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SOLVABILITY OF A SYSTEM OF FRACTIONAL HYBRID DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the existence of solutions of a system of fractional hybrid differential equations. The existence of solutions is derived from Leray-Schauder's alternative, while the uniqueness of solutions is established by means of Banach's contraction mapping principle. An illustrative example is also included.

Keywords. Caputo derivative; Coupled system; Hybrid differential equation; Fixed point; Existence.

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1. INTRODUCTION

In this paper, we discuss the existence and uniqueness of solutions for the following system of fractional hybrid differential equations:

$$\begin{cases} D^{\alpha_1}\left(\frac{u_1(t)}{g_1(t,u_1(t),u_2(t))}\right) = f_1\left(t,u_1\left(t\right),u_2\left(t\right)\right),\ t\in J, \\ \\ D^{\alpha_2}\left(\frac{u_2(t)}{g_2(t,u_1(t),u_2(t))}\right) = f_2\left(t,u_1\left(t\right),u_2\left(t\right)\right),\ t\in J, \end{cases} \tag{1.1}$$

supplemented with coupled integral boundary conditions:

$$\begin{cases} u_1\left(0\right) = \int_0^{\theta_1} A_1\left(s\right) u_1\left(s\right) ds,\ 0<\theta_1<1, \\ \\ u_2\left(0\right) = \int_0^{\theta_2} A_2\left(s\right) u_2\left(s\right) ds,\ 0<\theta_2<1, \end{cases} \tag{1.2}$$

where, for $i=1,2$, the symbols D^{α_i} denote the Caputo fractional derivatives, with $0<\alpha_i<1$, $J=[0,1]$, A_i are continuous functions on $[0,1]$, $f_i\in C\left(J\times\mathbb{R}^2,\mathbb{R}\right)$ and $g_i\in C\left(J\times\mathbb{R}^2,\mathbb{R}-\{0\}\right)$.

The differential equations of fractional orders arise in the mathematical modelling of systems and processes occurring in many engineering and scientific disciplines, such as chemistry, physics, control theory, biophysics and signal processing, see, for example, [1, 2, 3, 4, 5] and the references therein. Recently, by using fixed the point theory, many researchers have established existence results for various classes of equations. For more details, we refer the reader to [2, 3, 6, 7, 8, 9, 10] and the references

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therein. The study of systems of fractional orders is also found to be of great interest as such systems appear in a variety of problems of applied nature. For more details, one is referred to [11, 12, 13, 14] and the references therein. Moreover, fractional hybrid differential equations have also been studied by several authors, see, for instance, [15, 16, 17, 18, 19, 20]. Recently, the existence and uniqueness for solutions of some systems of coupled hybrid fractional differential equations has been intensively studied; see [21, 22, 23, 25] and references cited therein.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and lemmas. Section 3 is devoted to establish an existence and uniqueness result for system (1.1)-(1.2). The first result is based on the Leray Schauder alternative and the second one is based on the Banach contraction principle. In Section 4, an illustrative example is discussed.

2. Preliminaries

In the following, we give the necessary notation and basic de definitions and lemmas which will be used in this paper (see [4, 5, 26]).

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function h on $[0,\infty[$ is defined by:

$$I^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} h(\tau)\, d\tau,$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u}u^{\alpha-1}du$.

Definition 2.2. The fractional derivative of $h \in C^n([0,\infty[)$ in the Caputo's sense is defined as:

$$D^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1} h^{(n)}(\tau)\, d\tau, n-1 < \alpha < n, n \in N^*.$$

We also give the following lemmas.

Lemma 2.3. *For $\alpha > 0$, the general solution of the fractional differential equation $D^{\alpha}u(t) = 0$ is given by:*

$$u(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}, i = 0,1,2,..,n-1, n = [\alpha]+1$.

Lemma 2.4. *Let $\alpha > 0$. Then*

$$I^{\alpha}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1},$$

for some $c_i \in \mathbb{R}, i = 0,1,2,...,n-1, n = [\alpha]+1$.

