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EQUILIBRIUM AND MIXED EQUILIBRIUM PROBLEMS UNDER WEAK MONOTONICITY ON HADAMARD MANIFOLDS

S. JANA[1,*], C. NAHAK[2]

[1]Department of Mathematics, Kalinga Institute of Industrial Technology University, Bhubneswar 751024, Odisha, India

[2]Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur 721302, India

Abstract. In this paper, we develop the equilibrium theory on Hadamard manifolds. We study the existence of solutions of both equilibrium problems (EP) and mixed equilibrium problems (MEP) on Hadamard manifolds under generalized monotonicity assumptions.

Keywords. Hadamard manifold; Equilibrium problem; Monotonicity; KKM mapping.

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1. Introduction

The theory of equilibrium problems provides a unified way to research some nonlinear problems, for instance, optimization problems, variational inequality problems, complementarity problems. This problem contains many important problems as special cases, such as, Nash equilibrium, fixed point problems. In recent decades, many results concerned with the existence of solutions for equilibrium problems and mixed equilibrium problems have been established, see, for example, [1], [2], [8], [17] and the references therein.

On the other hand, recent interests of a number of researchers are focused on extending some concepts and techniques of nonlinear analysis in Euclidean spaces to Riemannian manifolds. There are some advantages for a generalization of optimization methods from Euclidean spaces to Riemannian manifolds, because nonconvex and nonsmooth constrained optimization problems can be seen as convex and smooth unconstrained optimization problems from the Riemannian geometry point of view; see, for example, [15], [11], [12]. Nemeth [10] and Wang *et al.* [16] studied monotone and accretive vector fields on Riemannian manifolds. Li *et al.* [5] extended maximal monotone vector fields from Banach spaces to Hadamard manifolds (simply connected complete Riemannian manifold with nonpositive sectional curvature). Nemeth [9] introduced variational inequality problems on Hadamard manifolds. Li *et al.* [6]

[*]Corresponding author.

E-mail address: shreyasi.janafma@kiit.ac.in (S. Jana).

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studied the variational inequality problems on Riemannian manifolds. Zhou and Huang [18] established the notion of the KKM mapping and proved a generalized KKM theorem on the Hadamard manifold. An existence result for equilibrium problems on Hadamard manifolds was first studied by Colao *et al.* [4], where the equilibrium problem was associated to a monotone bifunction. Zhou and Huang [19] investigated the relationship between a vector variational inequality problem and a vector optimization problem on a Hadamard manifold. Tang *et al.* [14] introduced the proximal point algorithm for pseudomonotone variational inequalities on Hadamard manifolds. Li and Huang [7], studied the generalized vector quasi-equilibrium problems.

Motivated by the research work mentioned above, we study the existence of solutions of both equilibrium and mixed equilibrium problems on Hadamard manifolds by using KKM techniques under weaker assumptions than monotonicity.

2. Preliminaries

2.1. **Riemannian Geometry.** We recall some fundamental definitions, basic properties and notations needed for a comprehensive reading of this paper. These can be found in any textbook on Riemannian geometry, for example, [13], [15].

Let M be an n-dimensional connected manifold. We denote by T_xM the n-dimensional tangent space of M at x and by $TM = \cup_{x\in M} T_xM$, the tangent bundle of M. When M is endowed with a Riemannian metric $< .,. >$ on the tangent space T_xM with corresponding norm denoted by $\|.\|$, M is a Riemannian manifold. The length of a piecewise smooth curve $\gamma : [a,b] \to M$ joining x to y such that $\gamma(a) = x$ and $\gamma(b) = y$, is defined by

$$L(\gamma) = \int_a^b \| \dot{\gamma}(t) \|_{\gamma(t)} \, dt.$$

Then for any $x, y \in M$ the Riemannian distance $d(x,y)$ which induces the original topology on M is defined by minimizing this length over the set of all curves joining x to y. On every Riemannian manifold there exists exactly one covariant derivation called Levi-Civita connection denoted by $\nabla_X Y$ for any vector fields X,Y on M. Let γ be a smooth curve in M. A vector field X is said to be parallel along γ if $\nabla_{\gamma'} X = 0$. If γ' itself is parallel along γ, we say that γ is a geodesic. A geodesic joining x to y in M is said to be a minimal geodesic if its length equals $d(x,y)$. A Riemannian manifold is complete if for any $x \in M$ all geodesics emanating from x are defined for all $t \in \mathbb{R}$. By the Hopf-Rinow theorem, we know that if M is complete, then any pair of points in M can be joined by a minimal geodesic. Moreover, (M,d) is a complete metric space and bounded closed subsets are compact. Assuming that M is complete the exponential mapping $\exp_x : T_xM \to M$ is defined by $\exp_x v = \gamma_v(1)$, where γ_v is the geodesic defined by its position x and velocity v at x.

