Commun. Optim. Theory 2019 (2019), Article ID 11 https://doi.org/10.23952/cot.2019.11

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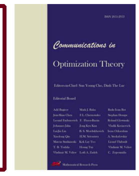

SOME MIXED INTEGRAL INEQUALITIES FOR DOUBLE VARIABLES ON TIME SCALES

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Abstract. In this paper, we establish some Gamidov type integral inequalities on time scales involving functions of two independent variables. Our results can act as a tool for studying properties of solutions of partial dynamic equations.

Keywords. Integral inequality, time scales, Double integrals.

2010 Mathematics Subject Classification. 26D15, 26D20, 39A12.

1. Introduction

Many phenomena in engineering, physics and other sciences can be described by using mathematical models in general and these models can be translated by differential equations (ordinary or partial), but most of these differential equations do not have exact analytic solutions, which brings us back to using other tools for the study of certain qualitative or quantitative properties of these solutions such as the approximations theory or numerical techniques etc. Among these tools are the integral inequalities which represent an easy and powerful means for the study of some qualitative properties of solutions such as the existence, uniqueness, boundedness, and, other properties. From the discovery of Gronwall-Bellman's inequality, a lot of efforts have been devoted by many researchers into developping variants, extensions and others types of inequalities; see, for instance, [1–15] and references therein.

Recently, Hilger [16] introduced the calculus on time scales in order to unify continuous and discrete analysis. This novel theory has attracted a considerable attention from many researchers. Many authors extended some fundamental integral inequalities used in the theory of differential and integral equations on time scales.

In this paper, we establish some Gamidov type integral inequalities on time scales involving functions of two independent variables. The obtained results can be used as a tool for the study of certain properties of solutions of partial dynamic equations.

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Received December 12, 2018; Accepted July 5, 2019.

2. Preliminaries

In this section, we recall without proof some fundamental definitions and results from the calculus on time scales, for more details about times scales, we refer the readers to refer to [17, 18].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The forward jump operator σ on $\mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \in \mathbb{T}$ for all $t \in \mathbb{T}$. C_{rd} denotes the set of rd-continuous functions and the set $\mathbb{T}^k$, which is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$. We say that function $f : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t) f(t) \neq 0$, $t \in \mathbb{T}^k$. We denote by $\mathscr{R}$ the set of all regressive and rd-continuous functions and $\mathscr{R}^+ = \{f \in \mathscr{R} : 1 + \mu(t) f(t) > 0\}$.

Definition 2.1. [18, Definition 1.10] Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^{\Delta}(t)$ to be the number (provided that it exists) with the property that given any $\varepsilon > 0$ there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| < \varepsilon\ |\sigma(t) - s| \text{ for all } s \in U.$$

We call $f^{\Delta}(t)$ the delta (or Hilger) derivative of f at t.

Lemma 2.2. *[18, Theorem 6.1] Let $u, b \in C_{rd}$ and $a \in \mathscr{R}^+$. If*

$$u^{\Delta}(t) \leq a(t) u(t) + b(t),$$

then, for all $t \in \mathbb{T}$,

$$u(t) \leq u(t_0) e_a(t, t_0) + \int_{t_0}^{t} b(\tau) e_a(t, \sigma(\tau)) \Delta\tau.$$

Let $\mathbb{T}_1$, $\mathbb{T}_2$ be two time scales. Te denote by σ_i and Δ_i the forward jump operator and the delta differentiation operator respectively, on $(\mathbb{T}_i)_{i=1,2}$. Suppose that $a < b$ are points in $\mathbb{T}_1$, and $c < d$ are points in $\mathbb{T}_2$. Let f be a real valued function on $\mathbb{T}_1 \times \mathbb{T}_2$.

At $(s,t) \in \mathbb{T}_1 \times \mathbb{T}_2$, we say that f has a Δ_1 partial derivative $f^{\Delta_1}(s,t)$ (the derivative with respect to s), if, for each $\varepsilon > 0$, there exists a neighborhood U_s (open in the relative topology of $\mathbb{T}_1$) of s such that

$$\left|f(\sigma_1(s), t) - f(\alpha, t) - f^{\Delta_1}(s,t)(\sigma_1(s) - \alpha)\right| \leq \varepsilon |\sigma_1(s) - \alpha|$$

for all $\alpha \in U_s$.

At $(s,t) \in \mathbb{T}_1 \times \mathbb{T}_2$, we say that f has a Δ_2 partial derivative $f^{\Delta_2}(s,t)$ (the derivative with respect to t), if for each $\varepsilon > 0$, there exists a neighborhood U_t (open in the relative topology of $\mathbb{T}_2$) of t such that

$$\left|f(s, \sigma_2(t)) - f(s, \beta) - f^{\Delta_2}(s,t)(\sigma_2(t) - \beta)\right| \leq \varepsilon |\sigma_2(t) - \beta|$$

for all $\beta \in U_t$.

Let f be a real valued function on $\mathbb{T}_1 \times \mathbb{T}_2$. The function f is said to be rd-continuous in t, if for every $\alpha \in \mathbb{T}_1$, the function $f(\alpha, t)$ is rd-continuous on $\mathbb{T}_2$. The function f is said to be rd-continuous in s, if for every $\beta \in \mathbb{T}_2$, the function $f(s, \beta)$ is rd-continuous on $\mathbb{T}_1$.

Lemma 2.3. *[19, Lemma 2] Assume that $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$. Then, for any $K > 0$,*

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}.$$

3. MAIN RESULTS

In what follows, we assume that $\mathbb{T}_1$, $\mathbb{T}_2$ are two time scales and $[\alpha,T]_{\mathbb{T}_1}=[\alpha,T]\cap\mathbb{T}_1$, $[\beta,S]_{\mathbb{T}_2}=[\beta,S]\cap\mathbb{T}_2$, $\Omega=[\alpha,M]_{\mathbb{T}_1}\times[\beta,N]_{\mathbb{T}_2}$, $\mathbb{R}^+=[0,\infty)$ and $\mathbb{R}_0^+=(0,\infty)$.

Lemma 3.1. *Let $u(x,y),m(x,y),n(x,y),l(x,y)\in C_{rd}(\Omega,\mathbb{R}^+)$, $\alpha,t,x,M\in\mathbb{T}_1$ and $\beta,s,y,N\in\mathbb{T}_2$. Assume that*

$$\int_\alpha^M\int_\beta^N n(t,s)l(t,s)\Delta s\Delta t<1. \tag{3.1}$$

If

$$u(x,y)\le m(x,y)+l(x,y)\int_\alpha^M\int_\beta^N n(t,s)u(t,s)\Delta s\Delta t,\quad \forall(x,y)\in\Omega \tag{3.2}$$

holds, then $u(x,y)$ has the following estimate

$$u(x,y)\le m(x,y)+\frac{l(x,y)\int_\alpha^M\int_\beta^N n(t,s)m(t,s)\Delta s\Delta t}{1-\int_\alpha^M\int_\beta^N n(t,s)l(t,s)\Delta s\Delta t}. \tag{3.3}$$

Proof. Let

$$\Gamma=\int_\alpha^M\int_\beta^N n(t,s)u(t,s)\Delta s\Delta t. \tag{3.4}$$

Clearly Γ is a constant. Substituting (3.4) in (3.2), we obtain

$$u(x,y)\le m(x,y)+l(x,y)\Gamma. \tag{3.5}$$

Multiplying both sides of (3.5) by $n(x,y)$, and then integrating the resulting inequality with respect to y and x over Ω, we get

$$\Gamma\le\int_\alpha^M\int_\beta^N n(x,y)m(x,y)\Delta y\Delta x+\Gamma\int_\alpha^M\int_\beta^N n(x,y)l(x,y)\Delta y\Delta x. \tag{3.6}$$

