x_n,a).$$

In particular,

$$\theta(r)\Omega^{-1}(\widetilde{d}(x_{2n-1},x_{2n})) \leq \widetilde{d}(x_{2n-1},a),$$

or

$$\theta(r)\Omega^{-1}(\widetilde{d}(x_{2n},x_{2n+1})) \leq \widetilde{d}(x_{2n},a).$$

In other words, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that (2.9) holds for each $k\in\mathbb{N}$. However, assumption (2.2) implies that

$$\widetilde{d}(Tx_{n_k},Ta) \leq r\Omega^{-2}(\widetilde{d}(x_{n_k},a)).$$

By Lemma 1.10, we get that

$$\begin{aligned}
\Omega^{-1}(\widetilde{d}(a,Ta)) &\leq \limsup_{n\longrightarrow\infty} \widetilde{d}(Tx_{n_k},Ta) \leq r \limsup_{n\longrightarrow\infty} \Omega^{-2}(\widetilde{d}(x_{n_k},a))\\
&\leq r\Omega^{-2}(\Omega(\widetilde{d}(a,a)))=0,
\end{aligned}$$

hence $\widetilde{d}(a,Ta)\leq 0$, which is possible only if $Ta=a$. Thus, we have proved that a is a fixed point of T. The uniqueness of the fixed point follows easily from (2.2). Indeed, if a,c are two fixed points of T,

$$\begin{aligned}
\theta(r)\Omega^{-1}(\widetilde{d}(a,Ta)) &= \theta(r)\Omega^{-1}(\widetilde{d}(a,a))=0\\
&\leq \widetilde{d}(a,c),
\end{aligned}$$

then (2.2) implies that

$$\widetilde{d}(a,c)=\widetilde{d}(Ta,Tc)\leq r\Omega^{-2}(\widetilde{d}(a,c))\leq r\widetilde{d}(a,c),$$

which implies that $a=c$. □

Corollary 2.2. *[7] Let (X,d) be a complete metric space. Let $T : X \longrightarrow X$ be a self-maps and let $\theta =: [0,1) \longrightarrow (\frac{1}{2},1]$ be defined by*

$$\theta(r) = \begin{cases} 1 & , \quad 0 \le r \le \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2} & , \quad \frac{\sqrt{5}-1}{2} \le r \le \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & , \quad \frac{1}{\sqrt{2}} \le r < 1. \end{cases}$$

If there exists $r \in [0,1)$ such that, for each $x,y \in X$,

$$\theta(r)d(x,Tx)) \le d(x,y) \Longrightarrow d(Tx,Ty) \le rd(x,y)).$$

Then

(i) T has a unique fixed point $a \in X$.

(ii) For each $x \in X$, the sequence $\{T^n x\}$ converges to a.

Proof. It is suffice to set $\Omega(t) = t$ in Theorem 2.1. □

Corollary 2.3. *[8] Let (X,d) be a complete b-metric space with $b \ge 1$. Let $T : X \longrightarrow X$ be a self-maps and let $\theta =: [0,1) \longrightarrow (\frac{1}{2},1]$ be defined by*

$$\theta(r) = \begin{cases} 1 & , \quad 0 \le r \le \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2} & , \quad \frac{\sqrt{5}-1}{2} \le r \le \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & , \quad \frac{1}{\sqrt{2}} \le r < 1. \end{cases}$$

If there exists $r \in [0,1)$ such that, for each $x,y \in X$,

$$\frac{1}{b}\theta(r)d(x,Tx)) \le d(x,y) \Longrightarrow d(Tx,Ty) \le \frac{1}{b^2}rd(x,y)). \tag{2.10}$$

Then

(i) T has a unique fixed point $a \in X$.

(ii) For each $x \in X$, the sequence $\{T^n x\}$ converges to a.

Proof. It is suffice to set $\Omega(t) = bt$ in Theorem 2.1. □

Now, we present an example which supports Corollary 2.3 and shows that the obtained fixed point results can be applied in the situations when some other known results fail.

Example 2.4. Let $X = [0,\infty)$ be equipped with the p-metric defined by

$$\widetilde{d}(x,y) = \begin{cases} 0, & x = y, \\ (x+y)^2, & x \ne y. \end{cases}$$

Consider the mapping $T : X \to X$ given as $Tx = Ln(1 + \frac{x}{5})$. For $b = 2$, choose $\Omega \in \Psi$ as $\Omega(t) = 2t$ and choose $r = \frac{1}{5}$. Hence $\theta(r) = 1$.

We check that the conditions of Corollary 2.3 are fulfilled. First of all, $Ln(1+x) \leq x$ holds for $x \in X$ this reduces to

$$\widetilde{d}(x,Tx) = (x + Ln(1+\frac{x}{5}))^2 \leq (x+\frac{x}{5})^2 = \frac{36}{25}x^2.$$

Since

$$\frac{1}{2}\theta(r)\widetilde{d}(x,Tx) \leq \frac{1}{2}.\frac{36}{25}x^2 \leq (x+y)^2 = \widetilde{d}(x,y), \quad \forall x \neq y.$$

Hence, the condition (2.10) reduces to

$$\widetilde{d}(Tx,Ty) \leq [Ln(1+\frac{x}{5}) + Ln(1+\frac{y}{5})]^2 \leq (\frac{x}{5}+\frac{y}{5})^2 \leq \frac{1}{25}(x+y)^2 \leq \frac{1}{4}r\widetilde{d}(x,y)$$

which is fulfilled for $x \neq y$. By Corollary 2.3, we obtain that mapping T has a fixed point (which is 0).

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