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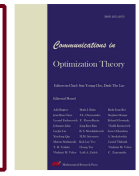

ON THE EXISTENCE OF SOLUTIONS OF A GENERAL VECTOR ALPHA OPTIMIZATION PROBLEM AND ITS APPLICATIONS

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Abstract. The general vector alpha optimization problems $(GVOP)_\alpha$ are formulated and sufficient conditions on the existence of solutions are shown. Firstly, under suitable assumptions, we obtain several results on the existence of solutions of $(GVOP)_I$, $(GVOP)_P$, $(GVOP)_{Pr}$ and $(GVOP)_W$. Secondly, as an application, several results on the existence of solutions of upper quasivariational inclusion problem $(UQVIP)$ and lower quasivariational inclusion problem $(LQVIP)$ are also provided.

Keywords. General vector alpha optimization problem; Quasivariational inclusion problem; Multivaluted mapping; Hausdorff local convex topological vector space.

1. Introduction

The general vector alpha optimization problem plays an important role in the vector optimization theory concerning multivaluted mappings and have any relations to the lower quasivariational inclusion problem and the upper quasivariational inclusion problem. In recent years, the existence of solutions of these problems have attracted attention by many authors, see, for example, Minh and Tan [1], Gurraggio and Tan [2], Lin, Yu and Kassay [3], Tan [4, 5], Luc and Tan [6], Park [7], Su and Dinh [8], Chang and Pang [9], Su [10], etc and the references therein. Tan [4] established several results on the existence of solutions of quasivariational inclusion problems in Hausdorff local convex topological vector spaces. Minh and Tan [1] established

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sufficient conditions on the existence of solutions of quasivariational inclusion problems of S-tampachia type in infinite-dimensional spaces. Tan [5] provided sufficient conditions for the existence of solutions to systems of vector quasi-optimization problems in Hausdorff local convex topological vector spaces, etc. The main purpose of us in this paper is to give several results on the existence of solutions of general vector alpha optimization problem and its applications to lower quasivariational inclusion problem and upper quasivariational inclusion problem.

The general vector alpha optimization problems $(GVOP)_\alpha$ (where $\alpha \in \{I, P, Pr, W\}$) are given as follows: Let X, Y and Z be Hausdorff local convex topological vector spaces, D and K be nonempty subsets on X and Z, respectively, and let C be a cone on Y. We consider the following multivaluted mappings

$$S : D \longrightarrow 2^D;$$

$$T : D \times D \longrightarrow 2^K;$$

$$F : K \times D \times D \longrightarrow 2^Y,$$

where $2^D, 2^K$ and 2^Y denote the family of all the subsets of D, K and Y, respectively. Our problem is finding $\bar{x} \in K$ and $\bar{y} \in D$ such that

$$(GVOP)_\alpha : \qquad F(\bar{x}, \bar{y}, \bar{y}) \cap \alpha Min(F(K \times D \times D) \,|\, C) \neq \emptyset.$$

This is called a general vector alpha optimization problem corresponding to K, D, F and C. The set of such points $(\bar{x}, \bar{y})$ is said to be a solution set of $(GVOP)_\alpha$. The elements of $\alpha Min(F(K \times D \times D)|C)$ are called alpha optimal values of $(GVOP)_\alpha$.

The upper and lower quasivariational inclusion problems are defined as follows (see [1, 4]):

(UQVIP), upper quasivariational inclusion problem. Finding $\bar{y} \in D$ such that

$$\bar{y} \in S(\bar{y})$$

and

$$F(x, \bar{y}, y) \subset F(x, \bar{y}, \bar{y}) + C, \text{ for all } y \in S(\bar{y}), x \in T(\bar{y}, y).$$

(LQVIP), lower quasivariational inclusion problem. Finding $\bar{y} \in D$ such that

$$\bar{y} \in S(\bar{y})$$

and

$$F(x,\bar{y},\bar{y}) \subset F(x,\bar{y},y) - C, \text{ for all } y \in S(\bar{y}), x \in T(\bar{y},y).$$

The organization of the paper is as follows. In Section 2 we collect definitions and preliminary facts for our later use. Section 3 is devoted to the existence of solutions of general vector alpha optimization problems and upper and lower quasivariational inclusion problems in Hausdorff local convex topological vector spaces.