From (3.1) and (3.6), we have

$$\Gamma\le\frac{\int_\alpha^M\int_\beta^N n(x,y)m(x,y)\Delta y\Delta x}{1-\int_\alpha^M\int_\beta^N n(x,y)l(x,y)\Delta y\Delta x}. \tag{3.7}$$

Now, substituting (3.7) into (3.5), we obtain the desired result. □

Theorem 3.2. *Let $u(x,y),a(x,y),b(x,y),f(x,y),g(x,y)\in C_{rd}(\Omega,\mathbb{R}^+)$. Let $f(x,y)$ and $g(x,y)$ be nondecreasing for each of its variables and c be a positive real constant. If*

$$u(x,y)\le c+f(x,y)\int_\alpha^x\int_\beta^y a(t,s)u(t,s)\Delta s\Delta t+g(x,y)\int_\alpha^M\int_\beta^N b(t,s)u(t,s)\Delta s\Delta t,\quad\forall(x,y)\in\Omega \tag{3.8}$$

holds, then $u(x,y)$ has the following estimate

$$u(x,y)\le P(x,y)+\frac{Q(x,y)\int_\alpha^M\int_\beta^N b(t,s)P(t,s)\Delta s\Delta t}{1-\int_\alpha^M\int_\beta^N b(t,s)Q(t,s)\Delta s\Delta t}, \tag{3.9}$$

where

$$P(x,y)=ce_{f(x,y)\int_{\beta}^{y}a(t,s)\Delta s}(x,\alpha) \tag{3.10}$$

and

$$Q(x,y)=g(x,y)e_{f(x,y)\int_{\beta}^{y}a(t,s)\Delta s}(x,\alpha) \tag{3.11}$$

provide that $\int_{\alpha}^{M}\int_{\beta}^{N}b(t,s)Q(t,s)\Delta s\Delta t<1$.

Proof. Fix arbitrary numbers $(x_1,y_1)\in\Omega$. For $\alpha\leq x\leq x_1$ and $\beta\leq y\leq y_1$ and (3.8) we have

$$u(x,y)\leq c+f(x_1,y_1)\int_{\alpha}^{x}\int_{\beta}^{y}a(t,s)u(t,s)\Delta s\Delta t+g(x_1,y_1)\int_{\alpha}^{M}\int_{\beta}^{N}b(t,s)u(t,s)\Delta s\Delta t. \tag{3.12}$$

Define a function $z(x,y)$ by

$$z(x,y)=c+f(x_1,y_1)\int_{\alpha}^{x}\int_{\beta}^{y}a(t,s)u(t,s)\Delta s\Delta t+g(x_1,y_1)\int_{\alpha}^{M}\int_{\beta}^{N}b(t,s)u(t,s)\Delta s\Delta t. \tag{3.13}$$

Clearly, $z(x,y)$ is positive, nondecreasing for each of its variables,

$$z(\alpha,y)=z(x,\beta)=z(\alpha,\beta)=c+g(x_1,y_1)\int_{\alpha}^{M}\int_{\beta}^{N}b(t,s)u(t,s)\Delta s\Delta t, \tag{3.14}$$

and

$$u(x,y)\leq z(x,y). \tag{3.15}$$

By differentiating (3.13) with respect to x, and using (3.15), we obtain

$$z^{\Delta x}(x,y)\leq f(x_1,y_1)\int_{\beta}^{y}a(x,s)z(x,s)\Delta s. \tag{3.16}$$

Using the fact that $z(x,y)$ is nondecreasing with respect to y, we get

$$z^{\Delta x}(x,y)\leq\left(f(x_1,y_1)\int_{\beta}^{y}a(x,s)\Delta s\right)z(x,y). \tag{3.17}$$

Applying Lemma 2.2 to (3.17), and using (3.14), we get

$$z(x,y)\quad\leq\quad\left(c+g(x_1,y_1)\int_{\alpha}^{M}\int_{\beta}^{N}b(t,s)z(t,s)\Delta s\Delta t\right)e_{f(x_1,y_1)\int_{\beta}^{y}a(t,s)\Delta s}(x,\alpha). \tag{3.18}$$

Since x_1 and y_1 are arbitrarily chosen and $\alpha\leq t\leq x\leq x_1$ and $\beta\leq s\leq y\leq y_1$, we find from (3.18) and (3.14) with $x_1=x$ and $y_1=y$ that

$$z(x,y)\leq\left(P(x,y)+Q(x,y)\int_{\alpha}^{M}\int_{\beta}^{N}b(t,s)z(t,s)\Delta s\Delta t\right),$$

where $P(x,y)$ and $Q(x,y)$ are given by (3.10) and (3.11) respectively. From Lemma 3.1, we obtain

$$z(x,y) \le P(x,y) + \frac{Q(x,y)\int_\alpha^M\int_\beta^N b(t,s)P(t,s)\Delta s\Delta t}{1-\int_\alpha^M\int_\beta^N b(t,s)Q(t,s)\Delta s\Delta t}. \tag{3.19}$$

Using (3.19) in (3.15), we obtain the desired result. □

Theorem 3.3. *Suppose that all the assumptions of Theorem 3.2 are satisfied, and let $c(x,y) \in C_{rd}(\Omega, \mathbb{R}_0^+)$ be nondecreasing function for each of its variables. If*

$$u(x,y) \le c(x,y) + f(x,y)\int_\alpha^x\int_\beta^y a(t,s)u(t,s)\Delta s\Delta t + g(x,y)\int_\alpha^M\int_\beta^N b(t,s)u(t,s)\Delta s\Delta t, \quad \forall (x,y)\in\Omega \tag{3.20}$$

holds, then $u(x,y)$ has the following estimate

$$u(x,y) \le c(x,y)\left(P_1(x,y) + \frac{Q(x,y)\int_\alpha^M\int_\beta^N b(t,s)P_1(t,s)\Delta s\Delta t}{1-\int_\alpha^M\int_\beta^N b(t,s)Q(t,s)\Delta s\Delta t}\right), \tag{3.21}$$

where

$$P_1(x,y) = e_{f(x,y)\int_\beta^y a(t,s)\Delta s}(x,\alpha) \tag{3.22}$$

and $Q(x,y)$ is given by (3.11) provide that $\int_\alpha^M\int_\beta^N b(t,s)Q(t,s)\Delta s\Delta t < 1$.

Proof. Since $c(x,y)$ is positive and monotonic nondecreasing for each of its variables, (3.20) can be restated as

$$\frac{u(x,y)}{c(x,y)} \le 1 + f(x,y)\int_\alpha^x\int_\beta^y a(t,s)\frac{u(t,s)}{c(t,s)}\Delta s\Delta t + g(x,y)\int_\alpha^M\int_\beta^N b(t,s)\frac{u(t,s)}{c(t,s)}\Delta s\Delta t. \tag{3.23}$$

Let

$$w(x,y) = \frac{u(x,y)}{c(x,y)}. \tag{3.24}$$

Substituting (3.24) into (3.23), we obtain

$$w(x,y) \le 1 + f(x,y)\int_\alpha^x\int_\beta^y a(t,s)w(t,s)\Delta s\Delta t + g(x,y)\int_\alpha^M\int_\beta^N b(t,s)w(t,s)\Delta s\Delta t. \tag{3.25}$$

