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GLOBAL EXISTENCE AND CONTROLLABILITY TO A STOCHASTIC INTEGRO-DIFFERENTIAL EQUATION WITH POISSON JUMPS

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Abstract. In this paper, we prove global existence and uniqueness results for a stochastic integro-differential equation with poisson jumps in Frechet spaces. The main results are obtained based on a resolvent operator combined with a nonlinear alternative of Leray-schauder type. As an application, we study the controllability of the corresponding control system.

Keywords. Stochastic integro-differential equation; Resolvent operator; Fixed point theorem; Frechet space; Controllability.

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1. INTRODUCTION

The problem of controllability of linear deterministic systems is well documented. It is well known that controllability of deterministic equations are widely used in analysis and the design of control system. Any control system is said to be controllable if every state corresponding to this process can be affected or controlled in respective time by some control signals. In many dynamical systems, it is possible to steer the dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls, that is, there are system which are completely controllable.

Stochastic differential and integro-differential equations have attracted much interest due to their applications in characterizing many problems in physics, biology, mechanics and so on. Qualitative properties, such as, existence, uniqueness and stability for various stochastic differential and integro-differential systems have been extensively studied by many researchers; see, for instance, [1, 2, 3, 4, 5, 6] and the references therein. The theory of nonlinear functional integro-differential equations with resolvent operators serves as an abstract formulation of partial integro-differential equations which arises in many physical phenomena [6, 7, 8, 9, 10]. As pointed out by Ouahab in [11], the investigation of many properties of solutions for a given equation, such as, stability, oscillation, often needs to guarantee its global existence. Thus it is of very importance to establish sufficient conditions for global existence results

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for functional differential equations. The existence of unique global solutions for deterministic functional differential evolution equations with infinite delays in Frechet spaces were studied by Baghli *et al.* [12, 13] and Benchohra *et al.* [14]. Our approach here is based on a recent Frigon and Granas nonlinear alternative of the Leray-Schauder type in Frechet spaces [15] combined with the resolvent operators theory.

In this paper, we consider the uniqueness of mild solutions on a semi-infinite positive real interval $J=[0,+\infty)$ for a class of stochastic integro-differential equations with poisson jumps in the abstract form

$$dx(t) = [Ax(t)+\int_0^t B(t-s)x(s)ds]dt+f(t,x(t))dw(t)+\int_Z h(t,x(t),v)\tilde{N}(dt,dv),\ t\in J, \quad (1.1)$$

$$x(0) = x_0 \quad (1.2)$$

where $A:D(A)\subset\mathbb{H}\to\mathbb{H}$, $B(t):D(B(t))\subset\mathbb{H}\to\mathbb{H}, t\geq 0$ are linear, closed, and densely defined operators in a Hilbert spaces $\mathbb{H}$, $f:J\times\mathbb{H}\to L_Q(\mathbb{K},\mathbb{H})$ and $h:J\times\mathbb{H}\times Z\to\mathbb{H}$ are appropriate functions specified later and $w(t), t\geq 0$ is a given $\mathbb{K}$-valued Brownian motion, which will be defined in Section 2. The initial data x_0 is an $\mathscr{F}_0$-adapted, $\mathbb{H}$-valued random variable independent of the Wiener process w. Further, as an applications, we study the controllability results with one parameter.

2. Preliminaries

This section is concerned with some basic concepts, notations, definitions, lemmas and preliminary facts which are used throughout this paper. For more details on this section, we refer the reader to [4, 16] and the references therein. Throughout the paper,$(\mathbb{H},\|\cdot\|,\langle\cdot,\cdot\rangle)$ and $(\mathbb{K},\|\cdot\|_{\mathbb{K}},\langle\cdot\cdot\rangle_{\mathbb{K}})$ denote two real separable Hilbert spaces. Let $(\Omega,\mathscr{F},\mathbb{P})$ be a complete probability space equipped with some filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of $\mathbb{K}$. We denote by $\{w(t), t\geq 0\}$ a cylindrical $\mathscr{K}$ valued Wiener process with a finite trace nuclear covariance operator $Q>0$, denote $Tr(Q)=\sum_{i=1}^{\infty}\lambda_i=\lambda<\infty$, which satisfies that $Qe_i=\lambda_i e_i, i=1,2,\ldots$. Actually, $w(t)$ is defined by

$$w(t)=\sum_{i=1}^{\infty}\sqrt{\lambda_i}w_i(t)e_i, t\geq 0,$$

where $\{w_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. We then let $\mathscr{F}_t=\sigma\{w(s):0\leq s\leq t\}$ be the σ algebra generated by w. Let $L(\mathbb{K},\mathbb{H})$ denote the space of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, equipped with the usual operator norm $\|.\|_{L(\mathbb{K},\mathbb{H})}$. For $\phi\in L(\mathbb{K},\mathbb{H})$, we define

$$\|\phi\|_Q^2=Tr(\phi Q\phi^*)=\sum_{i=1}^{\infty}\left\|\sqrt{\lambda_i\phi e_i}\right\|^2.$$