We also need the following auxiliary result. As notation, we put: $\overline{u} = (u_1, u_2)$ and $\overline{u}(t) = (u_1(t), u_2(t))$. It follows that

Lemma 2.5. *For $i = 1,2$, let $g_i \in C(\left(J \times \mathbb{R}^2, \mathbb{R} - \{0\}\right)$ and $h_i \in C(J,\mathbb{R})$. Then, the solution of the equation*

$$D^{\alpha_i}\left(\frac{u_i(t)}{g_i(t,\overline{u}(t))}\right) = h_i(t),\ t \in J, 0 < \alpha_i < 1, i = 1,2, \tag{2.1}$$

subject to the conditions:

$$u_i(0) = \int_0^{\theta_i} A_i(s)\, u_i(s)\, ds,\ 0 < \theta_i < 1, i = 1,2, \tag{2.2}$$

given by:

$$\begin{aligned} u_i(t) &= \frac{g_i(t,\overline{u}(t))}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1} h_i(\tau)\,d\tau \\ &+\frac{g_i(t,\overline{u}(t))}{\left[g_i(0,\overline{u}(0))-\int_0^{\theta_i} g_i(s,\overline{u}(s))A_i(s)\,ds\right]} \\ &\times\int_0^{\theta_i} g_i(s,\overline{u}(s))A_i(s)\left[\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1} h_i(\tau)\,d\tau\right]ds\Bigg), \end{aligned} \tag{2.3}$$

provided that $\int_0^{\theta_i} g_i(s,\overline{u}(s))A_i(s)\,ds \neq g_i(0,\overline{u}(0))$.

Proof. By Lemma 2.4, we have

$$\frac{u_i(t)}{g_i(t,\overline{u}(t))} = I^{\alpha_i}h_i(t)+c_0. \tag{2.4}$$

where $c_0 \in \mathbb{R}$ is an arbitrary constant. Using condition (2.2), we can write

$$\begin{aligned} c_0 &= \frac{1}{\left(g_i(0,\overline{u}(0))-\int_0^{\theta_i} g_i(s,\overline{u}(s))A_i(s)\,ds\right)} \\ &\times\int_0^{\theta_i} g_i(s,\overline{u}(s))A_i(s)I^{\alpha}(h_i(s))\,ds. \end{aligned} \tag{2.5}$$

Substituting the value of c_0 in (2.4), we obtain the desired result immediately. □

3. Main results

For $i=1,2$, we introduce the spaces

$$X_i = \{u_i(t), i=1,2 : u_i \in C(J,R)\},$$

endowed the norm defined by

$$\|u_i\|_{X_i} = \sup\{|u_i(t)| : t \in J\}.$$

It is clear that for each $i=1,2$, $(X_i,\|.\|_{X_i})$ is a Banach space. The product space $(X_1\times X_2,\|.\|_{X_1\times X_2})$ is also a Banach space with norm $\|\overline{u}\|_{X_1\times X_2} = \|u_1\|_{X_1}+\|u_2\|_{X_2}$.

In view of Lemma 2.5, we define the operator $T: X_1\times X_2 \to X_1\times X_2$ by

$$T\overline{u}(t) = (T_1\overline{u}(t), T_2\overline{u}(t)), \tag{3.1}$$

where

$$\begin{aligned} T_i\overline{u}(t) &= \frac{g_i(t,\overline{u}(t))}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1} f_i(\tau,\overline{u})\,d\tau \\ &+\frac{g_i(t,\overline{u}(t))}{\left[g_i(0,\overline{u}(0))-\int_0^{\theta_i} g_i(s,\overline{u}(s))A_i(s)\,ds\right]}\int_0^{\theta_i} g_i(s,\overline{u}(s))A_i(s) \\ &\times\left[\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1} f_i(\tau,\overline{u}(\tau))\,dr\right]ds, \end{aligned} \tag{3.2}$$

for $i=1,2$.

We impose the following hypotheses:

(H_1) : There exist nonnegative constants $k_{ij}, i, j = 1,2$, such that, for all $t \in J$ and $\overline{u}, \overline{v} \in \mathbb{R}^2$,

$$|f_i(t,\overline{u}) - f_i(t,\overline{v})| \leq \sum_{j=1}^{2} k_{ij} |u_j - v_j|.$$

(H_2) : There exist nonnegative constants $M_i, i = 1,2$ such that for each $t \in J$ and $\overline{u} \in \mathbb{R}^2$, $|g_i(t,\overline{u})| \leq M_i, i = 1,2$.

(H_3) : There exist real constants $a_i > 0$ and $b_{ij} \geq 0$ $(i, j = 1,2)$ such that, $|f_i(t,\overline{u})| \leq a_i + \sum_{j=1}^{2} b_{ij} |u_j|$ for all $t \in J, \overline{u} \in \mathbb{R}^2$.