(3.25) is similar to (3.8), according to Theorem 3.2 with $c = 1$, we have

$$w(x,y) \le P_1(x,y) + \frac{Q(x,y)\int_\alpha^M\int_\beta^N b(t,s)P_1(t,s)\Delta s\Delta t}{1-\int_\alpha^M\int_\beta^N b(t,s)Q(t,s)\Delta s\Delta t}, \tag{3.26}$$

where $P_1(x,y)$ and $Q(x,y)$ are defined as in (3.22) and (3.11) respectively. Combining (3.26) and (3.24), we obtain the desired result. □

Theorem 3.4. *Assume that all the assumptions of Theorem 3.3 are valid. Let K,p,q,r be a positive constants, where $p\geq q$ and $p\geq r$. If*

$$u^p(x,y)\leq c^p(x,y)+f(x,y)\int_{\alpha}^{x}\int_{\beta}^{y}a(t,s)u^q(t,s)\Delta s\Delta t+g(x,y)\int_{\alpha}^{M}\int_{\beta}^{N}b(t,s)u^r(t,s)\Delta s\Delta t,\quad\forall(x,y)\in\Omega \tag{3.27}$$

holds, then $u(x,y)$ has the following estimate

$$u(x,y)\leq c(x,y)\left\{L(x,y)\left(P_2(x,y)+\frac{\frac{r}{p}K^{\frac{r-p}{p}}Q_2(x,y)\int_{\alpha}^{M}\int_{\beta}^{N}b(t,s)c^{r-p}(t,s)P_2(t,s)\Delta s\Delta t}{1-\frac{r}{p}K^{\frac{r-p}{p}}\int_{\alpha}^{M}\int_{\beta}^{N}b(t,s)c^{r-p}(t,s)Q_2(t,s)\Delta s\Delta t}\right)\right\}^{\frac{1}{p}}, \tag{3.28}$$

where

$$\begin{aligned}L(x,y) &= 1+\tfrac{p-q}{p}K^{\frac{q}{p}}f(x,y)\int_{\alpha}^{x}\int_{\beta}^{y}a(t,s)c^{q-p}(t,s)\Delta s\Delta t\\ &\quad +\tfrac{p-r}{p}K^{\frac{r}{p}}g(x,y)\int_{\alpha}^{M}\int_{\beta}^{N}b(t,s)c^{r-p}(t,s)\Delta s\Delta t,\end{aligned} \tag{3.29}$$

and

$$P_2(x,y) = e_{f(x,y)\int_{\beta}^{y}\frac{q}{p}K^{\frac{q-p}{p}}a(t,s)c^{q-p}(t,s)\Delta s}(x,\alpha) \tag{3.30}$$

$$Q_2(x,y) = g(x,y)e_{f(x,y)\int_{\beta}^{y}\frac{q}{p}K^{\frac{q-p}{p}}a(t,s)c^{q-p}(t,s)\Delta s}(x,\alpha) \tag{3.31}$$

provide that $\frac{r}{p}K^{\frac{r-p}{p}}\int_{\alpha}^{M}\int_{\beta}^{N}b(t,s)c^{r-p}(t,s)Q_2(t,s)\Delta s\Delta t<1.$

Proof. Since $c(x,y)$ is positive and monotonic nondecreasing for each of its variables, (3.27) can be restated as

$$\begin{aligned}w^p(x,y) &\leq 1+f(x,y)\int_{\alpha}^{x}\int_{\beta}^{y}a(t,s)c^{q-p}(t,s)w^q(t,s)\Delta s\Delta t\\ &\quad +g(x,y)\int_{\alpha}^{M}\int_{\beta}^{N}b(t,s)c^{r-p}(t,s)w^r(t,s)\Delta s\Delta t,\end{aligned} \tag{3.32}$$

where

$$w(x,y)=\frac{u(x,y)}{c(x,y)}. \tag{3.33}$$

Define a function $z(x,y)$ as follows

$$\begin{aligned}z(x,y) &= 1+f(x,y)\int_{\alpha}^{x}\int_{\beta}^{y}a(t,s)c^{q-p}(t,s)w^q(t,s)\Delta s\Delta t\\ &\quad +g(x,y)\int_{\alpha}^{M}\int_{\beta}^{N}b(t,s)c^{r-p}(t,s)w^r(t,s)\Delta s\Delta t.\end{aligned} \tag{3.34}$$

Clearly $z(x,y)$ is positive nondecreasing for each of its variables,

$$w(x,y) \leq z^{\frac{1}{p}}(x,y). \tag{3.35}$$

Substituting (3.35) into (3.34), we obtain

$$\begin{aligned} z(x,y) \quad &\leq \quad 1+f(x,y)\int_{\alpha}^{x}\int_{\beta}^{y} a(t,s)c^{q-p}(t,s)z^{\frac{q}{p}}(t,s)\Delta s\Delta t \\ &\quad +g(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)c^{r-p}(t,s)z^{\frac{r}{p}}(t,s)\Delta s\Delta t. \end{aligned} \tag{3.36}$$

From Lemma 2.3, we have

$$\begin{cases} z^{\frac{q}{p}}(t,s) \leq \frac{q}{p}K^{\frac{q-p}{p}}z(t,s)+\frac{p-q}{p}K^{\frac{q}{p}} \\ z^{\frac{r}{p}}(t,s) \leq \frac{r}{p}K^{\frac{r-p}{p}}z(t,s)+\frac{p-r}{p}K^{\frac{r}{p}}. \end{cases} \tag{3.37}$$

Now, using (3.37) in (3.36), we get

$$\begin{aligned} z(x,y) \quad &\leq \quad L(x,y)+f(x,y)\int_{\alpha}^{x}\int_{\beta}^{y} \frac{q}{p}K^{\frac{q-p}{p}}a(t,s)c^{q-p}(t,s)z(t,s)\Delta s\Delta t \\ &\quad +g(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} \frac{r}{p}K^{\frac{r-p}{p}}b(t,s)c^{r-p}(t,s)z(t,s)\Delta s\Delta t, \end{aligned} \tag{3.38}$$

where $L(x,y)$ is defined by (3.29). Inequality (3.38) is similar to inequality (3.20). According to Theorem 3.3, we have

$$z(x,y) \leq L(x,y)\left(P_2(x,y)+\frac{\frac{r}{p}K^{\frac{r-p}{p}}Q_2(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)c^{r-p}(t,s)P_2(t,s)\Delta s\Delta t}{1-\frac{r}{p}K^{\frac{r-p}{p}}\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)c^{r-p}(t,s)Q_2(t,s)\Delta s\Delta t}\right), \tag{3.39}$$

where $P_2(x,y)$ and $Q_2(x,y)$ are given by (3.30) and (3.31) respectively. Combining (3.39), (3.35) and (3.33), we obtain the desired result. $\square$

Theorem 3.5. *Let $u(x,y), f(x,y), g(x,y) \in C_{rd}(\Omega,\mathbb{R}^+)$ and $a(x,y,t,s)$, $b(x,y,t,s) \in C_{rd}(\Omega\times\Omega,\mathbb{R}^+)$, and $c>0$. If*

$$u(x,y) \leq c+f(x,y)\int_{\alpha}^{x}\int_{\beta}^{y} a(x,y,t,s)u(t,s)\Delta s\Delta t+g(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(x,y,t,s)u(t,s)\Delta s\Delta t, \quad \forall (x,y)\in\Omega \tag{3.40}$$

holds, then $u(x,y)$ has the following estimate

$$u(x,y) \leq \frac{c\ P_3(x,y)}{1-g(x,y)P_3(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(x,y,t,s)\Delta s\Delta t}, \tag{3.41}$$

where

$$P_3(x,y) = e_{f(x,y)\int_{\beta}^{y} a(x,y,t,s)\Delta s}(x,\alpha) \tag{3.42}$$

provide that $g(x,y)P_3(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(x,y,t,s)\Delta s\Delta t < 1$.