If $\phi_Q^2<\infty$, then ϕ is called a Q-Hilbert-Schmidt operator. Let $L_Q(\mathbb{K},\mathbb{H})$ denote the space of all Q-Hillbert-Schmidt operator $\phi:\mathbb{K}\to\mathbb{H}$. The completion $L_Q(\mathbb{K},\mathbb{H})$ of $L(\mathbb{K},\mathbb{H})$ with respect to the topology induced by the norm $\|.\|_Q$, where $\|\phi\|_Q^2=\langle\phi,\phi\rangle$ is a Hillbert space with the above norm topology. The collection of all strongly measurable, square integrable, $\mathbb{H}$-valued random variables, denoted by $L_2(\Omega,\mathbb{H})$, is a Banach space equipped with norm $\|x\|_{L^2(\Omega;\mathbb{H})}=(E\|x\|^2)^{\frac{1}{2}}$, where $E[x]=\int_\Omega x(w)d\mathbb{P}(w)$. An important subspace is given by $L_2^0(\Omega,\mathbb{H})=\{f\in L_2(\Omega,\mathbb{H}): f \text{ is } \mathscr{F}_0-\text{measurable}\}$. Let $C_{\mathscr{F}_t(J,\mathbb{H})}$ denote the space of all continuous and $\mathscr{F}_t$-adapted measurable processes from J into $\mathbb{H}$. A measurable function $x : [0,+\infty) \to \mathbb{H}$ is Bochner integrable if $\|x\|$ is Lebesgue integrable. Let $L^1([0,+\infty),\mathbb{H})$ be the space of measurable functions $x : [0,+\infty) \to \mathbb{H}$, which are Bochner integrable, equipped with the norm

$$\|x\|_{L^1} = \int_0^{+\infty} \|x(t)\|\, dt.$$

Let $q = (q(t)), t \in D_q$, be a stationary $\mathscr{F}_t$-Poisson point process with characteristic measure λ. Let $N(dt,dv)$ be the Poisson counting measure associated with q, i.e., $N(t,Z) = \sum_{s\in D_q, s\leq t} I_Z(q(s))$ with measurable set $Z \in \bar{B}(y-0)$, which denotes the Borel σ-field of $Y-\{0\}$. Let $\tilde{N}(dt,dv) = N(dt,dv) - dt\lambda(dv)$ be the compensated Poisson measure that is independent of $w(t)$. Let $P^2([0,+\infty)) \times Z; \mathbb{H}$ be the space of all predictable mappings $h : [0,+\infty) \times Z \times \Omega \to \mathbb{H}$ for which $\int_0^t \int_Z E\,\|h(t,v)\|^2_{\mathbb{H}}\, dt\lambda(dv) < \infty$. Then, we can define the $\mathbb{H}$-valued stochastic integral $\int_0^t \int_Z h(t,v)\tilde{N}(dt,dv)$, which is a centred square-integral martingale. Consider the space

$$B_{+\infty} = \left\{ x : J \to \mathbb{H} \in C_{\mathscr{F}_t}(J,\mathbb{H}) : x_0 \in L_2^0(\Omega,\mathbb{H}) \right\}.$$

From now on, $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a resolvent operator $R(t), t \geq 0$ in the Hilbert space $\mathbb{H}$ and $B(t)$:$D(B(t)) \subset \mathbb{H} \to \mathbb{H}$,$t \geq 0$ is a bounded linear operator. To obtain our results, we assume that the abstract Cauchy problem

$$dx(t) = \left[Ax(t) + \int_0^t B(t-s)x(s)ds \right] dt, t \geq 0, \tag{2.1}$$

$$x(0) = x_0 \in \mathbb{H}, \tag{2.2}$$

has an associated resolvent operator of bounded linear operators $R(t), t \geq 0$ on $\mathbb{H}$.

