Commun. Optim. Theory 2018 (2018), Article ID 17 https://doi.org/10.23952/cot.2018.17

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org

OPTIMIZATION OF BOUNDARY VALUE PROBLEMS FOR THIRD ORDER POLYHEDRAL DIFFERENTIAL INCLUSIONS

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Abstract. We are concerned with sufficient conditions of optimality for third order polyhedral optimization described by polyhedral differential inclusions. The goal of this paper is to derive sufficient conditions of optimality for Lagrange and Bolza problem with boundary value constraint. Sufficient conditions, including distinctive transversality ones, are formulated by incorporating the Euler-Lagrange type of inclusions. The applications of these results are demonstrated by solving the problems with third order linear differential inclusions.

Keywords. Polyhedral; Differential Inclusions; Euler-Lagrange; Transversality.

2010 Mathematics Subject Classification. 49K15, 49J52, 90C25.

1. Introduction

We study a third order polyhedral optimization problem (P_C) given by third order polyhedral differential inclusions:

$$\text{minimize } J_1(x(\cdot)) = \int_0^1 g(x(t),t)dt \tag{1.1}$$

$$(P_C) \quad \text{subject to}$$

$$x'''(t) \in F(x(t),x'(t),x''(t)) \quad \text{a.e.} \quad t \in [0,1], \tag{1.2}$$

$$x(0) \in M_0,\ x'(0) \in M_1,\ x''(0) \in M_2,$$

$$x(1) \in N_0,\ x'(1) \in N_1,\ x''(1) \in N_2. \tag{1.3}$$

Here $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to P(\mathbb{R}^n)$ is the polyhedral set-valued mapping, $P(\mathbb{R}^n)$ is the set of all subsets of $\mathbb{R}^n$. Let us define $F(x,v_1,v_2) := \{v_3 : P_0x + P_1v_1 + P_2v_2 - Qv_3 \leq d\}$ where P_0, P_1, P_2 and Q are $m \times n$

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Received January 19, 2018; Accepted June 21, 2018.

matrices and d is a column vector of length m. Furthermore, M_i and N_i, $i = 0,1,2$, are polyhedral set and $g(\cdot,t) : \mathbb{R}^n \to \mathbb{R}$ is a polyhedral function, i.e. its epigraph epi $g(\cdot,t)$ is polyhedral set in $\mathbb{R}^{n+1}$.

We have to find a solution $\tilde{x}(t)$ of the problem $(1.1)-(1.3)$ for the third order differential inclusions satisfying (1.2) almost everywhere (a.e.) on $[0,1]$ and the initial and the end point conditions (1.3) minimizing the functional $J_1(x(\cdot))$. A feasible trajectory $x(t)$, $t \in [0,1]$ is an absolutely continuous function together with the second order derivatives for which $x^{'''}(t) \in L_1^n([0,1])$. Clearly such class of functions is a Banach space endowed with different equivalent norms. The principal method that we use to obtain the sufficient conditions of optimality for the problem (P_C) is a method based on the discrete approximations. The basic idea is to replace the continuous problem (P_C) with the discrete-approximation problem that can be studied effectively. To accomplish this goal, by converting the discrete problem into a problem with geometric constraints, necessary and sufficient conditions of optimality for a convex minimization problem with given by linear inequality constraints are formulated. Then via rewriting discrete-approximation problem associated with continuous problem to the form of the discrete problem, conditions of optimality for discrete-approximation problem are obtained. Finally by passing formally to the limit as $\delta \to 0$, where δ is a discrete step, the sufficient optimality conditions for the differential problem is established.

We approximate differential inclusions under consideration by differential (or even discrete-time) inclusions with simpler structure. We then solve the problem for the simplified dynamical systems and obtain the result for the original problem using a limiting procedure. This is the central idea of many proofs in the text that allows us to avoid complicated mathematical constructions. Most of research in optimization problems for differential inclusions consists of obtaining sufficient conditions for optimality [1, 2, 3]. Also optimal control problems for systems described by discrete inclusions have been studied by Clarke [4]. Moreover, optimal control problems with ordinary discrete and differential inclusions are one of the areas in mathematical theory of optimal processes being intensively developed [2, 3, 5, 6, 7, 8, 9]. The differential inclusions are not only models for many dynamical processes but also they provide a powerful tool for various branches of mathematical analysis [10].