Proof. Fixing any numbers $x_1 \in [\alpha, M]_{\mathbb{T}_1}$ and $y_1 \in [\beta, N]_{\mathbb{T}_2}$ with $\alpha \leq x \leq x_1$ and $\beta \leq y \leq y_1$, we have that (3.40) becomes

$$\begin{aligned} u(x,y) &\leq c + f(x_1,y_1)\int_{\alpha}^{x}\int_{\beta}^{y} a(x_1,y_1,t,s)u(t,s)\Delta s\Delta t \\ &\quad + g(x_1,y_1)\int_{\alpha}^{M}\int_{\beta}^{N} b(x_1,y_1,t,s)u(t,s)\Delta s\Delta t. \end{aligned} \tag{3.43}$$

Define a function $z(x,y)$ by

$$\begin{aligned} z(x,y) &= c + f(x_1,y_1)\int_{\alpha}^{x}\int_{\beta}^{y} a(x_1,y_1,t,s)u(t,s)\Delta s\Delta t \\ &\quad + g(x_1,y_1)\int_{\alpha}^{M}\int_{\beta}^{N} b(x_1,y_1,t,s)u(t,s)\Delta s\Delta t. \end{aligned} \tag{3.44}$$

Clearly $z(x,y)$ is positive, nondecreasing,

$$u(x,y) \leq z(x,y) \tag{3.45}$$

and

$$\begin{aligned} z(\alpha,y) &= z(x,\beta) = z(\alpha,\beta) \\ &= c + g(x_1,y_1)\int_{\alpha}^{M}\int_{\beta}^{N} b(x_1,y_1,t,s)u(t,s)\Delta s\Delta t. \end{aligned} \tag{3.46}$$

Differentiating (3.44) with respect to x and using (3.45), we obtain

$$z^{\Delta x}(x,y) \leq f(x_1,y_1)\int_{\beta}^{y} a(x_1,y_1,x,s)z(x,s)\Delta s. \tag{3.47}$$

Using the fact that $z(x,y)$ is nondecreasing with respect to y, (3.47) gives

$$z^{\Delta x}(x,y) \leq z(x,y)f(x_1,y_1)\int_{\beta}^{y} a(x_1,y_1,x,s)\Delta s. \tag{3.48}$$

Applying Lemma 2.2 to (3.48), and using (3.46) and (3.45), we get

$$\begin{aligned} z(x,y) &\leq c e_{f(x_1,y_1)\int_{\beta}^{y} a(x_1,y_1,t,s)\Delta s}(x,\alpha) \\ &\quad + g(x_1,y_1) e_{f(x_1,y_1)\int_{\beta}^{y} a(x_1,y_1,t,s)\Delta s}(x,\alpha)\int_{\alpha}^{M}\int_{\beta}^{N} b(x_1,y_1,t,s)z(t,s)\Delta s\Delta t. \end{aligned} \tag{3.49}$$

Since x_1 and y_1 are arbitrarily chosen and $\alpha \le x \le x_1$ and $\beta \le y \le y_1$, we have from (3.49) and (3.46) with $x_1 = x$ and $y_1 = y$ that

$$\begin{aligned} z(x,y) &\le cP_3(x,y) \\ &\quad + g(x,y)P_3(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(x_1,y_1,t,s)z(t,s)\Delta s\Delta t, \end{aligned} \tag{3.50}$$

where $P_3(x,y)$ is defined as in (3.42). Since $z(x,y)$ is monotonic nondecreasing in each of its variables, we get

$$z(x,y) \le cP_3(x,y) + z(x,y)g(x,y)P_3(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(x,y,t,s)\Delta s\Delta t. \tag{3.51}$$

So, from (3.51) and the fact that $g(x,y)P_3(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(x,y,t,s)\Delta s\Delta t < 1$, we have

$$z(x,y) \le \frac{c\ P_3(x,y)}{1 - g(x,y)P_3(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(x,y,t,s)\Delta s\Delta t}. \tag{3.52}$$

Combining (3.52) and (3.45), we obtain the desired result. □

Theorem 3.6. *Assume that all the hypothesis of Theorem 3.5 are valid. And let $\varphi(x,y) \in C_{rd}\left(\Omega, \mathbb{R}_0^+\right)$ such that $f(x,y), g(x,y)$ and $\varphi(x,y)$ are nondecreasing functions in each of its variables, and let K, p, q, r be positives constants, where $p \ge q$ and $p \ge r$. If*

$$\begin{aligned} u^p(x,y) &\le \varphi(x,y) + f(x,y)\int_{\alpha}^{x}\int_{\beta}^{y} a(x,y,t,s)u^q(t,s)\Delta s\Delta t \\ &\quad + g(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(x,y,t,s)u^r(t,s)\Delta s\Delta t, \quad \forall (x,y) \in \Omega \end{aligned} \tag{3.53}$$

holds, then $u(x,y)$ has the following estimate

$$u(x,y) \le \left(\frac{P_4(x,y)L_1(x,y)}{1 - \frac{r}{p}K^{\frac{r-p}{p}} g(x,y)P_4(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(x,y,t,s)\Delta s\Delta t} \right)^{\frac{1}{p}}, \tag{3.54}$$

where

$$\begin{aligned} L_1(x,y) &= \varphi(x,y) + \frac{p-q}{p}K^{\frac{q}{p}} f(x,y)\int_{\alpha}^{x}\int_{\beta}^{y} a(x,y,t,s)\Delta s\Delta t \\ &\quad + \frac{p-r}{p}K^{\frac{r}{p}} g(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(x,y,t,s)\Delta s\Delta t \end{aligned} \tag{3.55}$$

and

$$P_4(x,y) = e_{f(x,y)\int_{\beta}^{y}\frac{q}{p}K^{\frac{q-p}{p}} a(x,y,t,s)\Delta s}(x,\alpha) \tag{3.56}$$

provide that $\frac{r}{p}K^{\frac{r-p}{p}}g(x,y)P_4(x,y)\int_{\alpha}^{M}\int_{\beta}^{N}b(x,y,t,s)\Delta s\Delta t<1$.

Proof. Define a function $w(x,y)$ as follows

$$\begin{aligned} w(x,y) &= \varphi(x,y)+f(x,y)\int_{\alpha}^{x}\int_{\beta}^{y}a(x,y,t,s)u^{q}(t,s)\Delta s\Delta t \\ &+g(x,y)\int_{\alpha}^{M}\int_{\beta}^{N}b(x,y,t,s)u^{r}(t,s)\Delta s\Delta t. \end{aligned} \tag{3.57}$$

Clearly $w(x,y)$ is positive, nondecreasing for each of its variables, and

$$u(x,y)\leq w^{\frac{1}{p}}(x,y). \tag{3.58}$$

Substituting (3.58) into (3.57), and Applying Lemma 2.3 to the resulting inequality, we obtain

$$\begin{aligned} w(x,y) &\leq L_1(x,y)+f(x,y)\int_{\alpha}^{x}\int_{\beta}^{y}\frac{q}{p}K^{\frac{q-p}{p}}a(x,y,t,s)w(t,s)\Delta s\Delta t \\ &+g(x,y)\int_{\alpha}^{M}\int_{\beta}^{N}\frac{r}{p}K^{\frac{r-p}{p}}b(x,y,t,s)w(t,s)\Delta s\Delta t, \end{aligned} \tag{3.59}$$

where $L_1(x,y)$ is defined as in (3.55). Clearly $L_1(x,y)$ is positive and nondecreasing function for each of its variables. Dividing both sides of (3.59) by $L_1(x,y)$ we get

$$\begin{aligned} \frac{w(x,y)}{L_1(x,y)} &\leq 1+f(x,y)\int_{\alpha}^{x}\int_{\beta}^{y}\frac{q}{p}K^{\frac{q-p}{p}}a(x,y,t,s)\frac{w(t,s)}{L_1(t,s)}\Delta s\Delta t \\ &+g(x,y)\int_{\alpha}^{M}\int_{\beta}^{N}\frac{r}{p}K^{\frac{r-p}{p}}b(x,y,t,s)\frac{w(t,s)}{L_1(t,s)}\Delta s\Delta t. \end{aligned} \tag{3.60}$$