2. Preliminaries and Definitions

Let X,Y,Z,D,K,C be given as in Section 1. For each $M \subset Y$, as usual we denote by $\overline{M}$ instead of the closure of M, $IMin(M|C)$, $PMin(M|C)$, $PrMin(M|C)$ and $WMin(M|C)$ instead of the case of ideal, Pareto, proper, weak efficient points, respectively. The sets $IMax(M|C)$, $PMax(M|C)$, $PrMax(M|C)$ and $WMax(M|C)$ are duality given. Given a multivaluted mapping $L : D \longrightarrow 2^Y$. The definition domain and graph of L are defined respectively by

$$\text{dom}L = \{x \in D \,|\, L(x) \neq \emptyset\},$$

$$Gr(L) = \{(x,y) \in D \times Y \,|\, y \in L(x)\}.$$

We recall that L is said to be a compact mapping if the closure $\overline{L(D)}$ of its range $L(D)$ is a compact set in Y. Note that C is cone in Y if and only if $tc \in C$ for all $c \in C$ and $t \geq 0$, and in addition C is a convex set i.e. $ta + (1-t)b \in C$ for all $a,b \in C$ and $t \in [0;1]$, then cone C is called convex cone. If C is a closed set then cone C is called a closed cone. The following we introduce some new definitions of C-continuities of multivaluted mapping $L : D \longrightarrow 2^Y$ with a cone $C \subset Y$.

Definition 2.1. [1, 4, 6] Let $L : D \longrightarrow 2^Y$ be a multivalued mapping

(i) L is said to be upper C-continuous at $\bar{x} \in \text{dom}L$ if for all neighborhood V of the origin in Y there is a neighborhood U of $\bar{x}$ such that

$$L(x) \subset L(\bar{x}) + V + C$$

holds for all $x \in U \cap \text{dom}L$.

(ii) L is said to be upper C-continuous on D if L is upper C-continuous at any point of dom$L$.

(iii) L is said to be lower C-continuous at $\overline{x} \in$ dom$L$ if for all neighborhood V of the origin in Y there is a neighborhood U of $\overline{x}$ such that

$$L(\overline{x}) \subset L(x) + V - C$$

holds for all $x \in U \cap$ dom$L$.

(iv) L is said to be lower C-continuous on D if L is lower C-continuous at any point of dom$L$.

(v) L is said to be $C-$ continuous on D if L is simultaneously upper C-continuous and lower C-continuous on D.

(vi) L is said to be $C-$ convex if D is convex and for any $x, y \in D$, any $t \in [0,1]$ we have

$$tL(x) + (1-t)L(y) \subset L(tx + (1-t)y) + C.$$

(vii) L is said to be $C-$ concave if D is convex and for any $x, y \in D$, any $t \in [0,1]$ we have

$$tL(x) + (1-t)L(y) \subset L(tx + (1-t)y) - C.$$

Definition 2.2. [11] Let M be a nonempty subset of Y. We say that

(i) $x \in M$ is an ideal efficient (or ideal minimal) point of M with respect to C if $M \subset x + C$. The set of ideal minimal points of M is denoted by $IMin(M|C)$.

(ii) $x \in M$ is an efficient (or Pareto minimal or nondominated) point of M with respect to C if $M \cap (x - C) \subset x + C \cap (-C)$. The set of efficient points of M is denoted by $PMin(M|C)$.

(iii) $x \in M$ is a (global) proper efficient point of M with respect to C if there exists a convex cone $\widetilde{C}$ which is not the whole space and contains $C \setminus C \cap (-C)$ in its interior such that $x \in PMin(M|\widetilde{C})$. The set of proper efficient points of M is denoted by $PrMin(M|C)$.

(iv) Supposing that $intC \neq \emptyset$, point $x \in M$ is a weak efficient point of M with respect to C if $x \in PMin(M|intC \cup \{0\})$. The set of weak efficient points of M is denoted by $WMin(M|C)$.

Remark 2.3. It is well known that (see Luc [11])

$$PrMin(M|C) \subset PMin(M|C) \subset WMin(M|C)$$

and moreover

$$IMin(M|C) = PMin(M|C) \text{ if } IMin(M|C) \neq \emptyset.$$

If, in addition, $A \subset B$, then

$$IMin(B|C) \cap A \subset IMin(A|C).$$

3. Main results

In this subsection, we provide some existence theorems to general vector alpha optimization problems, see, for example, Theorem 3.1, Theorem 3.2 and Remark 3.3 below. As applications, sufficient conditions on the existence of solutions to upper and lower quasivariational inclusion problems are also established, see, for instance, Corollaries 3.4 and 3.5 below.