Definition 2.1. A family of bounded linear operators $R(t), t \geq o$ from $\mathbb{H}$ into $\mathbb{H}$ is a resolvent operator family for problem (2.1)-(2.2) if the following conditions are verified.
1. $R(0) = I$ (the identity operator on $\mathbb{H}$) and the map $t \to R(t)x$ is a continuous function on $[0,+\infty) \to \mathbb{H}$ for every $x \in \mathbb{H}$.
2. $AR(\cdot)x \in C([0,\infty],\mathbb{H})$ and $R(\cdot)x \in C^1([0,\infty],\mathbb{H})$ for every $x \in D(A)$.
3. For every $x \in D(A)$ and $t \geq 0$,

$$\frac{d}{dt}R(t)x = AR(t)x + \int_0^t B(t-s)R(s)xds,$$

$$\frac{d}{dt}R(t)x = R(t)Ax + \int_0^t R(t-s)B(s)xds.$$

For more details on the semigroup theory and resolvent operators, we refer to [6, 10, 17] and the references therein. Let X be a Frechet space with a family of semi-norms $\|\cdot\|_{n n\in\mathbb{N}}$. Let $Y \subset X$. We say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \leq \overline{M}_n, \quad \forall y \in Y.$$

With X, we associate a sequence of Banach spaces $(X^n, \|\cdot\|_n)$ as follows: For every $n \in \mathbb{N}$, we consider the equivalence relation $x \sim_n y$ if and only if $\|x-y\|_n = 0$ for all $x,y \in X$. We denote $X^n = (X\mid_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|$. To every $y \subset X$, we associate a sequence the $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $x \in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we define $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}, int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|.\|_n$ in X^n. We assume that the family of semi-norms $\{\|.\|_n\}$ verifies:

$$\|x\|_1 \leq \|x\|_2 \leq \|x\|_3 \leq \cdots, \quad \forall x \in X.$$

Definition 2.2. A function $f : J \times \mathbb{H} \to L_Q(\mathbb{K}, \mathbb{H})$ is said to be an L^2- Caratheodory function if it satisfies:
1. for each $t \in J$ the function $f(t,.) : \mathbb{H} \to L_Q(\mathbb{K}, \mathbb{H})$ is continuous;
2. for each $x \in \mathbb{H}$ the function $f(.,x) : J \to L_Q(\mathbb{K}, \mathbb{H})$ is $\mathscr{F}_t$- measurable;
3. for every positive integer k there exists $\alpha_k \in L^1_{loc}(J, \mathbb{R}_+)$ such that

$$E\|f(t,x)\|^2 \leq \alpha_k(t), \forall E\|x\|^2 \leq k$$

and for almost all $t \in J$.

Definition 2.3. [15] A function $G : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that

$$\|G(x) - G(y)\|_n \leq k_n \|x - y\|_n, \quad \forall x, y \in X.$$

Definition 2.4. [15] Let X be a Frechet space and $Y \subset X$ a closed subset and let $N : Y \to X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements hold:
(C1) N has a unique fixed point;
(C2) There exists $\lambda \in [0,1)$, $n \in \mathbb{N}$ and $x \in \partial_n Y^n$ such that $\|x - \lambda N(x)\|_n = 0$.

3. Main results

In this section, we prove that there is a unique global mild solution for the problem (1.1)-(1.2). We begin introducing the following concepts of mild solutions.

Definition 3.1. An $\mathscr{F}_t$ -adapted stochastic process $x : [0,+\infty) \to \mathbb{H}$ is called a mild solution of (1)-(2) if $x(0) = x_0 \in L^0_2(\Omega, \mathbb{H}), x(t)$ is continuous and satisfies the following integral equation

$$\begin{aligned} x(t) &= R(t)x_0 + \int_0^t R(t-s) f(s, x(s)) dw(s) \\ &+ \int_0^t R(t-s) \int_Z h(s, x(s), v) \tilde{N}(ds, dv), \text{ for each } t \in [0,+\infty). \end{aligned}$$

Let us list the following assumptions:
(H1) A is the infinitesimal generator of a resolvent operator $R(t), t \geq 0$ in the Hilbert space $\mathbb{H}$ and there exists a constant $M > 0$ such that

$$\|R(t)\|^2 \leq M, t \geq 0.$$

(H2) The function $f : J \times \mathbb{H} \to L_Q(\mathbb{K}, \mathbb{H})$ is L^2- Caratheodory and satisfies the following conditions:
(i) There exists a function $p \in L^1_{loc}(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi : J \to (0,+\infty)$ such that

$$E\|f(t,u)\|^2 \leq p(t)\psi(E\|u\|^2), \text{ for a.e. } t \in J \text{ and each } u \in \mathbb{H}.$$