In a wide range of mathematical problems the existence of a solution is equivalent to the existence of a fixed point for a suitable map. The existence of a fixed point is therefore of paramount importance in several areas of mathematics and other sciences [11, 12]. In Aubin and Cellina, the emphasis is given to differential problems, existence theorems and viability theory [1]. The results on the differentiability with respect to the boundary conditions were studied by many authors [13, 14]. It can be noticed that the works of Mahmudov devoted to optimization of ordinary discrete and differential inclusions, discrete and differential inclusions with distributed parameters and their duality problems [2, 8, 15, 16, 17]. In view of applications of the class of problems with polyhedral discrete and differential inclusions are very interesting. The present paper is dedicated to one of the interesting fields-optimization of the boundary value problems for third order ordinary differential inclusions. The posed problems and the corresponding optimality conditions are new.

The paper is organized as follows. In Section 2, we proved the sufficient conditions of optimality for the problem (P_C). In the present paper, one of the important problems is to formulate the transversality conditions at the end of the considered time interval $t = 0$ and $t = 1$ for Bolza problem with cost functional $J_2(x(\cdot))$. Therefore transversality conditions for mentioned Bolza problem are obtained in Corollary 2.2.

In Section 3, we are able to use the results presented in Section 2 to get sufficient conditions for optimality of third order differential inclusions. The applications of these results are demonstrated by solving the problems with third order linear differential inclusions. It is interesting to note that Euler-Poisson equation and the obtained conditions for a linear problem are of the same nature concerning the problems with third order derivatives.

2. Sufficient Conditions of Optimality for Third Order Differential Inclusions

In this section we present the main theorem of this paper. We will call the inclusion (a) in the following theorem as the Mahmudov's inclusion which coincides with the Euler-Lagrange inclusion for first order differential inclusions. We remind that our notation and terminology are generally consistent with those in Mahmudov [2], Mordukhovich [3] for optimization of first order differential inclusions.

Theorem 2.1. *In order for trajectory $\tilde{x}(t)$,$t \in [0,1]$ to be an optimal solution of the third order polyhedral differential inclusions of the problem (P_C), it is sufficient that there exists an absolutely continuous function $x^*(t)$ satisfying the following third order adjoint differential inclusion almost everywhere*

$$(a)\quad -\frac{d^3x^*(t)}{dt^3} \in P_0^*\lambda(t) - P_1^*\frac{d\lambda(t)}{dt} + P_2^*\frac{d^2\lambda(t)}{dt^2} + \partial g(\tilde{x}(t),t)\ ,\ a.e.\ \ t \in [0,1],$$
$$x^*(t) = Q^*\lambda(t),\ \textit{where}\ P_i^*\ (i=0,1,2)\ \textit{and}\ Q^*\ \textit{are transposed matrices,}$$
$$(b)\quad \left\langle P_0\tilde{x}(t) + P_1\frac{d\tilde{x}(t)}{dt} + P_2\frac{d^2\tilde{x}(t)}{dt^2} - Q\frac{d^3\tilde{x}(t)}{dt^3} - d\ ,\ \lambda(t)\right\rangle = 0\ ,\ a.e.\ t \in [0,1],$$

and transversality conditions at the initial and the end points, respectively

$$(c)\quad \frac{d^2x^*(0)}{dt^2} + P_2^*\frac{d\lambda(0)}{dt} - P_1^*\lambda(0) \in K_{M_0}^*\big(\tilde{x}(0)\big),$$
$$-\frac{dx^*(0)}{dt} - P_2^*\lambda(0) \in K_{M_1}^*\big(\tilde{x}'(0)\big)\ ,$$
$$x^*(0) \in K_{M_2}^*\big(\tilde{x}''(0)\big),$$
$$(d)\quad -\frac{d^2x^*(1)}{dt^2} - P_2^*\frac{d\lambda(1)}{dt} + P_1^*\lambda(1) \in K_{N_0}^*\big(\tilde{x}(1)\big),$$
$$\frac{dx^*(1)}{dt} + P_2^*\lambda(1) \in K_{N_1}^*\big(\tilde{x}'(1)\big)\ ,$$
$$-x^*(1) \in K_{N_2}^*\big(\tilde{x}''(1)\big).$$

Proof. By definition of subdifferential for all feasible solutions, inclusion (a) of theorem implies that

$$g(x(t),t) - g(\tilde{x}(t),t) \geq \left\langle -\frac{d^3x^*(t)}{dt^3} - P_0^*\lambda(t) + P_1^*\frac{d\lambda(t)}{dt} - P_2^*\frac{d^2\lambda(t)}{dt^2}\ ,x(t)-\tilde{x}(t)\right\rangle. \tag{2.1}$$