Then, we set the following

$$B_i := \frac{M_i}{\Gamma(\alpha_i + 1)} + \frac{M_i \theta_i^{\alpha_i + 1} \sup_{s \in J} |A_i(s)|}{\Gamma(\alpha_i + 2) \left| g_i(0, \overline{u}(0)) - \int_0^{\theta_i} g_i(s, \overline{u}(s)) A_i(s)\, ds \right|}, \tag{3.3}$$

where $i = 1,2$.

In our first result, we discuss the existence of solutions for problem (1.1)-(1.2) by means of Leray-Schauder alternative [24].

Lemma 3.1. *(Leray-Schauder alternative). Let $F : E \to E$ be a completely continuous operator (i.e.,a map that restricted to any bounded set in E is compact). Let*

$$\Theta(F) = \{u \in E : u = \lambda F(u) \text{ for some } 0 < \lambda < 1\}.$$

Then either $\Theta(F)$ is unbounded, or F has at least one fixed point.

Theorem 3.2. *Let $f_i : J \times \mathbb{R}^2 \to \mathbb{R}$ $(i = 1,2)$ be continuous functions satisfying conditions (H_2) and (H_3). In addition, it is assumed that*

$$\sum_{i=1}^{2} \left(B_i \sum_{j=1}^{2} b_{ij} \right) < 1, \tag{3.4}$$

where B_i $(i = 1,2)$ are given by (3.3). Then system (1.1)-(1.2) has at least one solution on J.

Proof. In the first step, we show that the operator $T : X_1 \times X_2 \to X_1 \times X_2$ is completely continuous. By continuity of the functions f_i, g_i $(i = 1,2)$, it follows that the operator T is continuous. Let $\Theta \subset X_1 \times X_2$ be bounded. Then we can find positive constants L_i such that

$$|f_i(t,\overline{u}(t))| \leq L_i, \forall\, \overline{u} \in \Theta.$$

Then for any $\overline{u} \in \Theta$ and for $i = 1,2$, we have

$$\begin{aligned} \|T_i \overline{u}\|_{X_i} \leq\ & \frac{M_i}{\Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i - 1} |f_i(\tau, \overline{u}(\tau))|\, d\tau \\ & + \frac{M_i}{\left| g_i(0, \overline{u}(0)) - \int_0^{\theta_i} g_i(s, \overline{u}(s)) A_i(s)\, ds \right|} \\ & \times \int_0^{\theta_i} |A_i(s)| \left[\frac{1}{\Gamma(\alpha_i)} \int_0^s (s-\tau)^{\alpha_i - 1} |f_i(\tau, \overline{u}(\tau))|\, d\tau \right] ds \Bigg), \end{aligned} \tag{3.5}$$

Hence, for all $i=1,2$, we obtain

$$\|T_i\overline{u}\|_{X_i} \leq L_i \times \left[\frac{M_i}{\Gamma(\alpha_i+1)} \right. \tag{3.6}$$
$$\left. \frac{M_i\theta_i^{\alpha_i+1}\sup_{s\in J}|A_i(s)|}{\left|f_i(0,\overline{u}(0))-\int_0^{\theta_i}f_i(s,\overline{u}(s))A_i(s)\,ds\right|\Gamma(\alpha_i+2)}\right] := L_iB_i.$$

It follows that

$$\|T\overline{u}\|_{X_1\times X_2} \leq \sum_{i=1}^{2} L_iB_i := L. \tag{3.7}$$

From the above inequalities, we see that operator T is uniformly bounded.

Next, we show that T is equi-continuous. Let $t_1,t_2\in J$ with $t_1<t_2$. Then we have

$$\begin{aligned}|T_i\overline{u}(t_1)-T_i\overline{u}(t_2)| \quad \leq \quad & \frac{M_i}{\Gamma(\alpha_i)}\int_0^{t_1}\left|(t_2-\tau)^{\alpha_i-1}(t_1-\tau)^{\alpha_i-1}\right||f_i(\tau,\overline{u}(\tau))|\,d\tau \\ & \frac{M_i}{\Gamma(\alpha_i)}\int_{t_1}^{t_2}\left|(t_2-\tau)^{\alpha_i-1}\right||f_i(\tau,\overline{u}(\tau))|\,d\tau. \end{aligned}\tag{3.8}$$

for $i=1,2$. Hence, by (H_2), we have

$$|T_i\overline{u}(t_1)-T_i\overline{u}(t_2)| \leq \frac{M_iL_i}{\Gamma(\alpha_i+1)}\left[(t_2-t_1)^{\alpha_i}+\left|t_2^{\alpha_i}-t_1^{\alpha_i}\right|\right], i=1,2. \tag{3.9}$$