(ii) For all $\mathfrak{R} > 0$, there exists a function $l_{\mathfrak{R}} \in L^1_{loc}(J, \mathbb{R}_+)$ such that

$$E\|f(t,u) - f(t,v)\|^2 \leq l_{\mathfrak{R}}(t) E\|u - v\|^2,$$

for all $u,v \in \mathbb{H}$ with $E\|u\|^2 \le \Re$ and $E\|v\|^2 \le \Re$.
(H3) The function $h : J \times \mathbb{H} \times z \to \mathbb{H}$ is Boral measurable function and satisfies the following conditions:
(i) There exists a function $p \in L^1_{loc}(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi : J \to (0,+\infty)$ such that

$$a).\int_0^t \int_Z \|h(t,x,v)-h(t,y,v)\|^2 \lambda(dv)ds \vee \left(\int_0^t \int_Z \|h(t,x,v)-h(t,y,v)\|^4 \lambda(dv)ds\right)^{\frac{1}{2}} \le \int_0^t l_{\Re}(t)E\|x-y\|^2,$$

$$b).\int_0^t \int_z \|h(t,x,v)\|^2 \lambda(dv)ds \le \int_0^t p(t)\psi(E\|x\|^2),$$

$$c).\left(\int_0^t \int_Z \|h(t,x,v)\|^4 \lambda(dv)ds\right)^{\frac{1}{2}} \le \int_0^t p(t)\psi(E\|x\|^2),$$

Theorem 3.2. *Assume the conditions $(H1)-(H3)$ are satisfied. For each $n \in \mathbb{N}$,*

$$\int_{c_n}^{+\infty} \frac{ds}{\psi(s)} \quad > \quad 3M[Tr(Q)+2]\int_0^n p(s)ds, \tag{3.1}$$

where $c_n = 3ME\|x_0\|^2$. Then problem (1.1)-(1.2) has a unique mild solution on J.

Proof. Fix $\tau > 1$. For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the semi-norms

$$\|x\|_n \quad = \quad \sup\left\{e^{-\tau L_n^*(t)}E\|x(t)\|^2 : t \in [0,n]\right\},$$

where $L_n^*(t) = \int_0^t \bar{l}_n(s)ds$, and $\bar{l}_n(t) = 2M[Tr(Q)+2]\,l_n(t)$ and l_n is the function from (H2). Then $B_{+\infty}$ is a Frechet space with the family of semi-norms $\|\cdot\|_{n\in\mathbb{N}}$. We transform (1.1)-(1.2) into a fixed point problem. Consider the operator $\Phi : B_{+\infty} \to B_{+\infty}$ defined by

$$\Phi(x)(t) \quad = \quad R(t)x_0 + \int_0^t R(t-s)f(s,x(s))dw(s) + \int_0^t R(t-s)\int_Z h(s,x(s),v)\tilde{N}(ds,dv)$$

for $t \in J$. Clearly fixed points of operator Φ are mild solutions of problem (1.1)-(1.2). For the sake of convenience, we set, for $n \in \mathbb{N}$,

$$\begin{aligned} c_n &= 3ME\|x_0\|^2, \\ m(t) &= 3M[Tr(Q)+2]\,p(t). \end{aligned}$$

Let $x \in B_{+\infty}$ be a possible fixed point the operator Φ. By hypotheses (H1), (H2) and (H3), we have, for each $t \in [0,n]$,

$$\begin{aligned} E\|x(t)\|^2 &= 3E\|R(t)x_0\|^2 + 3E\left\|\int_0^t R(t-s)f(s,x(s))dw(s)\right\|^2 \\ &+ 3E\left\|\int_0^t R(t-s)\int_Z h(s,x(s),v)\tilde{N}(ds,dv)\right\|^2 \\ &\le 3ME\|x_0\|^2 + 3Tr(Q)M\int_0^t E\|f(s,x(s))\|^2 ds \\ &+ 3M\int_0^t \int_Z \|h(s,x(s),v)\|^2 \lambda(dv)ds \\ &+ 3M\left(\int_0^t \int_Z \|h(s,x(s),v)\|^4 \lambda(dv)ds\right)^{\frac{1}{2}}. \\ &\le 3ME\|x_0\|^2 + 3M[Tr(Q)+2]\int_0^t p(s)\psi(E\|x(s)\|^2)ds. \end{aligned}$$