We rewrite the right hand side of the inequality (2.1) in the form

$$\left\langle -\frac{d^3x^*(t)}{dt^3} - P_0^*\lambda(t) + P_1^*\frac{d\lambda(t)}{dt} - P_2^*\frac{d^2\lambda(t)}{dt^2},\ x(t)-\tilde{x}(t)\right\rangle = \left\langle -\frac{d^3x^*(t)}{dt^3},\ x(t)-\tilde{x}(t)\right\rangle$$
$$-\left\langle \lambda(t), P_0\big(x(t)-\tilde{x}(t)\big)\right\rangle + \left\langle \frac{d\lambda(t)}{dt},\ P_1\big(x(t)-\tilde{x}(t)\big)\right\rangle - \left\langle \frac{d^2\lambda(t)}{dt^2},\ P_2\big(x(t)-\tilde{x}(t)\big)\right\rangle. \tag{2.2}$$

For all feasible solutions $x(\cdot)$ and for $\lambda(t) \geq 0, t \in [0,1]$, it is easy to observe that

$$\Big\langle P_0 x(t) + P_1 \frac{dx(t)}{dt} + P_2 \frac{d^2x(t)}{dt^2}, \lambda(t) \Big\rangle \leq \Big\langle Q \frac{d^3x(t)}{dt^3} + d, \lambda(t) \Big\rangle.$$

From the construction of the second condition (b) of theorem, we can write

$$\Big\langle P_0 \tilde{x}(t) + P_1 \frac{d\tilde{x}(t)}{dt} + P_2 \frac{d^2\tilde{x}(t)}{dt^2}, \lambda(t) \Big\rangle = \Big\langle Q \frac{d^3\tilde{x}(t)}{dt^3} + d, \lambda(t) \Big\rangle.$$

By using the equation $x^*(t) = Q^* \lambda(t)$ and the last two relations, we obtain that

$$\Big\langle P_0\big(x(t)-\tilde{x}(t)\big) + P_1 \frac{d\big(x(t)-\tilde{x}(t)\big)}{dt} + P_2 \frac{d^2\big(x(t)-\tilde{x}(t)\big)}{dt^2} \ , \ \lambda(t) \Big\rangle \leq \Big\langle \frac{d^3\big(x(t)-\tilde{x}(t)\big)}{dt^3} \ , \ x^*(t) \Big\rangle. \tag{2.3}$$

Then from (2.2) and (2.3), the latter relation yields

$$\begin{aligned}
\Big\langle -\frac{d^3x^*(t)}{dt^3} - P_0^*\lambda(t) + P_1^*\frac{d\lambda(t)}{dt} - P_2^*\frac{d^2\lambda(t)}{dt^2}, x(t)-\tilde{x}(t) \Big\rangle &\geq \Big\langle -\frac{d^3x^*(t)}{dt^3}, x(t)-\tilde{x}(t) \Big\rangle \\
-\Big\langle \frac{d^3\big(x(t)-\tilde{x}(t)\big)}{dt^3}, x^*(t) \Big\rangle + \Big\langle P_1 \frac{d\big(x(t)-\tilde{x}(t)\big)}{dt}, \lambda(t) \Big\rangle &+ \Big\langle P_2 \frac{d^2\big(x(t)-\tilde{x}(t)\big)}{dt^2}, \lambda(t) \Big\rangle \\
+\Big\langle \frac{d\lambda(t)}{dt}, P_1\big(x(t)-\tilde{x}(t)\big) \Big\rangle &- \Big\langle \frac{d^2\lambda(t)}{dt^2}, P_2\big(x(t)-\tilde{x}(t)\big) \Big\rangle.
\end{aligned}$$

This inequality consequently implies the following inequality

$$\begin{aligned}
\Big\langle -\frac{d^3x^*(t)}{dt^3} - P_0^*\lambda(t) + P_1^*\frac{d\lambda(t)}{dt} - P_2^*\frac{d^2\lambda(t)}{dt^2}, x(t)-\tilde{x}(t) \Big\rangle &\geq \Big\langle -\frac{d^3x^*(t)}{dt^3}, x(t)-\tilde{x}(t) \Big\rangle \\
-\Big\langle \frac{d^3\big(x(t)-\tilde{x}(t)\big)}{dt^3}, x^*(t) \Big\rangle &+ \frac{d}{dt}\Big[\Big\langle P_1\big(x(t)-\tilde{x}(t)\big), \lambda(t) \Big\rangle\Big] \\
+\frac{d}{dt}\Big[\Big\langle P_2 \frac{d\big(x(t)-\tilde{x}(t)\big)}{dt}, \lambda(t) \Big\rangle\Big] &- \frac{d}{dt}\Big[\Big\langle \frac{d\lambda(t)}{dt}, P_2\big(x(t)-\tilde{x}(t)\big) \Big\rangle\Big].
\end{aligned} \tag{2.4}$$