Theorem 3.1. *Let X, Y and Z be Hausdorff local convex topological vector spaces, D and K be nonempty compact subsets on X and Z, respectively, and let C be a closed convex cone on Y. We consider the following multivaluted mapping*

$$F : K \times D \times D \longrightarrow 2^Y.$$

Suppose, furthermore, that the following conditions are fulfilled

(i). $F(x,y,y)$ and $F(x,y,y)+C$ are closed and nonempty for all $x \in K$ and $y \in D$;

(ii). F is upper C-continuous on $K \times D \times D$;

(iii). $IMin(F(x,y,y)|C) \neq \emptyset$ for every $(x,y) \in K \times D$;

Then there exists $(\bar{x},\bar{y}) \in K \times D$ such that

$$F(\bar{x},\bar{y},\bar{y}) \cap \alpha Min\Big(F(K \times D \times D)|C\Big) \neq \emptyset, \text{ for all } \alpha \in \{P,I,W\}.$$

Proof. Making use of Proposition 2.2 in Luc [11], we have the following inclusions hold

$$IMin\Big(F(K \times D \times D)|C\Big) \subset PMin\Big(F(K \times D \times D)|C\Big)$$
$$\subset WMin\Big(F(K \times D \times D)|C\Big).$$

Thus, it suffices to show that the problem $(GVOP)_I$ has solution, which means that there exists $(\bar{x},\bar{y}) \in K \times D$ such that

$$F(\bar{x},\bar{y},\bar{y}) \cap IMin\Big(F(K \times D \times D)|C\Big) \neq \emptyset. \tag{3.1}$$

In fact, for every $z \in F(K \times D \times D))$, there exists sequence $\{z_\alpha\}_{\alpha \in I} \subset F(K \times D \times D))$, where I denotes the index set arbitrary, such that

$$z_\alpha \longrightarrow z \text{ as } \alpha \longrightarrow \infty. \tag{3.2}$$

Thus, for each index $\alpha \in I$, there exists a triple

$$(x_\alpha, y_\alpha, y_\alpha) \in K \times D \times D$$

such that

$$z_\alpha \in F((x_\alpha, y_\alpha, y_\alpha)).$$

By hypothesis, K and D are compact subsets on X and Z, respectively, by passing a subsequence, we may assume, without loss of generated, that

$$x_\alpha \longrightarrow \overline{x} \in K, \text{ as } \alpha \longrightarrow \infty$$

and

$$y_\alpha \longrightarrow \overline{y} \in D, \text{ as } \alpha \longrightarrow \infty.$$

Consequently, one has

$$(x_\alpha, y_\alpha, y_\alpha) \longrightarrow (\overline{x}, \overline{y}, \overline{y}), \text{ as } \alpha \longrightarrow \infty. \tag{3.3}$$

Since C is a cone which contains origin, it leads to the following result

$$z_\alpha \in F((x_\alpha, y_\alpha, y_\alpha)) + C \text{ for any } \alpha \in I.$$

Let us see that

$$z \in F(\overline{x}, \overline{y}, \overline{y}) + C. \tag{3.4}$$

Indeed, if (3.4) does not hold, then one can find a closed convex neighborhood of the zero in Y, say V_0, such that

$$(z + V_0) \cap \Big(F(\overline{x}, \overline{y}, \overline{y}) + C\Big) = \emptyset,$$

or

$$(z + \frac{V_0}{2}) \cap \Big(F(\overline{x}, \overline{y}, \overline{y}) + \frac{V_0}{2} + C\Big) = \emptyset. \tag{3.5}$$

Choosing $\beta_1 \geq 1$ such that $z_\alpha \in z + \frac{V_0}{2}$ for all $\beta \geq \beta_1$, because $z_\alpha \longrightarrow z$ $(\alpha \longrightarrow \infty)$. It follows from Assumption (ii) that F is upper C-continuous on $K \times D \times D$, which implies that F is upper