We consider the function u defined by

$$u(t) \quad = \quad \sup\left\{E\|x(s)\|^2 : 0 \leq s \leq t\right\}, 0 \leq t \leq +\infty.$$

Let $t^* \in [0,t]$ be such that $u(t) = E\|x(t^*)\|^2$. It follows that

$$u(t) \quad = \quad 3ME\|x_0\|^2 + 3M[Tr(Q)+2]\int_0^t p(s)\psi(E\|x(s)\|^2)ds.$$

Let us take the right-hand side of the above inequality as $v(t)$. It follows that $u(t) \leq v(t)$ for all $t \in [0,n]$ and $v(0) = c_n = 3ME\|x_0\|^2$ and

$$v'(t) \quad = \quad 3M[Tr(Q)+2]\,p(t)\psi(u(t)) \text{ a.e. } t \in [0,n].$$

Using the nondecreasing character of ψ, we get

$$v'(t) \quad = \quad 3[Tr(Q)+2]Mp(t)\psi(v(t)) \text{ a.e. } t \in [0,n].$$

This implies, for each $t \in [0,n]$, that

$$\int_{c_n}^{v(t)} \frac{ds}{\psi(s)} \quad \leq \quad \int_0^n m(s)ds < \int_{c_n}^{+\infty} \frac{ds}{\psi(s)}.$$

By (3.1), for every $t \in [0,n]$, there exists a constant $\wedge_n$, such that $v(t) \leq \wedge_n$. Hence $u(t) \leq \wedge_n$. Since $\|x\|_n \leq u(t)$, we have $\|x\|_n \leq \wedge_n$. Set

$$\Omega = \left\{x \in B_{+\infty} : \sup E\|x(t)\|^2 : 0 \leq t \leq n \leq \wedge_n + 1 \text{ for all } n \in \mathbb{N}\right\}.$$

Clearly, Ω is a closed subset of $B_{+\infty}$.

We next show that $\Phi : \Omega \to B_{+\infty}$ is a contraction operator. Indeed, we consider $x, y \in B_{+\infty}$ based on (H1), (H2) and (H3) for each $t \in [0,n]$ and $n \in \mathbb{N}$

$$\begin{aligned}
E\|\Phi(x)(t) - \Phi(y)(t)\|^2 \quad &= \quad 2E\left\|\int_0^t R(t-s)[f(s,x(s)) - f(s,y(s))]dw(s)\right\|^2 \\
&+ \quad 2E\left\|\int_0^t R(t-s)[\int_Z h(s,x(s),v) - h(s,y(s),v)]\tilde{N}(ds,dv)\right\|^2 \\
&\leq \quad 2Tr(Q)M\int_0^t E\|f(s,x(s)) - f(s,y(s))\|^2 ds \\
&+ \quad 2M\int_0^t\int_Z E\|h(s,x(s),v) - h(s,y(s),v)\|^2 \lambda(dv)ds \\
&+ \quad 2M\left(\int_0^t\int_Z E\|h(s,x(s),v) - h(s,y(s),v)\|^4 \lambda(dv)ds\right)^{\frac{1}{2}}. \\
&\leq \quad 2M[Tr(Q)+2]\int_0^t l_n(s)E\|x(s) - y(s)\|^2 ds. \\
&\leq \quad \int_0^t \left[\bar{l}_n(s)e^{\tau L_n^*(s)}\right]\left[e^{-\tau L_n^*(s)}E\|x(s) - y(s)\|^2\right]ds \\
&\leq \quad \int_0^t \left[\bar{l}_n(s)e^{\tau L_n^*(s)}\right]ds\,\|x-y\|_n \\
&\leq \quad \int_0^t \frac{1}{\tau}\left[e^{\tau L_n^*(s)}\right]' ds\,\|x-y\|_n \\
&\leq \quad \frac{1}{\tau}e^{\tau L_n^*(t)}\|x-y\|_n.
\end{aligned}$$

Therefore,

$$\|\Phi(x)-\Phi(y)\|_n \quad \leq \quad \frac{1}{\tau}\|x-y\|_n .$$

So, the operator ϕ is a contraction for all $n \in \mathbb{N}$. From the choice of Ω, there is no $x \in \partial\Omega^n$ such that $x = \lambda\Phi(x)$ for some $\lambda \in (0,1)$. The statement (C2) does not hold. A consequence of the nonlinear alternative of Frigon and Granas show that (C1) holds. We deduce that the operator Φ has a unique fixed point x, which is the unique mild solution of the problem (1.1)-(1.2). The proof is completed. □