Clearly from the inequalities (2.1) and (2.4), we have

$$\begin{aligned}
g(x(t),t) - g(\tilde{x}(t),t) &\geq \Big\langle -\frac{d^3x^*(t)}{dt^3}, x(t)-\tilde{x}(t) \Big\rangle - \Big\langle \frac{d^3\big(x(t)-\tilde{x}(t)\big)}{dt^3}, x^*(t) \Big\rangle \\
&+\frac{d}{dt}\Big[\Big\langle P_1^*\lambda(t) - P_2^*\frac{d\lambda(t)}{dt}, x(t)-\tilde{x}(t) \Big\rangle\Big] + \frac{d}{dt}\Big[\Big\langle P_2^*\lambda(t), \frac{d\big(x(t)-\tilde{x}(t)\big)}{dt} \Big\rangle\Big].
\end{aligned} \tag{2.5}$$

Then integrating the inequality (2.5), we obtain

$$\begin{aligned}
\int_0^1 \Big(g(x(t),t) - g(\tilde{x}(t),t)\Big)dt &\geq \int_0^1 \Big[\Big\langle -\frac{d^3x^*(t)}{dt^3}, x(t)-\tilde{x}(t) \Big\rangle - \Big\langle \frac{d^3\big(x(t)-\tilde{x}(t)\big)}{dt^3}, x^*(t) \Big\rangle\Big]dt \\
&+\Big\langle P_1^*\lambda(1) - P_2^*\frac{d\lambda(1)}{dt}, x(1)-\tilde{x}(1) \Big\rangle - \Big\langle P_1^*\lambda(0) - P_2^*\frac{d\lambda(0)}{dt}, x(0)-\tilde{x}(0) \Big\rangle \\
&+\Big\langle P_2^*\lambda(1), x'(1)-\tilde{x}'(1) \Big\rangle - \Big\langle P_2^*\lambda(0), x'(0)-\tilde{x}'(0) \Big\rangle.
\end{aligned} \tag{2.6}$$

Let us transform the following integral

$$\int_0^1 \Big[\Big\langle -\frac{d^3x^*(t)}{dt^3}, x(t)-\tilde{x}(t) \Big\rangle - \Big\langle \frac{d^3\big(x(t)-\tilde{x}(t)\big)}{dt^3}, x^*(t) \Big\rangle\Big]dt. \tag{2.7}$$

In the framework of the classical calculus of derivations we have

$$\Big[\Big\langle -\frac{d^3x^*(t)}{dt^3}, x(t)-\tilde{x}(t) \Big\rangle - \Big\langle \frac{d^3\big(x(t)-\tilde{x}(t)\big)}{dt^3}, x^*(t) \Big\rangle\Big] = -\frac{d}{dt}\Big\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t) \Big\rangle$$

$$-\frac{d}{dt}\Big\langle\frac{d^2(x(t)-\tilde{x}(t))}{dt^2},x^*(t)\Big\rangle+\frac{d}{dt}\Big\langle\frac{dx^*(t)}{dt},\frac{d(x(t)-\tilde{x}(t))}{dt}\Big\rangle,$$

and this transformation allows us to rigorously compute the integral in formula (2.7) as follows:

$$\int_0^1\Big[\Big\langle-\frac{d^3x^*(t)}{dt^3},x(t)-\tilde{x}(t)\Big\rangle-\Big\langle\frac{d^3\big(x(t)-\tilde{x}(t)\big)}{dt^3},x^*(t)\Big\rangle\Big]dt=-\Big\langle\frac{d^2x^*(1)}{dt^2},x(1)-\tilde{x}(1)\Big\rangle$$
$$+\Big\langle\frac{d^2x^*(0)}{dt^2},x(0)-\tilde{x}(0)\Big\rangle-\Big\langle\frac{d^2\big(x(1)-\tilde{x}(1)\big)}{dt^2},x^*(1)\Big\rangle+\Big\langle\frac{d^2\big(x(0)-\tilde{x}(0)\big)}{dt^2},x^*(0)\Big\rangle$$
$$+\Big\langle\frac{dx^*(1)}{dt},\frac{d\big(x(1)-\tilde{x}(1)\big)}{dt}\Big\rangle-\Big\langle\frac{dx^*(0)}{dt},\frac{d\big(x(0)-\tilde{x}(0)\big)}{dt}\Big\rangle. \tag{2.8}$$