Thus,

$$\|T\overline{u}(t_1)-T\overline{u}(t_2)\|_{X_1\times X_2} \leq \sum_{i=1}^{2}\frac{M_iL_i}{\Gamma(\alpha_i+1)}\left[(t_2-t_1)^{\alpha_i}+\left|t_2^{\alpha_i}-t_1^{\alpha_i}\right|\right]. \tag{3.10}$$

This implies that $\|T_i\overline{u}(t_1)-T_i\overline{u}(t_2)\|_{X_1\times X_2}\to 0$ as $t_2\to t_1$. Thus, by using the Arzela-Ascoli theorem one can conclude that the operator $T:X_1\times X_2\to X_1\times X_2$ is completely continuous.

Finally, it will be verified that the set $\Phi=\{\overline{u}\in X_1\times X_2,\overline{u}=\lambda T\overline{u},0<\lambda<1\}$ is bounded. Let $\overline{u}\in\Phi$. Then, for each $t\in J$, we can write

$$u_i(t)=\lambda T_i\overline{u}, i=1,2. \tag{3.11}$$

Then, for $i=1,2$, we have

$$\begin{aligned}|u_i(t)| \quad \leq \quad & \lambda\left(\frac{M_i}{\Gamma(\alpha_i)}\int_0^t(t-\tau)^{\alpha_i-1}|f_i(\tau,\overline{u}(\tau))|\,d\tau\right. \\ & +\frac{M_i}{\left|g_i(0,\overline{u}(0))-\theta_i\int_0^{\theta_i}g_i(s,\overline{u}(s))A_i(s)\,ds\right|} \\ & \left.\times\int_0^{\theta_i}|A_i(s)|\left[\frac{1}{\Gamma(\alpha_i)}\int_0^s(s-\tau)^{\alpha_i-1}|f_i(\tau,\overline{u}(\tau))|\,d\tau\right]ds\right).\end{aligned}\tag{3.12}$$

Thanks to (H_3), we can write

$$\begin{aligned}|u_i(t)| \quad \leq \quad & \left(\frac{M_i}{\Gamma(\alpha_i+1)}\right. \\ & \left.+\frac{M_i\theta_i^{\alpha_i+1}\sup_{t\in J}|A_i(s)|}{\left|g_i(0,\overline{u}(0))-\int_0^{\theta_i}g_i(s,\overline{u}(s))A_i(s)\,ds\right|\Gamma(\alpha_i+2)}\right) \\ & \times\left(a_i+\sum_{j=1}^{2}b_{ij}\left\|u_j\right\|\right).\end{aligned}\tag{3.13}$$

Hence, we have

$$\|u_i\|_{X_i} \leq B_i a_i + B_i \sum_{j=1}^{2} b_{ij} \|u_j\|, \ i = 1,2. \tag{3.14}$$

This implies that

$$\sum_{i=1}^{2} \|u_i\|_{X_i} \leq \sum_{i=1}^{2} B_i a_i + \sum_{i=1}^{2} B_i \sum_{j=1}^{2} b_{ij} \|u_j\|. \tag{3.15}$$

It follows that

$$\|\overline{u}\|_{X_1 \times X_2} \leq \frac{\sum_{i=1}^{2} B_i a_i}{\min_{1\leq j\leq 2} \left(1 - \sum_{i=1}^{2} \left(B_i \sum_{j=1}^{2} b_{ij}\right)\right)}. \tag{3.16}$$

This shows that the set Φ is bounded. Hence all the conditions of Lemma 3.1 are satisfied and consequently the operator has at least one fixed point, which corresponds to a solution of system (1.1)-(1.2). This completes the proof. □

Now, we are in a position to present the existence and uniqueness of solutions for (1.1)-(1.2). This result is based on the Banach's contraction mapping principle.

Theorem 3.3. *Let $f_i : J \times \mathbb{R}^2 \to \mathbb{R}; i = 1,2$ be continuous functions satisfying the conditions (H_1) and (H_2). If*

$$\sum_{i=1}^{2} \left(B_i \sum_{j=1}^{2} k_{ij} \right) < 1, \tag{3.17}$$

then hybrid system (1.1)-(1.2) has a unique solution on J.