$C-$ continuous at z_0 because $z_0 \in K \times D \times D$, which means that there exists a neighborhood U of z_0, where $z_0 := (\bar{x}, \bar{y}, \bar{y})$, such that

$$F(U \cap domF) \subset F(z_0) + \frac{V_0}{2} + C.$$

From condition (3.3), it yields that there exists $\beta_2 \geq 0$ such that $(x_\beta, y_\beta, y_\beta) \in U$ and moreover

$$z_\beta \in F((x_\beta, y_\beta, y_\beta)) + C \subset F(z_0) + \frac{V_0}{2} + C$$

for any $\beta \geq \beta_2$. This leads to the following result

$$z_{max\{\beta_1;\beta_2}\} \in (z + \frac{V_0}{2}) \cap (F(z_0) + \frac{V_0}{2} + C),$$

contradicting (3.5). From there it leads to condition (3.4) holds, meaning that $z \in F(\bar{x}, \bar{y}, \bar{y}) + C$. We now prove that

$$F(z_0) \cap IMin(F(U_0 \cap domF)|C) \neq \emptyset,$$

where U_0 is a neighborhood of z_0 in the product space $K \times D \times D$. In fact, let W be an any open neighborhood of origin in Y, it follows from Assumption (ii) that there exists an other neighborhood of z_0 in $K \times D \times D$, say U_0, such that

$$F(U_0 \cap domF) \subset F(z_0) + W + C = F((\bar{x}, \bar{y}, \bar{y})) + W + C.$$

Because W is an arbitrary open neighborhood, $F(z_0)$ is closed and cone C is so, therefore,

$$F(U_0 \cap domF) \subset F(z_0) + C. \tag{3.6}$$

From (3.6) it leads to the following

$$F(z_0) \cap IMin(F(U_0 \cap domF)|C) \neq \emptyset. \tag{3.7}$$

Posit to the contrary, that (3.7) is false, which means that for all $\bar{z} \in F(z_0)$, it follows that $F(U_0 \cap dom F) \not\subset \bar{z} + C$, which combines with (3.6)

$$F(z_0) + C \not\subset \bar{z} + C$$

and this leads to the following

$$F(z_0) \not\subset \bar{z} + C,$$

which conflicts with Assumption (iii). So, condition (3.7) holds. Again making use of Proposition 2.6 (1) in Luc [11], we have that

$$IMin\Big(F(\bar{x},\bar{y},\bar{y})|C\Big) \neq \emptyset.$$

So there exists $\widetilde{z} \in F(\bar{x},\bar{y},\bar{y})$ such that $F(\bar{x},\bar{y},\bar{y}) \subset \widetilde{z}+C$. Combining this with the fact that $z \in F(\bar{x},\bar{y},\bar{y})+C$, one obtains a result as follows

$$z \in \widetilde{z}+C+C \subset \widetilde{z}+C,$$

because C is convex cone. From (3.4) we have

$$F(K\times D\times D) \subset F(\bar{x},\bar{y},\bar{y})+C \subset \widetilde{z}+C,$$

which leads to $\widetilde{z} \in IMin\Big(F(K\times D\times D)|C\Big)$. From there we conclude that

$$F(\bar{x},\bar{y},\bar{y}) \cap IMin\Big(F(K\times D\times D)|C\Big) \neq \emptyset.$$

Thus the problems $(GVOP)_\alpha$ for every $\alpha \in \{P,I,W\}$ has solution, and the claim follows. □

Theorem 3.2. *Let X, Y and Z be Hausdorff local convex topological vector spaces, D and K be nonempty closed subsets on X and Z, respectively, and let C be a closed convex cone on Y. We consider the following multivaluted mappings*

$$S : D \longrightarrow 2^D,$$

$$T : D\times D \longrightarrow 2^K,$$

$$F : K\times D\times D \longrightarrow 2^Y.$$

Suppose, furthermore, that the following conditions are fulfilled

(i). $F(x,y,y)$ and $F(x,y,y)+C$ are closed and nonempty for every $x \in K$ and $y \in D$;

(ii). S and T are compact multivaluted mappings on D and K, respectively;

(iii). F is upper $C-$continuous on $K\times D\times D$;

(iv). $IMin(F(x,y,y)|C) \neq \emptyset$ for every $(x,y) \in K\times D$;

(v). $x \in S(x)$ for all $x \in D$, and $y \in T(z,z)$ for all $(y,z) \in K\times D$.