Recall that a Hadamard manifold is a simply connected complete Riemannian manifold with nonpositive sectional curvature. The exponential mapping exp and its inverse $\exp^{-1}$ are continuous on Hadamard manifold. Let M denote a finite dimensional Hadamard manifold.

2.2. **Convexity.**

Definition 2.1 ([12]). A subset K of M is said to be geodesic convex if and only if for any two points $x,y \in K$, the geodesic joining x to y is contained in K, that is, if $\gamma : [0,1] \to M$ is a geodesic with $x = \gamma(0)$ and $y = \gamma(1)$, then $\gamma(t) \in K,\ for\ 0 \le t \le 1$.

Definition 2.2 ([12]). A real-valued function $f : M \to \mathbb{R}$ defined on a geodesic convex set K is said to be geodesic convex if and only if for $0 \leq t \leq 1$,

$$f(\gamma(t)) \leq (1-t)f(\gamma(0)) + tf(\gamma(1)).$$

Definition 2.3 ([4]). For an arbitrary subset $C \subseteq M$ the minimal geodesic convex subset which contains C is called the convex hull of C and is denoted by $co(C)$. It is easy to check that $co(C) = \bigcup_{n=1}^{\infty} C_n$, where $C_0 = C$ and $C_n = \{z \in \gamma_{x,y} : x, y \in C_{n-1}\}$.

Definition 2.4 ([19]). Let $K \subset M$ be a nonempty closed geodesic convex set and let $G : K \to 2^K$ be a set-valued mapping. We say that G is a KKM mapping if for any $\{x_1, ..., x_m\} \subset K$, then

$$co(\{x_1, ..., x_m\}) \subset \bigcup_{i=1}^{m} G(x_i).$$

Lemma 2.5 ([4]). *Let K be a nonempty closed geodesic convex set and let $G : K \to 2^K$ be a set-valued mapping such that for each $x \in K$, $G(x)$ is closed. Suppose that*

(i) *there exists $x_0 \in K$ such that $G(x_0)$ is compact.*
(ii) $\forall x_1, .., x_m \in K$, $co(\{x_1, .., x_m\}) \subset \bigcup_{i=1}^{m} G(x_i)$.

Then $\bigcap_{x \in K} G(x) \neq \emptyset$.

Lemma 2.6 ([5]). *Let $x_0 \in M$ and $\{x_n\} \in M$ such that $x_n \to x_0$. Then the following assertions hold.*

(i) *For any $y \in M$*

$$\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y \text{ and } \exp_y^{-1} x_n \to \exp_y^{-1} x_0.$$

(ii) *If $\{v_n\}$ is a sequence such that $v_n \in T_{x_n}M$ and $v_n \to v_0$, then $v_0 \in T_{x_0}M$.*
(iii) *Given the sequence $\{u_n\}$ and $\{v_n\}$ with $u_n, v_n \in T_{x_n}M$, if $u_n \to u_0$ and $v_n \to v_0$ with $u_0, v_0 \in T_{x_0}M$, then $\langle u_n, v_n \rangle \to \langle u_0, v_0 \rangle$.*

Throughout the remaining part of the paper, we take M to be a finite dimensional Hadamard manifold and $K \subseteq M$ denote a nonempty closed geodesic convex set, unless explicitly stated otherwise.

3. Main results

This section includes the study of existence of solutions of both equilibrium problems (EP) and mixed equilibrium problems (MEP) on Hadamard manifolds.

3.1. **Existence results for equilibrium problems.** Let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying the property $F(x,x) = 0$ for all $x \in K$. Then the equilibrium problem introduced by Colao *et al.* [4] is to find a point $x \in K$, such that

$$(EP)\ F(x,y) \geq 0, \quad \forall y \in K. \tag{3.1}$$

A point that solves (EP) is called an equilibrium point. Some particular cases of (EP) is as follows.