Let

$$z(x,y)=\frac{w(x,y)}{L_1(x,y)}. \tag{3.61}$$

Substituting (3.61) into (3.60), we obtain

$$\begin{aligned} z(x,y) &\leq 1+f(x,y)\int_{\alpha}^{x}\int_{\beta}^{y}\frac{q}{p}K^{\frac{q-p}{p}}a(x,y,t,s)z(t,s)\Delta s\Delta t \\ &+g(x,y)\int_{\alpha}^{M}\int_{\beta}^{N}\frac{r}{p}K^{\frac{r-p}{p}}b(x,y,t,s)z(t,s)\Delta s\Delta t. \end{aligned} \tag{3.62}$$

The above inequality is similar to (3.40). According to Theorem 3.5 with $c=1$, we have

$$z(x,y)\leq \frac{P_4(x,y)}{1-\frac{r}{p}K^{\frac{r-p}{p}}g(x,y)P_4(x,y)\int_{\alpha}^{M}\int_{\beta}^{N}b(x,y,t,s)\Delta s\Delta t}, \tag{3.63}$$

where $P_4(x,y)$ is defined as in (3.56). Combining (3.63), (3.61) and (3.58) we obtain the desired result.

□

4. Further results

Theorem 4.1. *Let* $u(x,y), b(x,y), f(x,y), g(x,y) \in C_{rd}(\Omega, \mathbb{R}^+)$. *Let* $f(x,y)$ *and* $g(x,y)$ *be nondecreasing for each of its variables and let* $\varphi(x,y,\tau), \eta(x,y,\tau) \in C_{rd}(\Omega \times \mathbb{R}^+, \mathbb{R}^+)$ *and continuous on* $\mathbb{R}^+$ *such that for all* $\tau_1 \geq \tau_2$

$$0 \leq \varphi(x,y,\tau_1) - \varphi(x,y,\tau_2) \leq (\tau_1 - \tau_2)\,\eta(x,y,\tau_2) \tag{4.1}$$

and c *be a positive real constant. If*

$$u(x,y) \leq c + f(x,y)\int_{\alpha}^{x}\int_{\beta}^{y} \varphi(t,s,u(t,s))\Delta s\Delta t + g(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)u(t,s)\Delta s\Delta t, \quad \forall (x,y) \in \Omega \tag{4.2}$$

holds, then $u(x,y)$ *has the following estimate*

$$u(x,y) \leq c + \Phi(x,y) + \frac{\Psi(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)\Phi(t,s)\Delta s\Delta t}{1 - \int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)\Psi(t,s)\Delta s\Delta t}, \tag{4.3}$$

where

$$\Phi(x,y) = \int_{\alpha}^{x} f(x,y)\left(\int_{\beta}^{y} \varphi(t,s,c)\, e_{f(x,y)\int_{\beta}^{y}\eta(t,s,c)\Delta s}(x,\sigma(t))\right)\Delta t \tag{4.4}$$

and

$$\Psi(x,y) = g(x,y)\, e_{f(x,y)\int_{\beta}^{y}\eta(t,s,c)\Delta s}(x,\alpha) \tag{4.5}$$

provide $\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)\Psi(t,s)\Delta s\Delta t$.

Proof. Fix arbitrary numbers $(x_1,y_1) \in \Omega$. For $\alpha \leq x \leq x_1$ and $\beta \leq y \leq y_1$, and (4.2) we have

$$u(x,y) \leq c + f(x_1,y_1)\int_{\alpha}^{x}\int_{\beta}^{y} \varphi(t,s,u(t,s))\Delta s\Delta t + g(x_1,y_1)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)u(t,s)\Delta s\Delta t. \tag{4.6}$$

Define a function $z(x,y)$ by

$$z(x,y) = f(x_1,y_1)\int_{\alpha}^{x}\int_{\beta}^{y} \varphi(t,s,u(t,s))\Delta s\Delta t + g(x_1,y_1)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)u(t,s)\Delta s\Delta t. \tag{4.7}$$

Clearly, $z(x,y)$ is positive, nondecreasing for each of its variables,

$$z(\alpha,y) = z(x,\beta) = z(\alpha,\beta) = g(x_1,y_1)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)u(t,s)\Delta s\Delta t, \tag{4.8}$$

and

$$u(x,y) \leq c + z(x,y). \tag{4.9}$$

Differentiating (4.7) with respect to x, and using (4.9), we obtain

$$z^{\Delta x}(x,y) = f(x_1,y_1)\int_{\beta}^{y} \varphi(x,s,u(x,s))\Delta s \leq f(x_1,y_1)\int_{\beta}^{y} \varphi(x,s,c+z(x,s))\Delta s. \tag{4.10}$$

From (4.1), we can write

$$z^{\Delta x}(x,y) \leq f(x_1,y_1)\int_{\beta}^{y} z(x,s)\eta(x,s,c)\Delta s + f(x_1,y_1)\int_{\beta}^{y}\varphi(x,s,c)\Delta s. \tag{4.11}$$

Using the fact that $z(x,y)$ is nondecreasing with respect to y, we get

$$z^{\Delta x}(x,y) \leq \left(f(x_1,y_1)\int_{\beta}^{y}\eta(x,s,c)\Delta s\right) z(x,y) + f(x_1,y_1)\int_{\beta}^{y}\varphi(x,s,c)\Delta s. \tag{4.12}$$

Now, applying Lemma 2.2 to (4.12), and using (4.8), we obtain

$$\begin{aligned} z(x,y) \quad &\leq \quad \left(g(x_1,y_1)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)z(t,s)\Delta s\Delta t\right) e_{f(x_1,y_1)\int_{\beta}^{y}\eta(t,s,c)\Delta s}(x,\alpha) \\ &\quad + \int_{\alpha}^{x} f(x_1,y_1)\left(\int_{\beta}^{y}\varphi(t,s,c)e_{f(x_1,y_1)\int_{\beta}^{y}\eta(t,s,c)\Delta s}(x,\sigma(t))\right)\Delta t. \end{aligned} \tag{4.13}$$

Since x_1 and y_1 are arbitrarily chosen and $\alpha \leq t \leq x \leq x_1$ and $\beta \leq s \leq y \leq y_1$, we have from (4.13) and (4.8) with $x_1 = x$ and $y_1 = y$ that

$$z(x,y) \leq \Phi(x,y) + \Psi(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)z(t,s)\Delta s\Delta t, \tag{4.14}$$

where $\Phi(x,y)$ and $\Psi(x,y)$ are defined as in (4.4) and (4.5) respectively. Using Lemma 3.1 to (4.14), we obtain

$$z(x,y) \leq \Phi(x,y) + \frac{\Psi(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)\Phi(t,s)\Delta s\Delta t}{1-\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)\Psi(t,s)\Delta s\Delta t}. \tag{4.15}$$

Combining (4.15) and (4.9), we get the desired result. □

Theorem 4.2. *Under the hypotheses of Theorem 4.1, inequality (4.2) has the following estimate*

$$u(x,y) \leq c + \frac{\Phi(x,y)}{1-\Psi(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)\Delta s\Delta t}, \tag{4.16}$$

where $\Phi(x,y)$ *and* $\Psi(x,y)$ *are defined as in (4.4) and (4.5) respectively provide that* $\Psi(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)\Delta s\Delta t <$ 1.