Then there exists $(\bar{x},\bar{y}) \in K\times D$ such that

$$F(\bar{x},\bar{y},\bar{y}) \cap \alpha Min\Big(F(K\times D\times D)|C\Big) \neq \emptyset,\ \textit{for every}\ \alpha \in \{P,I,W\}.$$

Proof. Let I be given as in the proof of preceding Theorem 3.1. We arbitrarily take sequences

$$(x_\alpha, y_\alpha, y_\alpha)_{\alpha\in I} \subset K \times D \times D$$

and

$$(z_\alpha)_{\alpha\in I} \subset F(K \times D \times D)$$

such that

$$z_\alpha \in F(x_\alpha, y_\alpha, y_\alpha) \ \forall \alpha \in I.$$

It can easily be seen that for each $\alpha \in I$, $x_\alpha \in T(y_\alpha, y_\alpha) \subset T(D \times D) \subset \overline{T(D \times D)}$ and $y_\alpha \in S(y_\alpha) \subset S(D) \subset \overline{S(D)}$. Since $\overline{S(D)}$ is compact, hence there exists a subsequence $\{y_{\alpha_k}\} \subset \{y_\alpha\}_\alpha$ such that $y_{\alpha_k} \longrightarrow \bar{y} \in \overline{S(D)} \subset \overline{D}$. A consequence is $y_\alpha \longrightarrow \bar{y} \in D$ because $D = \overline{D}$ by the initial hypothesis. In the same way as above, we also get $x_\alpha \longrightarrow \bar{x} \in K$. By an argument analogous to that used for the proof of preceding Theorem 3.1 we deduce that the following inclusion holds

$$F(K \times D \times D) \subset F(\bar{x}, \bar{y}, \bar{y}) + C,$$

which yields that

$$F(\bar{x}, \bar{y}, \bar{y}) \cap \alpha Min\Big(F(K \times D \times D)|C\Big) \neq \emptyset, \text{ for every } \alpha \in \{P, I, W\}.$$

As was to be shown. □

Remark 3.3. If, in addition, $C \neq Y$ and $C \setminus C \cap (-C) = intC$ then there exists $(\bar{x}, \bar{y}) \in K \times D$ such that

$$F(\bar{x}, \bar{y}, \bar{y}) \cap PrMin\Big(F(K \times D \times D)|C\Big) \neq \emptyset.$$

In fact, by directly applying Definition 2.2 (iii) in Sec. 2 with $\widetilde{C} = C$, and the fact that (see Theorems 3.1 and 3.2)

$$F(\bar{x}, \bar{y}, \bar{y}) \cap PMin\Big(F(K \times D \times D)|\widetilde{C}\Big) \neq \emptyset,$$

we conclude.

Corollary 3.4. *We assume that all the conditions of Theorem 3.2 are fulfilled. Suppose, in addition, that K convex and for any* $x_1, x_2 \in K$, $y_0 \in D$ *and* $t \in [0,1]$, *one gets*

$$F(x_1, y_0, y_0) \subset F(tx_1 + (1-t)x_2, y_0, y_0) + C.$$

Then there exists $\overline{y} \in D$ such that $\overline{y} \in S(\overline{y})$ and

$$F(x,\overline{y},y) \subset F(x,\overline{y},\overline{y}) + C, \;\; \textit{for all } y \in S(\overline{y}), x \in T(\overline{y},y).$$

Proof. By directly applying the results of Theorem 3.2, there exist $\overline{x} \in K$ and $\overline{y} \in D$ such that

$$F(\overline{x},\overline{y},\overline{y}) \cap IMin\Big(F(K \times D \times D)|C\Big) \neq \emptyset.$$

Making use of the initial assumption, the following assertions hold

• K is a convex subset of X;

• For all $x_1, x_2 \in K, \;\; y_0 \in D$ and $t \in [0,1]$,

$$F(x_1,y_0,y_0) \subset F(tx_1 + (1-t)x_2, y_0, y_0) + C.$$

Taking $x_1 = \overline{x}, y_0 = \overline{y}$, $t = 0$, for all $y \in S(\overline{y}) \subset D$ and $x_2 = x \in T(\overline{y},y) \subset K$, which implies that

$$F(\overline{x},\overline{y},\overline{y}) \subset F(x,\overline{y},\overline{y}) + C. \tag{3.8}$$

Let us see that

$$F(x,\overline{y},y) \subset F(x,\overline{y},\overline{y}) + C. \tag{3.9}$$

In fact, if the left-hand side of (3.9) equals null then nothing to prove, otherwise, we arbitrarily take $s \in F(\overline{x},\overline{y},\overline{y}) \cap IMin\Big(F(K \times D \times D)|C\Big)$, one has

$$F(K \times D \times D) \subset s + C \subset F(\overline{x},\overline{y},\overline{y}) + C,$$

which combines with (3.8), yields that

$$F(x,\overline{y},y) \subset F(K \times D \times D) \subset F(x,\overline{y},\overline{y}) + C.$$

Thus condition (3.9) is valid, or the problem upper quasivariational inclusion problem (UQVIP) has solution.