(i) Variational inequality problem: Let $V : K \to TM$ be a vector field, that is, $V_x \in T_xM$ for each $x \in K$ and $\exp^{-1}$ denote the inverse of the exponential map. Then the problem introduced by Nemeth ([9]), is to find $x \in K$ such that

$$(VIP)\ \langle V_x, \exp_x^{-1} y \rangle \geq 0,\ \forall y \in K, \tag{3.2}$$

is called a variational inequality problem (VIP) on K. If we define

$$F(x,y) = \langle V_x, \exp_x^{-1} y \rangle,$$

then (EP) and the (VIP) are equivalent.

(ii) Optimization problem: Let $f : K \to \mathbb{R}$ be a function and consider the minimization problem

$$(P)\ find\ x \in K\ such\ that\ f(x) = \min_{y \in K} f(y).$$

If we set $F(x,y) = f(y) - f(x)$, for all $x, y \in K$. Then the optimization problem (P) and (EP) are equivalent.

Next, we study of existence of solutions of (EP)'s on Hadamard manifolds under weak pseudomonotonicity assumptions. First we define the notions of different types of monotonicities.

Definition 3.1 ([4]). We call the bifunction F to be monotone on K if for any $x, y \in K$, we have

$$F(x,y) + F(y,x) \leq 0. \tag{3.3}$$

Definition 3.2. We call the bifunction F to be pseudomonotone on K if for any $x, y \in K$

$$F(x,y) \geq 0 \Rightarrow F(y,x) \leq 0. \tag{3.4}$$

Definition 3.3 ([14]). The bifunction F is said to be weak monotone if for any two points $x, y \in K$ there exists a real number $\mu > 0$, such that

$$F(x,y) + F(y,x) \leq \mu d^2(x,y). \tag{3.5}$$

Definition 3.4. The bifunction F is said to be weak pseudomonotone if for any two points $x, y \in K$ there exists a real number $\mu > 0$, such that

$$F(x,y) \geq 0 \Rightarrow F(y,x) \leq \mu d^2(x,y). \tag{3.6}$$

We now give one example to show that weak pseudomonotonicity generalizes weak monotonicity.

Example 3.5. Let $H^1 = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 - x_2^2 = -1,\ x_2 > 0\}$ be the hyperbolic 1-space which forms a Hadamard manifold ([3]) endowed with the metric defined by

$$\langle x, y \rangle = x_1 y_1 - x_2 y_2,\ \forall x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

Let K be a subset of H^1 defined by $K = \{x = (x_1, x_2) \in H^1 : -1 \leq x_1 \leq 1\}$.

Now we define the bifunction $F : K \times K \to \mathbb{R}$ by

$$F(x,y) = x_2(x_1 - y_1). \tag{3.7}$$

To show that F is weak pseudomonotone on K but not weak monotone.
$F(x,y) \geq 0$ on K when $x_1 \geq y_1$ (as $x_2 > 0$),
then $F(y,x) = y_2(y_1 - x_1) \leq 0$, (as $y_2 > 0$),
$\leq \mu d^2(x,y)\ for\ any\ \mu.$
Therefore, F is weak pseudomonotone. Particularly if we take $x = (1, \sqrt{2}) \in K$ and $y = (0, 1) \in K$, then $F(x,y) + F(y,x) = \sqrt{2} - 1$ and and $d^2(x,y) = 1^2 - (\sqrt{2} - 1)^2 = 2(\sqrt{2} - 1)$. If we take $\mu = \frac{1}{4}$, then $F(x,y) + F(y,x) \geq \mu d^2(x,y)$, that is, F is not weak monotone. Therefore F is $\frac{1}{4}$-weak pseudomonotone but not $\frac{1}{4}$-weak monotone.

Also Example 3.1 of [14] shows that weak monotonicity generalizes monotonicity. Therefore the implication relationships between monotonicity and some generalized monotonicity are shown as follows:

$$\textit{monotonicity} \Rightarrow \textit{weak monotonicity} \Rightarrow \textit{weak pseudomonotonicity}.$$

Definition 3.6. Let K be a geodesic convex subset of M. A function $f : K \to \mathbb{R}$ is said to be hemicontinuous if for every geodesic $\gamma : [0,1] \to K$, whenever $t \to 0$, $f(\gamma(t)) \to f(\gamma(0))$.