Proof. Define $\sup_{t\in J} |f_i(t,0,0)| = N_i < \infty, i = 1,2$ such that

$$r \geq \frac{\sum_{i=1}^{2} B_i N_i}{1 - \sum_{i=1}^{2} B_i \sum_{j=1}^{2} k_{ij}}.$$

We first show that $TB_r \subset B_r$, where $B_r = \left\{\overline{u} \in X_1 \times X_2 : \|\overline{u}\|_{X_1\times X_2} < r\right\}$. Thanks to (H_1), for $\overline{u} \in B_r$, we observe that

$$\begin{aligned} |f_i(t,\overline{u}(t))| &\leq |f_i(t,\overline{u}(t)) - f_i(t,0,0)| + |f_i(t,0,0)| \\ &\leq \sum_{j=1}^{2} k_{ij} r + N_i. \end{aligned} \tag{3.18}$$

Hence, we get

$$\begin{aligned} \|T_i(\overline{u})\|_{X_i} \leq\ & M_i \times \sup_{t\in J} \left\{ \left(\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i - 1} |f_i(\tau,\overline{u}(\tau))| \, d\tau \right. \right. \\ & + \frac{1}{\left| g_i(0,\overline{u}(0)) - \int_0^{\theta_i} g_i(s,\overline{u}(s)) A_i(s)\, ds \right|} \\ & \left. \left. \times \int_0^{\theta_i} |A_i(s)| \left[\frac{1}{\Gamma(\alpha_i)} \int_0^s (s-\tau)^{\alpha_i - 1} |f_i(\tau,\overline{u}(\tau))| \, d\tau \right] ds \right) \right\}. \end{aligned} \tag{3.19}$$

Using (3.18), we can write

$$\begin{aligned}\|T_i(\overline{u})\|_{X_i} &\leq \left[\frac{M_i}{\Gamma(\alpha_i+1)}+\right. \\ &\left.\frac{M_i\theta_i^{\alpha_i+1}\sup_{s\in J}|A_i(s)|}{\Gamma(\alpha_i+2)\left|g_i(0,\overline{u}(0))-\int_0^{\theta_i}g_i(s,\overline{u}(s))A_i(s)\,ds\right|}\right]\times\left(\sum_{j=1}^{2}k_{ij}r+N_i\right) \\ &= B_i\left(\sum_{j=1}^{2}k_{ij}r+N_i\right),\end{aligned} \tag{3.20}$$

for $i=1,2$. Therefore, it yields that

$$\|T_i(\overline{u})\|_{X_i}\leq B_i\left(\sum_{j=1}^{2}k_{ij}r+N_i\right),\ i=1,2. \tag{3.21}$$

Consequently,

$$\|T\overline{u}\|_{X_1\times X_2}\leq r, \tag{3.22}$$

which implies that $TB_r\subset B_r$. For $\overline{u},\overline{v}\in X$ and, for each $t\in J$, we have

$$\begin{aligned}&\|T_i(\overline{u})-T_i(\overline{v})\|_{X_i} \\ \leq\ & M_i\times\sup_{t\in J}\left\{\left(\frac{1}{\Gamma(\alpha_i)}\int_0^t(t-\tau)^{\alpha_i-1}|f_i(\tau,\overline{u}(\tau))-f_i(\tau,\overline{v}(\tau))|\,d\tau\right.\right. \\ &+\frac{1}{\left|g_i(0,\overline{u}(0))-\int_0^{\theta_i}g_i(s,\overline{u}(s))A_i(s)\,ds\right|}\int_0^{\theta_i}|A_i(s)| \\ &\left.\left.\times\left[\frac{1}{\Gamma(\alpha_i)}\int_0^s(s-\tau)^{\alpha_i-1}|f_i(\tau,\overline{u}(\tau))-f_i(\tau,\overline{v}(\tau))|\,d\tau\right]ds\right)\right\}.\end{aligned} \tag{3.23}$$