As was to be shown. □

Corollary 3.5. *Let $X,Y,Z,D,K,C,\ S,T,F$ be given as in Theorem 3.2. Assume, furthermore, that the following conditions are satified*

(i). $F(x,y,y)$ and $F(x,y,y) - C$ are closed and nonempty for every $(x,y) \in K \times D$;

(ii). S and T are compact multivaluted mappings on D and K, respectively;

(iii). F is upper $(-C)$- continuous on $K \times D \times D$;

(iv). $IMax(F(x,y,y)|C) \neq \emptyset$ for every $(x,y) \in K \times D$;

(v). $x \in S(x)$ for all $x \in D$, and $y \in T(z,z)$ for all $(y,z) \in K \times D$;

(vi). $K \times D$ convex and for any $x_1, x_2 \in K,\ y_0, y_1, y_2 \in D$ and $t \in [0,1]$, one gets

$$F(tx_1 + (1-t)x_2, y_0, ty_1 + (1-t)y_2) \subset F(x_1, y_0, y_1) - C.$$

Then there exists $\bar{y} \in D$ such that $\bar{y} \in S(\bar{y})$ and

$$F(x, \bar{y}, \bar{y}) \subset F(x, \bar{y}, y) - C, \text{ for all } y \in S(\bar{y}), x \in T(\bar{y}, y).$$

Proof. We note that

$$IMin(F(x,y,y)|(-C)) = IMax(F(x,y,y)|C)$$

for all $(x,y) \in K \times D$. By using the results obtained of preceding Theorem 3.2, with $(-C)$ is replace by C, it follows that there exists $(\bar{x}, \bar{y}) \in K \times D$ such that

$$F(\bar{x}, \bar{y}, \bar{y}) \cap IMin\Big(F(K \times D \times D)|(-C)\Big) \neq \emptyset,$$

which is equivalent to

$$F(\bar{x}, \bar{y}, \bar{y}) \cap IMax\Big(F(K \times D \times D)|C\Big) \neq \emptyset.$$

Consequently, we have

$$F(K \times D \times D) \subset F(\bar{x}, \bar{y}, \bar{y}) - C. \tag{3.10}$$

Indeed, we take $s \in F(\bar{x}, \bar{y}, \bar{y}) \cap IMax\Big(F(K \times D \times D)|C\Big)$, which means that $s \in F(\bar{x}, \bar{y}, \bar{y})$ and moreover

$$F(K \times D \times D) \subset s - C \subset F(\bar{x}, \bar{y}, \bar{y}) - C.$$

Thus, condition (3.10) is verified.

On the one hand, it follows from the hypotheses (vi) that the set $K \times D$ convex, and for any $x_1, x_2 \in K,\ y_0, y_1, y_2 \in D$ and $t \in [0,1]$,

$$F(tx_1 + (1-t)x_2, y_0, ty_1 + (1-t)y_2) \subset F(x_1, y_0, y_1) - C.$$

By taking $x_2 = \bar{x}, y_0 = y_2 = \bar{y}$, $t = 0$, for all $y_1 = y \in S(\bar{y}) \subset D$ and $x_1 = x \in T(\bar{y}, y) \subset K$, one obtains the following result

$$F(\bar{x}, \bar{y}, \bar{y}) \subset F(x, \bar{y}, y) - C. \tag{3.11}$$

Combining (3.10)-(3.11), yields that

$$F(K \times D \times D) \subset F(x, \bar{y}, y) - C. \tag{3.12}$$

From (3.12), we deduce that there exists $\bar{y} \in D$ such that $\bar{y} \in S(\bar{y})$ and

$$F(x, \bar{y}, \bar{y}) \subset F(x, \bar{y}, y) - C, \text{ for all } y \in S(\bar{y}), x \in T(\bar{y}, y),$$

meaning that the problem lower quasivariational inclusion problem (LQVIP) has solution. As was to be shown. □

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