Next we give the following lemma which will be needed in the sequel.

Lemma 3.7. *Let $F : K \times K \to \mathbb{R}$ be weak pseudomonotone and hemicontinuous in the first argument. Let for fixed $x \in K$ the mapping $z \mapsto F(x,z)$ be geodesic convex also for a geodesic $\gamma : [0,1] \to K$, with $\gamma(0) = x$, $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. Then $x \in K$ is a solution of (EP) if and only if*

$$F(y,x) \leq \mu d^2(x,y), \quad \forall y \in K.$$

Proof. Let $x \in K$ is a solution of (EP). Then

$$F(x,y) \geq 0, \quad \forall y \in K. \tag{3.8}$$

Since F is weak pseudomonotone, we have

$$F(y,x) \leq \mu d^2(x,y), \quad \forall y \in K. \tag{3.9}$$

Conversely, let $x \in K$ be a solution of (3.9). Let $\gamma(t)$ be a geodesic joining x and y such that $\gamma(0) = x$. As K is geodesic convex, then $F(\gamma(t),x) \leq \mu d^2(x,\gamma(t))\ for\ 0 \leq t \leq 1$. Now
$0 = F(\gamma(t),\gamma(t))$
$\leq tF(\gamma(t),y) + (1-t)F(\gamma(t),x)$, (as $z \mapsto F(x,z)$ is geodesic convex.)
$\Rightarrow t[F(\gamma(t),x) - F(\gamma(t),y)] \leq F(\gamma(t),x) \leq \mu d^2(x,\gamma(t))$.
$\Rightarrow F(\gamma(t),x) - F(\gamma(t),y) \leq \mu \frac{d^2(x,\gamma(t))}{t}$, *as* $t \geq 0$.
Since F is hemicontinuous in the first argument taking $t \to 0$, we have

$$F(x,x) - F(x,y) \leq 0 \Rightarrow F(x,y) \geq 0, \quad \forall y \in K.$$

This completes the proof. □

We are now in the position of proving the main existence theorem. First we take K to be bounded. So in this case K is compact.

Theorem 3.8. *Let K be compact and let $F : K \times K \to \mathbb{R}$ be weak pseudomonotone and hemicontinuous in the first argument. Suppose for fixed $x \in K$ the mapping $z \mapsto F(x,z)$ be geodesic convex and lower semicontinuous, also for a geodesic $\gamma : [0,1] \to K$, with $\gamma(0) = x$, $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. Then (EP) has a solution.*

Proof. Consider the two set-valued mappings $G_1 : K \to 2^K$ and $G_2 : K \to 2^K$ such that

$$G_1(y) = \{x \in K : F(x,y) \geq 0\}, \quad \forall y \in K,$$

$$G_2(y) = \{x \in K : F(y,x) \leq \mu d^2(x,y)\}, \quad \forall y \in K.$$

It is easy to see that $x \in K$ solves (EP) if and only if $x \in \cap_{y \in K} G_1(y)$. Thus it suffices to prove that $\cap_{y \in K} G_1(y) \neq \emptyset$. First we show G_1 is a KKM mapping. So we have to prove that for any choice of $x_1, ..., x_m \in K$

$$co(\{x_1, ..., x_m\}) \subset \bigcup_{i=1}^{m} G_1(x_i). \tag{3.10}$$

Suppose on the contrary that there exists a point x_0 *in* K, such that $x_0 \in co(\{x_1, ..., x_m\})$ but $x_0 \notin \bigcup_{i=1}^{m} G_1(x_i)$. That is

$$F(x_0, x_i) < 0, \ \forall i \in \{1, ..., m\}. \tag{3.11}$$

This implies that for any $i \in \{1, ..., m\}$, $x_i \in \{y \in K : F(x_0, y) < 0\}$. Since the function $y \mapsto F(x_0, y)$ is geodesic convex, the set $\{y \in K : F(x_0, y) < 0\}$ is a geodesic convex set. Then

$$x_0 \in co(\{x_1, ..., x_m\}) \subseteq \{y \in K : F(x_0, y) < 0\}.$$

Therefore $F(x_0, x_0) < 0$. But we have $F(x_0, x_0) = 0$, a contradiction. Hence G_1 is a KKM mapping. From Lemma 3.7, we have $G_1(y) \subset G_2(y), \ \forall y \in K$. That is,