4. Controllability results

As an application of Theorem 3.1, we consider the following controllability for stochastic functional integro-differential evolution equations of the form

$$dx(t) = \left[Ax(t)+\int_0^t B(t-s)x(s)ds\right]dt + Cu(t)dt + f(t,x(t))dw(t) + \int_Z h(t,x(t),v)\tilde{N}(dt,dv), \tag{4.1}$$

$$x(0) = x_0, \tag{4.2}$$

where the control function $u(\cdot)$ is given in $L^2(J,\mathbb{U})$, the Banach space of admissible control functions with $\mathbb{U}$ is real separable Hilbert space with the norm $|\cdot|$, C is a bounded linear operator from $\mathbb{U}$ into $\mathbb{H}$ and functions $A, B(t-s), f$ and x_0 are as in problem (1.1)-(1.2). For more results on the controllability defined on a compact interval, we refer to [18, 19, 20, 21] and the references therein.

Definition 4.1. An $\mathscr{F}_t$-adapted stochastic process $x:[0,+\infty)\to\mathbb{H}$ is called a mild solution of the problem (4.1)-(4.2) if $x(0)=x_0\in L_2^0(\Omega,\mathbb{H}), x(t)$ is continuous and satisfies the following integral equation

$$\begin{aligned} x(t) \quad &= \quad R(t)x_0+\int_0^t R(t-s)Cu(s)ds+\int_0^t R(t-s)f(s,x(s))dw(s) \\ &+ \quad \int_0^t R(t-s)\int_Z h(s,x(s),v)\tilde{N}(ds,dv),\ t\in J=[0,+\infty). \end{aligned} \tag{4.3}$$

Definition 4.2. The system (4.1)-(4.2) is said to controllable if for every initial random variable $x_0 \in L_2^0(\Omega,\mathbb{H}), x^*\in\mathbb{H}$, and $n\in\mathbb{N}$, there is some $\mathscr{F}_t$-adapted stochastic control $u\in L^2([0,n],\mathbb{U})$ such that the mild solution $x(\cdot)$ of (4.1)-(4.2) satisfies the terminal condition $x(n)=x^*$.

In addition to conditions (H1)-(H3), we need the following assumption

(H4) For each $n\in\mathbb{N}$, the linear operator $W: L^2([0,n],\mathbb{U})\to L_2(\Omega,\mathbb{H})$, which is defined by

$$Wu \quad = \quad \int_0^n R(n-s)Cu(s)ds \tag{4.4}$$

has a pseudo invertible operator $\tilde{W}^{-1}$, which takes values in $L^2([0,n],\mathbb{U})/KerW$ and there exist positive constants M_1 and M_2 such that $\|C\|^2\leq M_1$, and $\left\|\tilde{W}^{-1}\right\|^2\leq M_2$.

Theorem 4.3. *Assume conditions (H1)-(H4) are satisfied and moreover for each $n\in\mathbb{N}$, there exists a constant $\wedge_n>0$ such that*

$$\frac{\wedge_n}{\beta_n+4M\left(Tr(Q+2)\right)\left[4MM_1M_2n^2+1\right]\psi(\wedge_n)\|p\|_{L^1_{[0,n]}}}>1, \tag{4.5}$$

with

$$\beta_n \quad = \quad \beta_n(x^*,x_0)=4ME\|x_0\|^2+16MM_1M_2n^2\left[E\|x^*\|^2+ME\|x_0\|^2\right].$$

Then (4.1)-(4.2) is controllable on J.

Proof. Fix $\tau > 1$. For every $n \in \mathbb{N}$, we define, in $B_{+\infty}$, the semi-norms

$$\|x\|_n = \sup\left\{e^{-\tau L_n^*(t)} E\|x(t)\|^2 : t \in [0,n]\right\},$$

where $L_n^*(t) = \int_0^t \bar{l}_n(s)ds$, and $\bar{l}_n(t) = 3m[Tr(Q+2)]\, l_n(t)[2MM_1M_2n^2+1]$ and l_n is the function from (H2). Then $B_{+\infty}$ is a Frechet space with the family of semi-norms we transform (4.1)-(4.2) into a fixed point problem. Consider the operator $\Gamma : B_{+\infty} \to B_{+\infty}$ defined by

$$\begin{aligned}\Gamma(x)(t) &= R(t)x_0 + \int_0^t R(t-s)Cu_x(s)ds \\ &+ \int_0^t R(t-s)f(s,x(s))dw(s) + \int_0^t R(t-s)\int_Z h(s,x(s),v)\tilde{N}(ds,dv),\ t \in J.\end{aligned}$$