Hence, taking into account (2.8) from the inequality (2.6), we conclude

$$\int_0^1\Big(g(x(t),t)-g(\tilde{x}(t),t)\Big)dt\geq\Big\langle-\frac{d^2x^*(1)}{dt^2}-P_2^*\frac{d\lambda(1)}{dt}+P_1^*\lambda(1),\ x(1)-\tilde{x}(1)\Big\rangle$$
$$+\Big\langle\frac{d^2x^*(0)}{dt^2}+P_2^*\frac{d\lambda(0)}{dt}-P_1^*\lambda(0),\ x(0)-\tilde{x}(0)\Big\rangle+\Big\langle\frac{dx^*(1)}{dt}+P_2^*\lambda(1),\ x'(1)-\tilde{x}'(1)\Big\rangle$$
$$-\Big\langle\frac{dx^*(0)}{dt}+P_2^*\lambda(0),\ x'(0)-\tilde{x}'(0)\Big\rangle-\Big\langle x^*(1),\ x''(1)-\tilde{x}''(1)\Big\rangle+\Big\langle x^*(0),x''(0)-\tilde{x}''(0)\Big\rangle. \tag{2.9}$$

By using definition of the dual to a cone of tangent vectors and from the condition (c) of theorem, we have

$$\Big\langle\frac{d^2x^*(0)}{dt^2}+P_2^*\frac{d\lambda(0)}{dt}-P_1^*\lambda(0),\ x(0)-\tilde{x}(0)\Big\rangle\geq 0\,,\quad\Big\langle-\frac{dx^*(0)}{dt}-P_2^*\lambda(0),\ x'(0)-\tilde{x}'(0)\Big\rangle\geq 0,$$
$$\Big\langle x^*(0),\ x''(0)-\tilde{x}''(0)\Big\rangle\geq 0,$$

and from the condition (d), we write

$$\Big\langle-\frac{d^2x^*(1)}{dt^2}-P_2^*\frac{d\lambda(1)}{dt}+P_1^*\lambda(1),\ x(1)-\tilde{x}(1)\Big\rangle\geq 0\,,\quad\Big\langle\frac{dx^*(1)}{dt}+P_2^*\lambda(1),\ x'(1)-\tilde{x}'(1)\Big\rangle\geq 0,$$
$$\Big\langle-x^*(1),\ x''(1)-\tilde{x}''(1)\Big\rangle\geq 0.$$

Therefore, by rewriting inequality (2.9), we deduce that

$$\int_0^1\Big(g(x(t),t)-g(\tilde{x}(t),t)\Big)dt\geq 0, \tag{2.10}$$

i.e., $J_1(x(\cdot))-J_1(\tilde{x}(\cdot))\geq 0$ for all feasible solutions $x(t)$ and so $\tilde{x}(t)$ is optimal. □

Corollary 2.2. *Let us consider the Bolza problem with cost functional*

$$J_2(x(\cdot))=\int_0^1 g(x(t),t)dt+\varphi_0(x(1),x'(1),x''(1))$$

and differential inclusion (1.2) for boundary value conditions (1.3), where $g(\cdot,t)$ and φ_0 are polyhedral functions. Then for optimality of the trajectory $\widetilde{x}(t)$ in the Bolza problem the transversality conditions at points 0 and 1 should be as follows:

$$(e)\qquad \frac{d^2x^*(0)}{dt^2}+P_2^*\frac{d\lambda(0)}{dt}-P_1^*\lambda(0)\in K_{M_0}^*\big(\widetilde{x}(0)\big),$$

$$-\frac{dx^*(0)}{dt}-P_2^*\lambda(0)\in K^*_{M_1}\left(\widetilde{x}'(0)\right),\quad x^*(0)\in K^*_{M_2}\left(\widetilde{x}''(0)\right),$$

$$(f)\ -P_2^*\frac{d\lambda(1)}{dt}+P_1^*\lambda(1)\in K^*_{N_0}\left(\widetilde{x}(1)\right),\quad P_2^*\lambda(1)\in K^*_{N_1}\left(\widetilde{x}'(1)\right),\quad -x^*(1)\in K^*_{N_2}\left(\widetilde{x}''(1)\right),$$

$$\Big(\frac{d^2x^*(1)}{dt^2},-\frac{dx^*(1)}{dt},x^*(1)\Big)\in\partial\varphi_0\left(\widetilde{x}(1),\widetilde{x}'(1),\widetilde{x}''(1)\right).$$

Proof. Indeed, since $\Big(\frac{d^2x^*(1)}{dt^2},-\frac{dx^*(1)}{dt},x^*(1)\Big)\in\partial\varphi_0(\widetilde{x}(1),\widetilde{x}'(1),\widetilde{x}''(1))$, we have

$$\varphi_0\left(x(1),x'(1),x''(1)\right)-\varphi_0\left(\tilde{x}(1),\tilde{x}'(1),\tilde{x}''(1)\right)\geq\Big\langle\frac{d^2x^*(1)}{dt^2},x(1)-\tilde{x}(1)\Big\rangle$$
$$+\Big\langle-\frac{dx^*(1)}{dt},\,x'(1)-\tilde{x}'(1)\Big\rangle+\Big\langle x^*(1),\,x''(1)-\tilde{x}''(1)\Big\rangle.\tag{2.11}$$