$$co(\{x_1, ..., x_m\}) \subset \bigcup_{i=1}^{m} G_2(x_i).$$

Hence G_2 is also a KKM mapping. Since $F(y, .)$ is lower semicontinuous, we find that $G_2(y)$ is closed for all $y \in K$. Now $G_2(y)$ is a closed subset of a compact set K. So $G_2(y)$ is compact for all $y \in K$. Hence by Lemma 2.5, there exists a point $x \in K$ such that $x \in \bigcap_{y \in K} G_2(y)$. By Lemma 3.7, we have $\bigcap_{y \in K} G_1(y) = \bigcap_{y \in K} G_2(y)$, that is, $x \in \bigcap_{y \in K} G_1(y)$. So there exists a point $x \in K$ such that

$$F(x, y) \geq 0, \ \forall y \in K.$$

Therefore, $x \in K$ solves (EP). □

As weak pseudomonotonicity is a generalization of monotonicity. The following corollary is obvious.

Corollary 3.9. *Let K be compact and let $F : K \times K \to \mathbb{R}$ be monotone and hemicontinuous in the first argument. Let for fixed $x \in K$ the mapping $z \mapsto F(x, z)$ be geodesic convex and lower semicontinuous. Also for a geodesic $\gamma : [0, 1] \to K$, with $\gamma(0) = x$, $\lim_{t \to 0} \frac{d^2(x, \gamma(t))}{t} = 0$. Then (EP) has a solution.*

Next we consider the case when K is not bounded i.e. in this case K is noncompact.

Theorem 3.10. *Let K be noncompact and let $F : K \times K \to \mathbb{R}$ be weak pseudomonotone and hemicontinuous in the first argument. Let for fixed $x \in K$ the mapping $z \mapsto F(x, z)$ be geodesic convex and lower semicontinuous. Also assume that for a geodesic $\gamma : [0, 1] \to K$, with $\gamma(0) = x$, $\lim_{t \to 0} \frac{d^2(x, \gamma(t))}{t} = 0$ and there exists a point $x_0 \in K$, such that*

$$F(x, x_0) < 0, \ \textit{whenever} \ d(\mathbf{0}, x) \to +\infty, \ x \in K. \tag{3.12}$$

Then (EP) has a solution.

Proof. Given a point $\mathbf{0} \in M$, we denote $\Sigma_R = \{x \in M : d(\mathbf{0}, x) \leq R\}$ to be the closed geodesic ball of radius R and center $\mathbf{0}$. Let $K_R = K \cap \Sigma_R$. If $K_R \neq \emptyset$, then there exists at least one $x_R \in K_R$ such that

$$F(x_R, y) \geq 0, \ \forall y \in K_R, \tag{3.13}$$

by Theorem 3.8. We now take a point $x_0 \in K$ satisfying (3.12) with $d(\mathbf{0}, x_0) < R$, so $x_0 \in K_R$.

Hence by (3.13) we have

$$F(x_R, x_0) \geq 0. \tag{3.14}$$

If $d(\mathbf{0}, x_R) = R$ for all R, we may choose R large enough so that $d(\mathbf{0}, x_R) \to +\infty$. Hence by (3.12) $F(x_R, x_0) < 0$, contradicts (3.14). So there exists an R such that $d(\mathbf{0}, x_R) < R$. Given $y \in K$, let $\gamma(t)$ be a geodesic joining x_R to y. Now since $d(\mathbf{0}, x_R) < R$, we can choose $0 < t < 1$ sufficiently small so that $\gamma(t) \in K_R$. Hence

$$\begin{aligned} 0 &\leq F(x_R, \gamma(t)) \\ &\leq tF(x_R, y) + (1-t)F(x_R, x_R) \\ &= tF(x_R, y), \end{aligned}$$

or $F(x_R, y) \geq 0$, for $y \in K$, that is, x_R solves (EP). □

The following corollary is obvious.

Corollary 3.11. *Let K be noncompact and let $F : K \times K \to \mathbb{R}$ be monotone and hemicontinuous in the first argument. Let for fixed $x \in K$ the mapping $z \mapsto F(x,z)$ be geodesic convex and lower semicontinuous. Also assume that for a geodesic $\gamma : [0,1] \to K$, with $\gamma(0) = x$, $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$ and there exists a point $x_0 \in K$, such that*

$$F(x, x_0) < 0, \ \textit{whenever}\ d(\mathbf{0}, x) \to +\infty,\ x \in K. \tag{3.15}$$

Then (EP) has a solution.