Proof. By a similar process, we can easily lead to inequality (4.14). So, by using the fact that $z(x,y)$ is nondecreasing for each of its variables, (4.14) gives

$$z(x,y) \leq \Phi(x,y) + \Psi(x,y)\left(\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)\Delta s\Delta t\right) z(x,y). \tag{4.17}$$

Since $1-\Psi(x,y)\int\limits_{\alpha}^{M}\int\limits_{\beta}^{N} b(t,s)\Delta s\Delta t > 0$, we have from (4.17)

$$z(x,y) \le \frac{\Phi(x,y)}{1-\Psi(x,y)\int\limits_{\alpha}^{M}\int\limits_{\beta}^{N} b(t,s)\Delta s\Delta t}. \tag{4.18}$$

Combining (4.18) and (4.9), we obtain the desired result. □

Theorem 4.3. *Let* $u(x,y), b(x,y), f(x,y), g(x,y) \in C_{rd}(\Omega,\mathbb{R}^{+})$ *and* $c(x,y) \in C_{rd}\left(\Omega,\mathbb{R}_0^{+}\right)$. *Let* $c(x,y), f(x,y)$ *and* $g(x,y)$ *be nondecreasing for each of its variables, and let* $\varphi(x,y,\tau)$ *and* $\eta(x,y,\tau)$ *be defined as in Theorem 4.1. If*

$$u(x,y) \le c(x,y) + f(x,y)\int\limits_{\alpha}^{x}\int\limits_{\beta}^{y} \varphi(t,s,u(t,s))\Delta s\Delta t + g(x,y)\int\limits_{\alpha}^{M}\int\limits_{\beta}^{N} b(t,s)u(t,s)\Delta s\Delta t, \quad \forall (x,y) \in \Omega \tag{4.19}$$

holds, then $u(x,y)$ *has the following estimate*

$$u(x,y) \le c(x,y)\left(1+\Phi(x,y)+\frac{\Psi(x,y)\int\limits_{\alpha}^{M}\int\limits_{\beta}^{N} b(t,s)\Phi(t,s)\Delta s\Delta t}{1-\int\limits_{\alpha}^{M}\int\limits_{\beta}^{N} b(t,s)\Psi(t,s)\Delta s\Delta t}\right), \tag{4.20}$$

where $\Phi(x,y)$ *and* $\Psi(x,y)$ *are defined as in* (4.4) *and* (4.5) *respectively provide that* $\int\limits_{\alpha}^{M}\int\limits_{\beta}^{N} b(t,s)\Psi(t,s)\Delta s\Delta t <$ 1.

Proof. Taking into account that $c(x,y)$ is positive and monotonic nondecreasing for each of its variables and (4.1), (4.19) can be restated as follows

$$\frac{u(x,y)}{c(x,y)} \le 1 + f(x,y)\int\limits_{\alpha}^{x}\int\limits_{\beta}^{y} \varphi(t,s,\frac{u(t,s)}{c(t,s)})\Delta s\Delta t + g(x,y)\int\limits_{\alpha}^{M}\int\limits_{\beta}^{N} b(t,s)\frac{u(t,s)}{c(t,s)}\Delta s\Delta t. \tag{4.21}$$

Let

$$w(x,y) = \frac{u(x,y)}{c(x,y)}. \tag{4.22}$$

Using (4.22) in (4.21), we get

$$w(x,y) \le 1 + f(x,y)\int\limits_{\alpha}^{x}\int\limits_{\beta}^{y} \varphi(t,s,w(t,s))\Delta s\Delta t + g(x,y)\int\limits_{\alpha}^{M}\int\limits_{\beta}^{N} b(t,s)w(t,s)\Delta s\Delta t. \tag{4.23}$$

Now, applying Theorem 4.1 with $c=1$ to (4.23), we get

$$w(x,y) \le 1+\Phi(x,y)+\frac{\Psi(x,y)\int\limits_{\alpha}^{M}\int\limits_{\beta}^{N} b(t,s)\Phi(t,s)\Delta s\Delta t}{1-\int\limits_{\alpha}^{M}\int\limits_{\beta}^{N} b(t,s)\Psi(t,s)\Delta s\Delta t}, \tag{4.24}$$

where $\Phi(x,y)$ and $\Psi(x,y)$ are defined as in (4.4) and (4.5) respectively. Combining (4.24) and (4.22), we get the desired result. □

Theorem 4.4. *Under the hypotheses of Theorem 4.3, inequality (4.19) has the following estimate*

$$u(x,y) \leq c(x,y) + \frac{c(x,y)\Phi(x,y)}{1-\Psi(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)\Delta s\Delta t}, \tag{4.25}$$

where $\Phi(x,y)$ *and* $\Psi(x,y)$ *are defined as in (4.4) and (4.5) respectively provided*

$$\Psi(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)\Delta s\Delta t < 1.$$

Proof. By a similar process, we can easily be led to inequality (4.23). Applying Theorem 4.2 with $c = 1$ to (4.23), and combining the resulting inequality with (4.22), we get the desired result. □

Theorem 4.5. *Let* $u(x,y), b(x,y), f(x,y), g(x,y) \in C_{rd}(\Omega, \mathbb{R}^+)$. *Let* $f(x,y)$ *and* $g(x,y)$ *are nondecreasing for each of its variables, and let* $\varphi(x,y,\tau), \eta(x,y,\tau) \in C_{rd}(\Omega \times \mathbb{R}^+, \mathbb{R}^+)$ *and continuous satisfying (4.1), and* c, p, K *be a positive real constants such that* $p \geq 1$. *If*

$$u^p(x,y) \leq c + f(x,y)\int_{\alpha}^{x}\int_{\beta}^{y} \varphi(t,s,u(t,s))\Delta s\Delta t + g(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)u(t,s)\Delta s\Delta t, \quad \forall (x,y) \in \Omega \tag{4.26}$$

holds, then $u(x,y)$ *has the following estimate*

$$u(x,y) \leq \left(P(x,y) + \frac{Q(x,y)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)P(t,s)\Delta s\Delta t}{1-\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)Q(t,s)\Delta s\Delta t} \right)^{\frac{1}{p}}, \tag{4.27}$$

where

$$\begin{aligned} P(x,y) = {} & \tfrac{p-1}{p}K^{\frac{1}{p}} g(x,y) e_{f(x,y)\left(\int_{\beta}^{y}\frac{1}{p}K^{\frac{1-p}{p}}\eta\left(t,s,\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s\right)}(x,\alpha)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)\Delta s\Delta t \\ & + c e_{f(x,y)\left(\int_{\beta}^{y}\frac{1}{p}K^{\frac{1-p}{p}}\eta\left(t,s,\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s\right)}(x,\alpha) + \int_{t_0}^{t}\left(f(x,y)\int_{\beta}^{y}\varphi\left(t,s,\tfrac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s \right. \\ & \left. \times\, e_{f(x,y)\left(\int_{\beta}^{y}\frac{1}{p}K^{\frac{1-p}{p}}\eta\left(t,s,\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s\right)}(x,\sigma(t)) \right)\Delta t. \end{aligned} \tag{4.28}$$

and

$$Q(x,y) = \tfrac{1}{p}K^{\frac{1-p}{p}} g(x,y) e_{f(x,y)\left(\int_{\beta}^{y}\frac{1}{p}K^{\frac{1-p}{p}}\eta\left(t,s,\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s\right)}(x,\alpha) \tag{4.29}$$

provide that $\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)Q(t,s)\Delta s\Delta t < 1$.