Adding the inequalities (2.9) and (2.11) and taking into account the other transversality conditions, we have

$$\int_0^1 g(x(t),t)dt+\varphi_0(x(1),x'(1),x''(1))\geq\int_0^1 g(\tilde{x}(t),t)dt+\varphi_0(\tilde{x}(1),\tilde{x}'(1),\tilde{x}''(1)).$$

Therefore for all feasible solutions $x(t)$, we get $J_2(x(\cdot))-J_2(\tilde{x}(\cdot))\geq 0$. Therefore $\tilde{x}(t)$ is optimal. □

Corollary 2.3. *Let us consider Bolza problem with cost functional $J_2(x(\cdot))$ and differential inclusion (1.2) for initial conditions*

$$x(0)=\alpha_0,\ x'(0)=\alpha_1,\ x''(0)=\alpha_2,$$

where $\alpha_0,\alpha_1,\alpha_2$ are fixed vectors. Then for optimality of the trajectory $\widetilde{x}(t)$ in the Bolza problem the transversality conditions will be as follows:

$$\Big(\frac{d^2x^*(1)}{dt^2}+P_2^*\frac{d\lambda(1)}{dt}-P_1^*\lambda(1)\,,\,-\frac{dx^*(1)}{dt}-P_2^*\lambda(1)\,,\,x^*(1)\Big)\in\partial\varphi_0\left(\tilde{x}(1),\tilde{x}'(1),\tilde{x}''(1)\right).$$

Proof. Taking into account the formula (2.9) and by using $x(\cdot),\tilde{x}(\cdot)$ are feasible solutions, we have

$$\int_0^1\Big(g(x(t),t)-g(\tilde{x}(t),t)\Big)dt\ \geq\ \Big\langle-\frac{d^2x^*(1)}{dt^2}-P_2^*\frac{d\lambda(1)}{dt}+P_1^*\lambda(1),\ x(1)-\tilde{x}(1)\Big\rangle$$

$$+\Big\langle\frac{dx^*(1)}{dt}+P_2^*\lambda(1),\ x'(1)-\tilde{x}'(1)\Big\rangle-\Big\langle x^*(1),\ x''(1)-\tilde{x}''(1)\Big\rangle.\tag{2.12}$$

By definition of subdifferential for all feasible solutions, we write

$$\varphi_0\left(x(1),x'(1),x''(1)\right)-\varphi_0\left(\tilde{x}(1),\tilde{x}'(1),\tilde{x}''(1)\right)\geq\Big\langle\frac{d^2x^*(1)}{dt^2}+P_2^*\frac{d\lambda(1)}{dt}-P_1^*\lambda(1)\,,x(1)-\tilde{x}(1)\Big\rangle$$
$$+\Big\langle-\frac{dx^*(1)}{dt}-P_2^*\lambda(1)\,,\,x'(1)-\tilde{x}'(1)\Big\rangle+\Big\langle x^*(1)\,,\,x''(1)-\tilde{x}''(1)\Big\rangle.\tag{2.13}$$

By summing the inequalities (2.12) and (2.13), we conclude that

$$\int_0^1 g(x(t),t)dt+\varphi_0(x(1),x'(1),x''(1))\geq\int_0^1 g(\tilde{x}(t),t)dt+\varphi_0(\tilde{x}(1),\tilde{x}'(1),\tilde{x}''(1))$$

i.e., $J_2(x(\cdot))-J_2(\tilde{x}(\cdot))\geq 0$ for all feasible solutions $x(t)$. So, $\tilde{x}(t)$ is optimal. □