3.2. **Existence of solutions of mixed equilibrium problems.** In this section we study the existence of solutions of mixed equilibrium problems under weak monotonicity assumptions.

Let $\psi : K \to \mathbb{R}$ be a mapping and let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying the property $F(x,x) = 0$ for all $x \in K$. Then the problem is to find $x \in K$ such that

$$(MEP)\ F(x,y) + \psi(y) - \psi(x) \geq 0,\ \forall y \in K, \tag{3.16}$$

is called a mixed equilibrium problem on K.

Some particular cases of (MEP) is as follows.

(i) Equilibrium problem: If we take $\psi \equiv 0$, then (MEP) reduces to (EP).

(ii) Mixed variational inequality problem: Let $\psi : K \to \mathbb{R}$ be a mapping. Then the mixed variational inequality problem ([4]) is to find $x \in K$ such that

$$(MVIP)\ \left\langle V_x, \exp_x^{-1} y \right\rangle + \psi(y) - \psi(x) \geq 0,\ \forall y \in K. \tag{3.17}$$

If we take

$$F(x,y) = \left\langle V_x, \exp_x^{-1} y \right\rangle,$$

then (MEP) and (MVIP) are equivalent.

Next we give the following lemma which will be needed to prove the main existence theorem.

Lemma 3.12. *Let the bifunction $F : K \times K \to \mathbb{R}$ be weak monotone and hemicontinuous in the first argument and for fixed $x \in K$ the mapping $z \mapsto F(x,z)$ be geodesic convex. Also assume that the map $\psi : K \to \mathbb{R}$ is geodesic convex and for a geodesic $\gamma : [0,1] \to K$, with $\gamma(0) = x$, $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. Then $x \in K$ is a solution of (MEP), if and only if*

$$F(y,x) + \psi(x) - \psi(y) \leq \mu d^2(x,y), \quad \forall y \in K. \tag{3.18}$$

Proof. Let $x \in K$ is a solution of (MEP). Then

$$F(x,y) + \psi(y) - \psi(x) \geq 0, \ for\ all\ y \in K. \tag{3.19}$$

Since F is weak monotone with respect to the function, we have

$$F(y,x) \leq \mu d^2(x,y) - F(x,y).$$

$$\Rightarrow F(y,x) + \psi(x) - \psi(y) \leq \mu d^2(x,y) - [F(x,y) + \psi(y) - \psi(x)] \leq \mu d^2(x,y)\ (by\ (3.19)).$$

Conversely, let $x \in K$ be a solution of (3.18). Let $\gamma(t)$ be a geodesic joining x and y such that $\gamma(0) = x$. As K is geodesic convex, we have

$$F(\gamma(t),x) + \psi(x) - \psi(\gamma(t)) \leq \mu d^2(x,\gamma(t))\ for\ 0 \leq t \leq 1. \tag{3.20}$$

As ψ is geodesic convex then

$$\psi(\gamma(t)) \leq t\psi(y) + (1-t)\psi(x)$$

$$\Rightarrow \psi(\gamma(t)) - \psi(x) \leq t[\psi(y) - \psi(x)].$$

Also as $z \mapsto F(x,z)$ is geodesic convex, $0 = F(\gamma(t),\gamma(t)) \leq tF(\gamma(t),y) + (1-t)F(\gamma(t),x)$,
$\Rightarrow \psi(\gamma(t)) - \psi(x) \leq tF(\gamma(t),y) + (1-t)F(\gamma(t),x) + \psi(\gamma(t)) - \psi(x)$
$\leq tF(\gamma(t),y) + (1-t)F(\gamma(t),x) + t[\psi(y) - \psi(x)]$
$\Rightarrow t[F(\gamma(t),x) - F(\gamma(t),y) - \psi(y) + \psi(x)] \leq [F(\gamma(t),x) + \psi(x) - \psi(\gamma(t))] \leq \mu d^2(x,\gamma(t))$ (by (3.20)).
That is,

$$F(\gamma(t),x) - F(\gamma(t),y) - \psi(y) + \psi(x) \leq \mu \frac{d^2(x,\gamma(t))}{t},$$

as $t \geq 0$. Since F is hemicontinuous in the first argument taking $t \to 0$, we have

$$F(x,x) - F(x,y) - \psi(y) + \psi(x) \leq 0 \Rightarrow F(x,y) + \psi(y) - \psi(x) \geq 0, \quad \forall y \in K.$$

This completes the proof. □

Next we prove the main existence theorem. First we consider the case when the set K is bounded. So in this case K is compact.