Proof. Fix arbitrary numbers $(x_1,y_1) \in \Omega$. For $\alpha \leq x \leq x_1$ and $\beta \leq y \leq y_1$, and (4.26) gives

$$u^p(x,y) \leq c + f(x_1,y_1)\int_{\alpha}^{x}\int_{\beta}^{y} \varphi(t,s,u(t,s))\Delta s\Delta t + g(x_1,y_1)\int_{\alpha}^{M}\int_{\beta}^{N} b(t,s)u(t,s)\Delta s\Delta t. \tag{4.30}$$

Define a function $z(x,y)$ by

$$z(x,y)=c+f(x_1,y_1)\int_{\alpha}^{x}\int_{\beta}^{y}\varphi(t,s,u(t,s))\Delta s\Delta t+g(x_1,y_1)\int_{\alpha}^{M}\int_{\beta}^{N}b(t,s)u(t,s)\Delta s\Delta t. \tag{4.31}$$

Clearly, $z(x,y)$ is positive, nondecreasing for each of its variables,

$$z(\alpha,y)=z(x,\beta)=z(\alpha,\beta)=c+g(x_1,y_1)\int_{\alpha}^{M}\int_{\beta}^{N}b(t,s)u(t,s)\Delta s\Delta t, \tag{4.32}$$

and

$$u(x,y)\le z^{\frac{1}{p}}(x,y). \tag{4.33}$$

Differentiating (4.31) with respect to x, and using (4.33), we obtain

$$z^{\Delta x}(x,y)\le f(x_1,y_1)\int_{\beta}^{y}\varphi(x,s,z^{\frac{1}{p}}(x,s))\Delta s. \tag{4.34}$$

From (4.1), Lemma 2.3 and the fact that $z(x,y)$ is nondecreasing for each of its variables, (4.34) gives

$$\begin{aligned}
z^{\Delta x}(x,y) &\le f(x_1,y_1)\int_{\beta}^{y}\varphi\left(x,s,\tfrac{1}{p}K^{\frac{1-p}{p}}z(x,s)+\tfrac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s\\
&\le f(x_1,y_1)\left(\int_{\beta}^{y}\tfrac{1}{p}K^{\frac{1-p}{p}}\eta\left(x,s,\tfrac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s\right)z(x,y)\\
&\quad +f(x_1,y_1)\int_{\beta}^{y}\varphi\left(x,s,\tfrac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s.
\end{aligned} \tag{4.35}$$

The application of Lemma 2.2 to (4.35) gives

$$\begin{aligned}
z(x,y) &\le z(\alpha,y)e_{f(x_1,y_1)\left(\int_{\beta}^{y}\frac{1}{p}K^{\frac{1-p}{p}}\eta\left(t,s,\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s\right)}(x,\alpha)\\
&\quad+\int_{t_0}^{t}\left(f(x_1,y_1)\int_{\beta}^{y}\varphi\left(t,s,\tfrac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s\right.\\
&\quad\left.\times\, e_{f(x_1,y_1)\left(\int_{\beta}^{y}\frac{1}{p}K^{\frac{1-p}{p}}\eta\left(t,s,\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s\right)}(x,\sigma(t))\right)\Delta t.
\end{aligned} \tag{4.36}$$

Substituting (4.32) into (4.36) and using (4.33), we get

$$\begin{aligned} z(x,y) \leq\ & g(x_1,y_1)e_{f(x_1,y_1)\left(\int_\beta^y \frac{1}{p}K^{\frac{1-p}{p}}\eta\left(t,s,\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s\right)}(x,\alpha)\int_\alpha^M\int_\beta^N b(t,s)z^{\frac{1}{p}}(t,s)\Delta s\Delta t \\ & ce_{f(x_1,y_1)\left(\int_\beta^y \frac{1}{p}K^{\frac{1-p}{p}}\eta\left(t,s,\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s\right)}(x,\alpha)+ \\ & +\int_{t_0}^t \Bigg(f(x_1,y_1)\int_\beta^y \varphi\left(t,s,\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s \\ & \times e_{f(x_1,y_1)\left(\int_\beta^y \frac{1}{p}K^{\frac{1-p}{p}}\eta\left(t,s,\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s\right)}(x,\sigma(t))\Bigg)\Delta t. \end{aligned} \tag{4.37}$$

Apply Lemma 2.3 to (4.37), we obtain

$$\begin{aligned} z(x,y) \leq\ & \frac{1}{p}K^{\frac{1-p}{p}}g(x_1,y_1)e_{f(x_1,y_1)\left(\int_\beta^y \frac{1}{p}K^{\frac{1-p}{p}}\eta\left(t,s,\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s\right)}(x,\alpha)\int_\alpha^M\int_\beta^N b(t,s)z(t,s)\Delta s\Delta t \\ & +\frac{p-1}{p}K^{\frac{1}{p}}g(x_1,y_1)e_{f(x_1,y_1)\left(\int_\beta^y \frac{1}{p}K^{\frac{1-p}{p}}\eta\left(t,s,\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s\right)}(x,\alpha)\int_\alpha^M\int_\beta^N b(t,s)\Delta s\Delta t \\ & +ce_{f(x_1,y_1)\left(\int_\beta^y \frac{1}{p}K^{\frac{1-p}{p}}\eta\left(t,s,\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s\right)}(x,\alpha) \\ & +\int_{t_0}^t \Bigg(f(x_1,y_1)\int_\beta^y \varphi\left(t,s,\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s \\ & \times e_{f(x_1,y_1)\left(\int_\beta^y \frac{1}{p}K^{\frac{1-p}{p}}\eta\left(t,s,\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta s\right)}(x,\sigma(t))\Bigg)\Delta t. \end{aligned} \tag{4.38}$$

Since x_1 and y_1 are arbitrarily chosen and $\alpha \leq t \leq x \leq x_1$ and $\beta \leq s \leq y \leq y_1$, we have from (4.38) and (4.32) with $x_1 = x$ and $y_1 = y$ that

$$z(x,y) \leq P(x,y) + Q(x,y)\int_\alpha^M\int_\beta^N b(t,s)z(t,s)\Delta s\Delta t, \tag{4.39}$$

where $P(x,y)$ and $Q(x,y)$ are defined as in (4.28) and (4.29) respectively. Applying Lemma 3.1 to (4.39), we obtain

$$z(x,y) \leq P(x,y) + \frac{Q(x,y)\int_\alpha^M\int_\beta^N b(t,s)P(t,s)\Delta s\Delta t}{1-\int_\alpha^M\int_\beta^N b(t,s)Q(t,s)\Delta s\Delta t}. \tag{4.40}$$

Combining (4.40) and (4.33), we get the desired result. □

5. Applications

In this section, we present an applications of the results. Consider the following partial dynamic equation on time scales

$$u^p(x,y)=\Phi\left(x,y,\int_\alpha^x\int_\beta^y F(x,y,t,s,u(t,s))\Delta s\Delta t,\int_\alpha^M\int_\beta^N G(x,y,t,s,u(t,s))\Delta s\Delta t\right), \tag{5.1}$$

where p is a positive constant, $\Phi:\Omega\times\mathbb{R}^2\to\mathbb{R}$ is right-dense continuous function on Ω and continuous on $\mathbb{R}^2$. $F,G:\Omega^2\times\mathbb{R}\to\mathbb{R}$ are right-dense continuous functions on Ω^2 and continuous on $\mathbb{R}$.