3. Applications of Third Order Optimization for Boundary Value Problem and Cauchy Problem

Now an application of these results is demonstrated by solving the variational problem of a functional with a single function, but containing its second order derivations. Accordingly, boundary value problem is this:

$$(P_L)\quad \begin{aligned} &\text{minimize } J_3(x(\cdot)) = \int_0^1 L(x(t),x'(t),x''(t),t)dt \\ &\text{subject to} \\ &x'''(t) \in F(x(t),x'(t),x''(t)) \quad \text{a.e.} \quad t \in [0,1], \\ &x(0) = \alpha_0, \quad x'(0) = \alpha_1, \quad x''(0) = \alpha_2, \\ &x(1) = \beta_0, \quad x'(1) = \beta_1, \quad x''(1) = \beta_2, \end{aligned}$$

where F is a polyhedral set-valued mapping, the Lagrangian L is a real-valued function with continuous first partial derivatives and $x(\cdot) \in C^2([0,1])$, α_i, β_i, $i = 0,1,2$ are fixed vectors. Let us denote

$$\frac{\partial L}{\partial x^{(k)}} = \frac{\partial L\big(\widetilde{x}(t),\widetilde{x}'(t),\widetilde{x}''(t)\big)}{\partial x^{(k)}},\ k = 0,1,2,$$

and suppose that

$$L(x,v_1,v_2) - L\big(\widetilde{x}(t),\widetilde{x}'(t),\widetilde{x}''(t)\big) \geq \Big\langle \frac{\partial L}{\partial x}\ ,\ x - \widetilde{x}(t) \Big\rangle + \Big\langle \frac{\partial L}{\partial x'}\ ,\ v_1 - \widetilde{x}'(t) \Big\rangle + \Big\langle \frac{\partial L}{\partial x''}\ ,\ v_2 - \widetilde{x}''(t) \Big\rangle \quad ,\forall\ (x,v_1,v_2) \in \mathbb{R}^{3n}.$$

Then by Theorem 4.3 [17] for optimality of $\widetilde{x}(t)$ the Euler-Lagrange inclusion is valid:

$$\left(-\frac{d^3x^*(t)}{dt^3} + \frac{d\psi_2^*(t)}{dt} + \frac{\partial L}{\partial x}\ ,\ \psi_2^*(t) + \frac{d\psi_1^*(t)}{dt} + \frac{\partial L}{\partial x'}\ ,\ \psi_1^*(t) + \frac{\partial L}{\partial x''} \right) \in F^*(x^*(t);(x(t),x'(t),x''(t),x'''(t)),t). \tag{3.1}$$

Suppose now that in the problem (P_L), dom $F(\cdot,t) \equiv (\mathbb{R}^n)^3$ and $F(\cdot,t) \equiv \mathbb{R}^n$. Then, obviously, $F^* \equiv \{(0,0,0)\}$ so that $x^* = v_1^* = v_2^* = 0$. It means that $x^{*(k)}(t) \equiv 0$, $t \in [0,1]$, $k = 0,1,2,3$. Therefore, it follows from (3.1) that

$$\frac{d\psi_2^*(t)}{dt} + \frac{\partial L}{\partial x} = 0\,, \quad \psi_2^*(t) + \frac{d\psi_1^*(t)}{dt} + \frac{\partial L}{\partial x'} = 0\,, \quad \psi_1^*(t) + \frac{\partial L}{\partial x''} = 0. \tag{3.2}$$

Now, beginning from the last relation by sequentially substitution in (3.2), we derive the Euler-Poisson equation for problem (P_L)

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\Big(\frac{\partial L}{\partial x'}\Big) + \frac{d^2}{dt^2}\Big(\frac{\partial L}{\partial x''}\Big) = 0. \tag{3.3}$$

It is well known that in calculus of variations for optimality of trajectory $x(t)$ the Euler-Poisson equation (3.3) is necessary condition.

Remark 3.1. We note that for the problem of calculus of variations (P_L) the transversality conditions of Theorem 2.1 is superfluous. Indeed for problem (P_L), $M_k = \{\alpha_k\}$, $N_k = \{\beta_k\}$, $k = 0,1,2$ and also

$$K_{M_k}\Big(\widetilde{x}^{(k)}(0)\Big) = \{0\} \text{ and } K_{N_k}\Big(\widetilde{x}^{(k)}(1)\Big) = \{0\},\ k = 0,1,2.$$

Consequently,

$$K^*_{M_k}\left(\widetilde{x}^{(k)}(0)\right)=\mathbb{R}^n \text{ and } K^*_{N_k}\left(\widetilde{x}^{(k)}(1)\right)=\mathbb{R}^n,\ k=0,1,2.$$

Therefore, the transversality conditions at the end points $t=0$ and $t=1$ for problem (P_L) is superfluous.