Thanks to (H_1), we can write

$$\begin{aligned}&\|T_i(\overline{u})-T_i(\overline{v})\|_{X_i} \\ \leq\ & \sum_{j=1}^{2}k_{ij}M_i\times\sup_{t\in J}\left\{\left(\frac{1}{\Gamma(\alpha_i)}\int_0^t(t-\tau)^{\alpha_i-1}\sum_{j=1}^{2}\|u_j-v_j\|\,d\tau\right.\right. \\ &+\frac{1}{\left|g_i(0,\overline{u}(0))-\int_0^{\theta_i}g_i(s,\overline{u}(s))A_i(s)\,ds\right|}\int_0^{\theta_i}|A_i(s)| \\ &\left.\left.\times\left[\frac{1}{\Gamma(\alpha_i)}\int_0^s(s-\tau)^{\alpha_i-1}\sum_{j=1}^{2}\|u_j-v_j\|\,d\tau\right]ds\right)\right\} \\ \leq\ & \sum_{j=1}^{2}k_{ij}\left[\frac{M_i}{\Gamma(\alpha_i+1)}\right. \\ &\left.+\frac{M_i\theta_i^{\alpha_i+1}\sup_{s\in J}|A_i(s)|}{\left|g_i(0,\overline{u}(0))-\int_0^{\theta_i}g_i(s,\overline{u}(s))A_i(s)\,ds\right|\Gamma(\alpha_i+2)}\right]\left(\sum_{j=1}^{2}\|u_j-v_j\|\right) \\ =\ & \sum_{j=1}^{2}k_{ij}B_i\times\left(\sum_{j=1}^{2}\|u_j-v_j\|\right),\end{aligned} \tag{3.24}$$

for $i=1,2$. This implies that

$$\|T(\overline{u})-T(\overline{v})\|_{X_1\times X_2}\leq\sum_{i=1}^{2}\left(B_i\sum_{j=1}^{2}k_{ij}\right)\times\left(\sum_{j=1}^{2}\|u_j-v_j\|\right). \tag{3.25}$$

Consequently by (3.17), we conclude that T is contractive. As a consequence of the Banach fixed point theorem, we deduce that T has a unique fixed point which is a solution of hybrid system (1.1)-(1.2). □

4. Example

Example 4.1. Consider the following system of hybrid fractional differential equations:

$$\begin{cases} D^{\frac{1}{4}}\left(\frac{u_1(t)}{\frac{|\cos u_1(t)|+|\sin u_2(t)|+3}{9}}\right)=\frac{e^{-t}(\cos u_1(t)+\sin u_2(t))}{\pi e^t+5}+e^{1+t^2},t\in[0,1],\\ D^{\frac{2}{5}}\left(\frac{u_2(t)}{\frac{|\cos(u_1(t)+u_2(t))|+2}{3}}\right)=\frac{\sin(2\pi u_1(t))}{32\pi}+\frac{\tan^{-1}u_2(t)}{16}+\ln\left(1+t^2\right),t\in[0,1],\\ u_1(0)=\int_0^{\frac{\pi}{4}}\frac{\sin s}{62}u_1(s)\,ds,\\ u_2(0)=\int_0^{\frac{\pi}{6}}\frac{\cos s}{62}u_2(s)\,ds. \end{cases} \tag{4.1}$$

For $u_1,u_2\in\mathbb{R},t\in J=[0,1]$, we have

$$\begin{aligned} f_1(t,u_1,u_2) &= \frac{e^{-t}(\cos u_1(t)+\sin u_2(t))}{\pi e^t+5}+e^{1+t^2},\\ f_1(t,u_1,u_2) &= \frac{\sin(2\pi u_1(t))}{32\pi}+\frac{\tan^{-1}u_2(t)}{16}+\ln\left(1+t^2\right),\\ g_1(t,u_1,u_2) &= \frac{|\cos u_1(t)|+|\sin u_2(t)|+3}{9},\\ g_2(t,u_1,u_2) &= \frac{|\cos(u_1(t)+u_2(t))|}{3}, \end{aligned}$$

and

$$A_1(t)=\frac{\sin t}{62},A_2(t)=\frac{\cos t}{62}.$$

It is clear that

$$\begin{aligned} |h_1(t,u_1,u_2)-h_1(t,v_1,v_2)| &\leq \frac{1}{\pi+5}(|u_1-v_1|+|u_2-v_2|),\\ |h_2(t,u_1,u_2)-h_2(t,v_1,v_2)| &\leq \frac{1}{16}(|u_1-v_1|+|u_2-v_2|), \end{aligned}$$

$$|f_1(t,u_1,u_2)|\leq\frac{5}{9},|f_2(t,u_1,u_2)|\leq\frac{1}{3}.$$

So, we take $\omega_{11}=\omega_{12}=\frac{1}{\pi+5},\omega_{21}=\omega_{22}=\frac{1}{16}$ and found $B_1\simeq 0.6262,B_2\simeq 0.3810$. Hence, we obtain

$$\sum_{i=1}^{2}\left(B_i\sum_{j=1}^{2}k_{ij}\right)\simeq 0.20145<1.$$

Thus all the conditions are satisfied. Consequently, there exists a unique solution for problem (4.1) on $[0,1]$.

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