Theorem 3.13. *Let K be a compact subset of M and let $F : K \times K \to \mathbb{R}$ be weak monotone and hemicontinuous in the first argument. Let for fixed $x \in K$, the mapping $z \mapsto F(x,z)$ be geodesic convex, lower semicontinuous and $\psi : K \to \mathbb{R}$ be geodesic convex and lower semicontinuous. Also assume that for a geodesic $\gamma : [0,1] \to K$, with $\gamma(0) = x$, $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. Then (MEP) has a solution.*

Proof. Consider the two set-valued mappings $G_1 : K \to 2^K$ and $G_2 : K \to 2^K$ such that

$$G_1(y) = \{x \in K : F(x,y) + \psi(y) - \psi(x) \geq 0\}, \quad \forall y \in K,$$

$$G_2(y) = \{x \in K : F(y,x) + \psi(x) - \psi(y) \leq \mu d^2(x,y)\}, \quad \forall y \in K.$$

It is easy to see that $x \in K$ solves (MEP) if and only if $x \in \cap_{y\in K} G_1(y)$. Thus it suffices to prove that $\cap_{y\in K} G_1(y) \neq \emptyset$. First we show G_1 is a (KKM) mapping. So we have to prove that for any choice of $x_1,\ldots,x_m \in K$

$$co(\{x_1,\ldots,x_m\}) \subset \bigcup_{i=1}^{m} G_1(x_i). \tag{3.21}$$

Suppose on the contrary that there exists a point x_0 *in* K, such that $x_0 \in co(\{x_1,...,x_m\})$ but $x_0 \notin \bigcup_{i=1}^{m} G_1(x_i)$. That is,

$$F(x_0,x_i)+\psi(x_i)-\psi(x_0)<0,\ \forall i\in\{1,...,m\}. \tag{3.22}$$

This implies that for any $i \in \{1,...,m\}$, $x_i \in \{y \in K : F(x_0,y)+\psi(y)-\psi(x_0) < 0\}$. Now the function $y \mapsto F(x_0,y)$ is geodesic convex, also ψ is geodesic convex. Being the sum of two geodesic convex function, $y \mapsto F(x_0,y)+\psi(y)$ is geodesic convex. Hence the set $\{y \in K : F(x_0,y)+\psi(y)-\psi(x_0) < 0\}$ is a geodesic convex set. Then

$$x_0 \in co(\{x_1,...,x_m\}) \subseteq \{y \in K : F(x_0,y)+\psi(y)-\psi(x_0) < 0\}.$$

Therefore $F(x_0,x_0)+\psi(x_0)-\psi(x_0) < 0$. But we have $F(x_0,x_0) = 0$, a contradiction. Hence G_1 is a (KKM) mapping. From Lemma 3.12, we have $G_1(y) \subset G_2(y)$, $\forall y \in K$. That is,

$$co(\{x_1,...,x_m\}) \subset \bigcup_{i=1}^{m} G_2(x_i).$$

Hence G_2 is also a (KKM) mapping. Since $F(y,.)$ and ψ are lower semicontinuous, $G_2(y)$ is closed for all $y \in K$. Now $G_2(y)$ is a closed subset of a compact set K. So $G_2(y)$ is compact for all $y \in K$. Hence by Lemma 2.5, there exists a point $x \in K$ such that $x \in \bigcap_{y\in K} G_2(y)$. By Lemma 3.12, we have $\bigcap_{y\in K} G_1(y) = \bigcap_{y\in K} G_2(y)$. That is $x \in \bigcap_{y\in K} G_1(y)$. So there exists a point $x \in K$, such that

$$F(x,y)+\psi(y)-\psi(x) \geq 0,\ \forall y \in K.$$

Therefore, $x \in K$ solves (MEP). □

The following corollary follows immediately.