Using condition (H3), for arbitrary function $x(\cdot)$, we define the control

$$u_x(t) = \tilde{W}^{-1}\left[x^* - R(n)x_0 - \int_0^n R(n-s)f(s,x(s))dw(s) - \int_0^n R(n-s)\int_Z h(s,x(s),v)\tilde{N}(ds,dv)\right](t)$$

Note that

$$\begin{aligned}E\|u_x(t)\|^2 \le \|\tilde{W}^{-1}\|E\|x^* - R(t)x_0 - \int_0^n R(n-\tau)f(\tau,x(\tau))dw(\tau) \\ - \int_0^n R(n-\tau)\int_Z h(\tau,x(\tau),v)\tilde{N}(d\tau,dv)\|^2.\end{aligned}$$

Applying (H1)-(H3), we get

$$E\|u_x(t)\|^2 \le 4M_2\left[E\|x^*\|^2 + ME\|x_0\|^2 + M[Tr(Q)+2]\int_o^n p(\tau)\psi(E\|x(\tau)\|^2)d\tau\right].$$

We shall show that Γ has a fixed point $x(\cdot)$. Then $x(\cdot)$ is a mild solution of system (4.1)-(4.2). Let $x \in B_{+\infty}$ be a possible fixed point of the operator Γ. By conditions (H1)-(H3), we have, for each $t \in [0,n]$, that

$$\begin{aligned}&E\|x(t)\|^2 \\ &\le 4E\|R(t)x_0\|^2 + 4E\left\|\int_0^t R(t-s)Cu_x(s)ds\right\|^2 \\ &\quad + 4E\left\|\int_0^t R(t-s)f(s,x(s))dw(s)\right\|^2 + 4E\left\|\int_0^t R(t-s)\int_Z h(s,x(s),v)\tilde{N}(ds,dv)\right\|^2 \\ &\le 4ME\|x_0\|^2 + 16MM_1M_2n\int_0^t\left[E\|x^*\|^2 + ME\|x_0\|^2 + M[Tr(Q)+2]\int_0^n p(\tau)\psi(E\|x(\tau)\|^2)d\tau\right]ds \\ &\quad + 4Tr(Q)M\int_0^t p(s)\psi(E\|x(s)\|^2)ds + 8M\int_0^t p(s)\psi(E\|x(s)\|^2)ds \\ &\le 4ME\|x_0\|^2 + 16MM_1M_2n^2E\|x^*\|^2 + 16M^2M_1M_2n^2E\|x_0\|^2 \\ &\quad + 16M^2M_1M_2n^2[Tr(Q)+2]\int_0^n p(s)\psi(E\|x(s)\|^2)ds + 8M[Tr(Q)+2]\int_0^t p(s)\psi(E\|x(s)\|^2)ds.\end{aligned}$$

Set $\beta_n = 4ME\|x_0\|^2 + 16MM_1M_2n^2\left[E\|x^*\|^2 + ME\|x_0\|^2\right]$. It follows that

$$\begin{aligned}E\|x(t)\|^2 &\le \beta_n + 16M^2M_1M_2n^2[Tr(Q)+2]\int_o^n p(s)\psi(E\|x(s)\|^2)ds \\ &+ 8M[Tr(Q)+2]\int_o^n p(s)\psi(E\|x(s)\|^2)ds.\end{aligned}$$

We consider the function μ defined by $\mu(t)=\sup\left\{E\left\|x(s)\right\|^2 : 0\leq s\leq t\right\}$, $0\leq t\leq +\infty$. Let $t^*\in[0,t]$ be such that $\mu(t)=E\left\|x(t^*)\right\|^2$. If $t^*\in[0,n]$, by the previous inequality, we have, for $t\in[0,n]$,

$$\mu(t) \quad \leq \quad \beta_n+16M^2M_1M_2n^2\left[Tr(Q)+2\right]\int_0^n p(s)\psi(\mu(s))ds+8M\left[Tr(Q)+2\right]\int_0^n p(s)\psi(\mu(s))ds.$$