Proposition 5.1. *Assume that*

$$\begin{aligned}&\Phi\left(x,y,\int_\alpha^x\int_\beta^y F(x,y,t,s,u(t,s))\Delta s\Delta t,\int_\alpha^M\int_\beta^N G(x,y,t,s,u(t,s))\Delta s\Delta t\right)\\ \leq\ &\chi(x,y)+f(x,y)\int_\alpha^x\int_\beta^y F(x,y,t,s,u(t,s))\Delta s\Delta t\\ &+g(x,y)\int_\alpha^M\int_\beta^N G(x,y,t,s,u(t,s))\Delta s\Delta t,\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}|\chi(x,y)|&\leq c^p(x,y)\\ |F(x,y,t,s,u(t,s))|&\leq a(t,s)\,|u(t,s)|^q\\ |G(x,y,t,s,u(t,s))|&\leq b(t,s)\,|u(t,s)|^r,\end{aligned} \tag{5.3}$$

with $p\geq q>0$ and $p\geq r>0$, and $f(x,y),g(x,y),a(x,y),b(x,y)$ and $c(x,y)$ satisfy the hypotheses of Theorem 3.3. If $u(x,y)$ is any solution of (5.1)-(5.3), then $u(x,y)$ satisfies the following estimate

$$|u(x,y)|\leq c(x,y)\left\{L(x,y)\left(P_2(x,y)+\frac{\frac{r}{p}K^{\frac{r-p}{p}}Q_2(x,y)\int_\alpha^M\int_\beta^N b(t,s)c^{r-p}(t,s)P_2(t,s)\Delta s\Delta t}{1-\frac{r}{p}K^{\frac{r-p}{p}}\int_\alpha^M\int_\beta^N b(t,s)c^{r-p}(t,s)Q_2(t,s)\Delta s\Delta t}\right)\right\}^{\frac{1}{p}}, \tag{5.4}$$

where $L(x,y)$, $P_2(x,y)$ and $Q_2(x,y)$ are defined as in (3.29)-(3.31) respectively, and K is a positive real number provide that $\frac{r}{p}K^{\frac{r-p}{p}}\int_\alpha^M\int_\beta^N b(t,s)c^{r-p}(t,s)Q_2(t,s)\Delta s\Delta t<1$.

Proof. Let $u(x,y)$ be a solution of (5.1). Using the modulus, we obtain

$$|u(x,y)|^p=\left|\Phi\left(x,y,\int_\alpha^x\int_\beta^y F(x,y,t,s,u(t,s)))\Delta s\Delta t,\int_\alpha^M\int_\beta^N G(x,y,t,s,u(t,s))\Delta s\Delta t\right)\right|. \tag{5.5}$$

Substituting (5.2) and (5.3) into (5.5), we get

$$\begin{aligned} |u(x,y)|^p &\leq c^p(x,y)+f(x,y)\int_\alpha^x\int_\beta^y a(t,s)\,|u(t,s)|^q\,\Delta s\Delta t \\ &+g(x,y)\int_\alpha^M\int_\beta^N b(t,s)\,|u(t,s)|^r\,\Delta s\Delta t. \end{aligned} \tag{5.6}$$

Now, an application of Theorem 3.4 for (5.6) gives the estimate (5.4). □

Proposition 5.2. *Assume that*

$$\begin{aligned} |\chi(x,y)| &\leq \varphi(x,y) \\ |F(x,y,t,s,u(t,s))| &\leq a(x,y,t,s)\,|u(t,s)|^q \\ |G(x,y,t,s,u(t,s))| &\leq b(x,y,t,s)\,|u(t,s)|^r, \end{aligned} \tag{5.7}$$

with $p \geq q > 0$ and $p \geq r > 0$, where $f(x,y), g(x,y), a(x,y,t,s), b(x,y,t,s)$ and $\varphi(x,y)$ satisfy the hypotheses of Theorem 3.6. If $u(x,y)$ is any solution of (5.1),(5.2) and (5.7), then $u(x,y)$ has the following estimate

$$|u(x,y)| \leq \left(\frac{P_4(x,y)L_1(x,y)}{1-\frac{r}{p}K^{\frac{r-p}{p}}g(x,y)P_4(x,y)\int_\alpha^M\int_\beta^N b(x,y,t,s)\Delta s\Delta t}\right)^{\frac{1}{p}}, \tag{5.8}$$

where $L_1(x,y)$ and $P_4(x,y)$ are defined as in (3.55) and (3.56) respectively, and K is a positive real number provide that $\frac{r}{p}K^{\frac{r-p}{p}}g(x,y)P_4(x,y)\int_\alpha^M\int_\beta^N b(x,y,t,s)\Delta s\Delta t < 1$.

Proof. Let $u(x,y)$ be a solution of (5.1). Using the modulus, we obtain

$$|u(x,y)|^p = \left|\Phi\left(x,y,\int_\alpha^x\int_\beta^y F(x,y,t,s,u(t,s)))\,\Delta s\Delta t,\int_\alpha^M\int_\beta^N G(x,y,t,s,u(t,s))\,\Delta s\Delta t\right)\right|. \tag{5.9}$$

Substituting (5.2) and (5.7) into (5.9), we get

$$\begin{aligned} |u(x,y)|^p &\leq \varphi(x,y)+f(x,y)\int_\alpha^x\int_\beta^y a(x,y,t,s)\,|u(t,s)|^q\,\Delta s\Delta t \\ &+g(x,y)\int_\alpha^M\int_\beta^N b(x,y,t,s)\,|u(t,s)|^r\,\Delta s\Delta t. \end{aligned} \tag{5.10}$$

Now, an application of Theorem 3.6 to (5.10) gives the estimate (5.8). □

Proposition 5.3. *Assume that*

$$\begin{aligned} \chi(x,y) &= c \\ F(x,y,t,s,u(t,s)) &= \varphi(t,s,u(t,s)) \\ G(x,y,t,s,u(t,s)) &= b(t,s)u(t,s), \end{aligned} \tag{5.11}$$

where $b(x,y), f(x,y), g(x,y), c$ and $\varphi(t,s,u(t,s))$ satisfy the hypotheses of Theorem 4.3. Then the problem (5.1),(5.2) and (5.11), has at most one solution on Ω.

Proof. Let $u_1(x,y)$ and $u_2(x,y)$ be tow solutions of (5.1)-(5.2) with (5.11). Then

$$u_1^p(x,y) = c + f(x,y)\int_\alpha^x\int_\beta^y \varphi(t,s,u_1(t,s))\Delta s\Delta t + g(x,y)\int_\alpha^M\int_\beta^N b(t,s)u_1(t,s)\Delta s\Delta t \tag{5.12}$$

and

$$u_2^p(x,y) = c + f(x,y)\int_\alpha^x\int_\beta^y \varphi(t,s,u_2(t,s))\Delta s\Delta t + g(x,y)\int_\alpha^M\int_\beta^N b(t,s)u_2(t,s)\Delta s\Delta t. \tag{5.13}$$

Making the difference between (5.12) and (5.13), and taking the absolute value at both sides of the resulting equality and using (5.11), we get

$$\begin{aligned}\left|u_1^p(x,y) - u_2^p(x,y)\right| \le\ & f(x,y)\int_\alpha^x\int_\beta^y |u_1(t,s) - u_2(t,s)|\Delta s\Delta t \\ & + g(x,y)\int_\alpha^M\int_\beta^N b(t,s)\,|u_1(t,s) - u_2(t,s)|\Delta s\Delta t. \end{aligned} \tag{5.14}$$

An application of Theorem 4.3 or Theorem 4.4 to (5.14) gives $\left|u_1^p(x,y) - u_2^p(x,y)\right| = 0$, which implies that the problem (5.1)-(5.2) with (5.11) admits a unique solution. □

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