Corollary 3.14. *Let K be compact and $F : K \times K \to \mathbb{R}$ be monotone and hemicontinuous in the first argument. Suppose for fixed $x \in K$, the mapping $z \mapsto F(x,z)$ is geodesic convex, lower semicontinuous and $\psi : K \to \mathbb{R}$ is geodesic convex, lower semicontinuous . Also for a geodesic $\gamma : [0,1] \to K$, with $\gamma(0) = x$, $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. Then (MEP) has a solution.*

Next we take K to be unbounded. So K is a noncompact subset of M.

Theorem 3.15. *Let $F : K \times K \to \mathbb{R}$ be weak monotone and hemicontinuous in the first argument and for fixed $x \in K$ the mapping $z \mapsto F(x,z)$ be geodesic convex, lower semicontinuous. Also assume that $\psi : K \to \mathbb{R}$ is geodesic convex, lower semicontinuous and for a geodesic $\gamma : [0,1] \to K$, with $\gamma(0) = x$, $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. If there exists a point $x_0 \in K$, such that*

$$F(x,x_0)+\psi(x_0)-\psi(x) < 0,\ \textit{whenever}\ d(\mathbf{0},x) \to +\infty,\ x \in K, \tag{3.23}$$

holds, then (MEP) has a solution.

Proof. For a point $\mathbf{0} \in M$, let $\Sigma_R = \{x \in M : d(\mathbf{0},x) \leq R\}$ be the closed geodesic ball of radius R and center $\mathbf{0}$. Let $K_R = K \cap \Sigma_R$. If $K_R \neq \emptyset$, then there exists at least one $x_R \in K_R$ such that

$$F(x_R,y)+\psi(y)-\psi(x_R) \geq 0,\ \forall y \in K_R, \tag{3.24}$$

by Theorem 3.13. We now take a point $x_0 \in K$ satisfying (3.23) with $d(\mathbf{0},x_0) < R$, so $x_0 \in K_R$. Hence by (3.24), we have

$$F(x_R,x_0)+\psi(x_0)-\psi(x_R) \geq 0. \tag{3.25}$$

If $d(\mathbf{0},x_R) = R$ for all R, we may choose R large enough so that $d(\mathbf{0},x_R) \to +\infty$. Hence by (3.23), $F(x_R,x_0) + \psi(x_0) - \psi(x_R) < 0$ contradicts (3.25). So there exists an R such that $d(\mathbf{0},x_R) < R$. Given $y \in K$, let $\gamma(t)$ be a geodesic joining x_R to y with $\gamma(0) = x_R$. Now since $d(\mathbf{0},x_R) < R$, we can choose $0 < t < 1$, sufficiently small so that $\gamma(t) \in K_R$. Hence

$$\begin{aligned} 0 &\leq F(x_R,\gamma(t)) + \psi(\gamma(t)) - \psi(x_R) \\ &\leq tF(x_R,y) + (1-t)F(x_R,x_R) + t[\psi(y) - \psi(x_R)] \\ &= t[F(x_R,y) + \psi(y) - \psi(x_R)], \end{aligned}$$

or

$$F(x_R,y) + \psi(y) - \psi(x_R) \geq 0, \ for\ y \in K.$$

That is x_R is a solution of (MEP). □

The following corollary is obvious.

Corollary 3.16. *Let K be noncompact and $F : K \times K \to \mathbb{R}$ be monotone and hemicontinuous in the first argument. Let for fixed $x \in K$ the mapping $z \mapsto F(x,z)$ be geodesic convex, lower semicontinuous and $\psi : K \to \mathbb{R}$ is geodesic convex, lower semicontinuous. Also for a geodesic $\gamma : [0,1] \to K$, with $\gamma(0) = x$, $\lim_{t\to 0} \frac{d^2(x,\gamma(t))}{t} = 0$. If there exists a point $x_0 \in K$, such that*

$$F(x,x_0) < 0, \ whenever\ d(\mathbf{0},x) \to +\infty,\ x \in K. \tag{3.26}$$

Then (MEP) has a solution.

4. Conclusions

This paper is devoted to the study of existence of solutions of equilibrium problems (EP) under weak pseudomonotonicity and existence of solutions of mixed equilibrium problems (MEP) under weak monotonicity assumptions on Hadamard manifolds. The results presented in this paper are completely new and some existing results followed as a special case of our results.

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