It follows that

$$\mu(t) \quad \leq \quad \beta_n+4M\left(Tr(Q+2)\right)\left[4MM_1M_2n^2+1\right]\int_0^n p(s)\psi(\mu(s))ds.$$

Consequently, one has

$$\frac{\|x\|_n}{\beta_n+4M\left(Tr(Q+2)\right)\left[4MM_1M_2n^2+1\right]\psi(\|x\|_n)\|p\|_{L^1_{[0,n]}}}\leq 1.$$

From (4.3), there exists $\wedge_n$ such that $\mu(t)\leq\wedge_n$. Since $\|x\|_n\leq\mu(t)$, we have $\|x\|_n\leq\wedge_n$. Set

$$\Omega=\left\{x\in B_{+\infty} : \sup E\left\|x(t)\right\|^2 : 0\leq t\leq n\leq\wedge_n+1 \text{ for all } n\in\mathbb{N}\right\}.$$

Clearly, Ω is a closed subset of $B_{+\infty}$. We shall show that $\Gamma:\Omega\to B_{+\infty}$ is a contraction operator. Indeed, consider $x,y\in B_{+\infty}$. By (H1)-(H3), we find, for each $t\in[0,n]$ and $n\in\mathbb{N}$,

$$\begin{aligned}
E\left\|\Gamma(x)(t)-\Gamma(y)(t)\right\|^2 \quad &\leq \quad 3E\left\|\int_0^t R(t-s)C[u_x(s)-u_y(s)]ds\right\|^2\\
&+ \quad 3E\left\|\int_0^t R(t-s)[f(s,x(s))-f(s,y(s))]dw(s)\right\|^2\\
&+ \quad 3E\left\|\int_0^t R(t-s)\int_Z[h(s,x(s),v)-h(s,y(s),v)]\tilde{N}(ds,dv)\right\|^2\\
&\leq \quad 3MM_1n\int_0^t E\left\|\tilde{W}^{-1}\left[x^*-R(n)x_0-\int_0^n R(n-s)f(\tau,x(\tau))dw(\tau)\right.\right.\\
&- \quad \left.\int_0^n R(t-s)\int_Z h(\tau,x(\tau),v)\tilde{N}(d\tau,dv)\right]-\tilde{W}\left[x^*-R(n)x_0\right.\\
&- \quad \left.\left.\int_0^n R(n-s)f(\tau,y(\tau))dw(\tau)-\int_0^n R(n-s)\int_Z h(\tau,y(\tau),v)\tilde{N}(ds,dv)\right]\right\|^2 ds\\
&+ \quad 3M\left[Tr(Q+2)\right]\int_0^t l_n(s)E\left\|x(s)-y(s)\right\|^2 ds.\\
&\leq \quad 6M^2M_1M_2n^2\left[Tr(Q+2)\right]\int_0^t l_n(s)E\left\|x(s)-y(s)\right\|^2 ds\\
&+ \quad 3M\left[Tr(Q+2)\right]\int_0^t l_n(s)E\left\|x(s)-y(s)\right\|^2 ds\\
&\leq \quad \int_0^t [l_n(s)e^{\tau L_n^*(s)}][e^{-\tau L_n^*(s)}E\left\|x(s)-y(s)\right\|^2]ds\\
&\leq \quad \int_0^t [l_n(s)e^{\tau L_n^*(s)}]ds\left\|x-y\right\|_n\\
&\leq \quad \int_0^t \frac{1}{\tau}[e^{\tau L_n^*(s)}]'ds\left\|x-y\right\|_n\\
&\leq \quad \int_0^t \frac{1}{\tau}e^{\tau L_n^*(t)}\left\|x-y\right\|_n.
\end{aligned}$$

Therefore

$$\|\Gamma(x)-\Gamma(y)\|_n \quad \leq \quad \frac{1}{\tau}\|x-y\|_n .$$

So, Γ is a contraction for all $n \in \mathbb{N}$. From the choice of Ω, we see that there is no $x \in \partial\Omega^n$ such that $x = \lambda\Omega(x)$ for some $\lambda \in (0,1)$. Then the statement (C2) does not hold. A consequence of the nonlinear alternative of Frigon and Granas show that (C1) holds. We deduce that the operator Γ has a unique fixed point x. This fixed point is the solution of system (4.1)-(4.2). Clearly, $x(n) = (\Gamma x)(n) = x^*$ which implies that system (4.1)-(4.2) is controllable on J. The proof is completed. □

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