Suppose now we have so-called linear continuous problem for third order differential inclusions:

$$\text{minimize } J_2(x(\cdot))=\int_0^1 g(x(t),t)dt+\varphi(x(1),x'(1),x''(1)) \tag{3.4}$$

subject to

$$x'''(t)\in F(x(t),x'(t),x''(t)) \quad \text{a.e.} \quad t\in[0,1],$$
$$F(x,v_1,v_2)=\{v_3:\ v_3=A_0x+A_1v_1+A_2v_2+Bu,\ u\in U\},$$
$$x(0)=\alpha_0,\ x'(0)=\alpha_1,\ x''(0)=\alpha_2,$$

where $g(\cdot,t)$ and φ are continuously differentiable functions, A_i, $i=0,1,2$ and B are $n\times n$ and $n\times r$ matrices, respectively, U is a polyhedral subset of $\mathbb{R}^r$. The problem is to find a controlling parameter $\widetilde{u}(t)\in U$ such that the arc $\widetilde{x}(t)$ corresponding to it minimizes $J_2[x(\cdot)]$.

Corollary 3.2. *The arc $\widetilde{x}(t)$ corresponding to the controlling parameter $\widetilde{u}(t)$ minimizes $J_2[x(\cdot)]$ in the problem (3.4) if there exists an absolutely continuous function $x^*(t)$ satisfying third order adjoint differential inclusion(equation) (3.5), the transversality (3.6) and Weierstrass-Pontryagin conditions (3.7):*

$$-\frac{d^3x^*(t)}{dt^3}+A_0^*x^*(t)-A_1^*\frac{dx^*(t)}{dt}+A_2^*\frac{d^2x^*(t)}{dt^2}=g'(\tilde{x}(t),t)\ ,\ a.e.\ t\in[0,1], \tag{3.5}$$

$$\frac{d^2x^*(1)}{dt^2}+A_1^*x^*(1)-A_2^*\frac{dx^*(1)}{dt}=\varphi'_x(\tilde{x}(1),\tilde{x}'(1),\tilde{x}''(1)),$$
$$-\frac{dx^*(1)}{dt}+A_2^*x^*(1)=\varphi'_{v_1}(\tilde{x}(1),\tilde{x}'(1),\tilde{x}''(1))\ , \tag{3.6}$$
$$x^*(1)=\varphi'_{v_2}(\tilde{x}(1),\tilde{x}'(1),\tilde{x}''(1))\ ,$$

$$\left\langle B\widetilde{u}(t),\ x^*(t)\right\rangle=\sup_{u\in U}\left\langle Bu,\ x^*(t)\right\rangle. \tag{3.7}$$

Proof. By elementary computations, we find that

$$F^*\left(v_3^*;(x,v_1,v_2)\right)=\begin{cases}\left(A_0^*v_3^*\ ,\ A_1^*v_3^*\ ,\ A_2^*v_3^*\right)\ , & -B^*v_3^*\in K_U^*(u),\\ \emptyset\ , & -B^*v_3^*\notin K_U^*(u),\end{cases} \tag{3.8}$$

where $v_3=A_0x+A_1v_1+A_2v_2+Bu$, $u\in U$, $A_i^*(i=0,1,2)$ and B^* are transposed matrices. Then by using the definition of LAM, we have the following formula

$$\left(-P_0^*\lambda(t)\ ,\ -P_1^*\lambda(t)\ ,\ -P_2^*\lambda(t)\right)\in F^*\left(x^*(t);\left(\tilde{x}(t),\tilde{x}'(t),\tilde{x}''(t),\tilde{x}'''(t)\right)\right)$$

where $\lambda(t)\geq 0$ and taking into account inclusion (a) of Theorem 2.1, we conclude

$$-\frac{d^3x^*(t)}{dt^3}+A_0^*x^*(t)-A_1^*\frac{dx^*(t)}{dt}+A_2^*\frac{d^2x^*(t)}{dt^2}=g'(\tilde{x}(t),t)\ ,\ \text{a.e. } t\in[0,1].$$

Then by using Corollary 2.3, we can establish the following results

$$\frac{d^2x^*(1)}{dt^2}+A_1^*x^*(1)-A_2^*\frac{dx^*(1)}{dt}=\varphi'_x(\tilde{x}(1),\tilde{x}'(1),\tilde{x}''(1))\ ,$$

$$-\frac{dx^*(1)}{dt}+A_2^*x^*(1)=\varphi'_{v_1}(\widetilde{x}(1),\widetilde{x}'(1),\widetilde{x}''(1))\ ,$$
$$x^*(1)=\varphi'_{v_2}(\widetilde{x}(1),\widetilde{x}'(1),\widetilde{x}''(1))\ .$$

Besides $-B^*v_3^*\in K_U^*(u)$ means that the Weierstrass-Pontryagin maximum condition

$$\left\langle B\widetilde{u}(t),\ x^*(t)\right\rangle=\sup_{u\in U}\left\langle Bu,\ x^*(t)\right\rangle$$

satisfied.

Acknowledgements

The first author was supported by Scientific Research Projects Coordination Unit of Istanbul University, under grant Project number IRP-52983.

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