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# GENERALIZED $f$-VECTOR EQUILIBRIUM PROBLEM

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**Abstract.** In this paper, we consider a generalized $f$-vector equilibrium problem and prove some existence results in the setting of Hausdorff topological vector spaces and reflexive Banach spaces. Our results extend and improve some known results in the literature. Some examples are given.

**Keywords:** vector equilibrium problem; existence; algorithm; convexity.

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## 1. Introduction

In order to describe the real world and economic behavior better, recently, much attention has been attracted to multicriteria equilibrium models. It is well known that equilibrium problem is closely related to game theory, mechanics and physics, economics and finance, transportation and operation research, variational inequality and complementarity problem, optimization and control problem, etc., see, for example, [1-7,9-19] and references therein. Recently, the vector equilibrium problems have studied by many authors, see, for example [9,14,17-19]. It includes

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as special cases vector variational inequality problems, vector variational-like inequality problems, vector complementarity problems, etc.

Motivated by the applications of vector equilibrium problems, in this paper, we introduce and study generalized $f$-vector equilibrium problem and prove some existence results in the setting of Hausdorff topological vector spaces and reflexive Banach spaces. Some special cases are also discussed.

## 2. Preliminaries

We need the following definitions and results for the presentation of this paper.

**Definition 2.1.** Let $\eta : K \times K \to X$, $f : K \to K$ and $g : K \times X \to Y$ be the mappings. Then $g$ is said to be

(i) $\eta$-$f$-monotone with respect to $C$, if and only if for all $x, y \in K$

$$g(f(x), \eta(y,x)) + g(f(y), \eta(x,y)) \in -C;$$

(ii) $\eta$-hemicontinuous, if and only if for all $x, y \in K$, $t \in [0,1]$, the mapping $t \longrightarrow g(ty + (1-t)x, \eta(y,x))$ is continuous at $0^+$;

(iii) $\eta$-$f$-pseudomonotone with respect to $C$, if and only if for all $x, y \in K$, $g(f(x), \eta(y,x)) \notin -intC$ implies $g(f(y), \eta(y,x)) \notin -intC$;

(iv) $\eta$-$f$-generally convex, if and only if for all $x, y, w \in K$, $g(f(x), \eta(x,w)) \notin -intC$ and $g(f(x), \eta(y,w)) \notin -intC$,

$$\Rightarrow g(f(x), \eta(\lambda x + (1-\lambda)y), w) \notin -intC.$$

**Definition 2.2.** A mapping $\eta : K \times K \to X$ is said to be affine in the first argument, if and only if for all $x, y, z \in K$ and $t \in [0,1]$,

$$\eta(tx + (1-t)y, z) = t\eta(x,z) + (1-t)\eta(y,z).$$

Similarly one can define the affine property of $\eta$ with respect to the second argument.

**Definition 2.3.** Let $K$ be a subset of a topological vector space $X$. A set-valued mapping $A : K \to 2^X$ is said to be KKM-mapping if, for each finite subset $\{x_1, x_2, x_3, ...x_n\}$ of K, $Co\{x_1, x_2, x_3, ...x_n\} \subseteq \bigcup_{i=1}^{n} A(x_i)$, where $Co\{x_1, x_2, x_3, ...x_n\}$ denotes the convex hull of $\{x_1, x_2, x_3, ...x_n\}$.

The following theorem is important for us to prove the existence results of this paper.

**Theorem 2.1. (KKM Theorem)** [8] *Let $K$ be a subset of Hausdorff topological vector space $X$ and let $A : K \to 2^X$ be a KKM mapping. If for each $x \in K$, $A(x)$ is closed, and if for at least one point $x \in K$, $A(x)$ is compact, then $\bigcap_{x \in K} A(x) \neq \phi$.*

An ordered topological space is a pair $(Y, C)$, where $Y$ is Hausdorff topological vector space and $C$ is a pointed closed convex cone with the linear order induced by $C$. The partial order $\leq$ on $Y$ induced by $C$ is defined by

$$\forall \ x, y \in Y, y \leqq x \Leftrightarrow x - y \in C,$$

$$\forall \ x, y \in Y, y \leq x \Leftrightarrow x - y \in C \setminus \{0\},$$

$$\forall \ x, y \in Y, y \nleq x \Leftrightarrow x - y \notin C \setminus \{0\}.$$

If the interior of $C$, $intC \neq \phi$, then the weak ordering relations in $Y$ are also defined as follows:

$$\forall \ x, y \in Y, y < x \Leftrightarrow x - y \in intC,$$

$$\forall \ x, y \in Y, y \nless x \Leftrightarrow x - y \notin intC.$$

Let $X$ and $Y$ be Hausdorff topological vector spaces and $K$ be a nonempty closed convex subset of $X$. Let $C$ be a pointed convex cone in $Y$ with $intC \neq \phi$. Let $g : K \times X \to Y$ be vector valued mapping, $f : K \to K$ and $\eta : K \times K \to X$ be the mappings. We introduce the following generalized $f$-vector equilibrium problem. Find $x_o \in K$ such that

$$g(f(x_o), \eta(y, x_o)) \notin -intC, \forall \ y \in K. \tag{2.1}$$

If $f = I$, the identity mapping and $\eta(y, x_o) = y \in X$, then the problem (2.1) reduces to the vector equilibrium problem of finding $x_o \in K$ such that

$$g(x_o, y) \notin -intC, \forall \ y \in K. \tag{2.2}$$

Problem (2.2) was introduced and studied by Tan and Tinh [22].

In addition, if $Y = R$ and $C = R_+$, then problem (2.1) reduces to the equilibrium problem of finding $x_o \in K$ such that

$$g(x_o, y) \geq 0, \forall \; y \in K. \tag{2.3}$$

Problem (2.3) was introduced and studied by Blum and Oettli [3].

**Example 2.1** Let $X = R$, $K = R_+$, $Y = R^2$, and $C = \{(x,y) : x \leq 0, y \leq 0\}$.
Let $g : K \times X \to Y$, $f : K \to K$ and $\eta : K \times K \to X$ be the mappings such that

$$g(x,y) = (y, x^2), f(x) = \sin x,$$

and

$$\eta(x,y) = x^2 + y^2, \forall \; x, y \in K.$$

Then

$$\begin{aligned} g(f(x), \eta(y,x)) + g(f(y), \eta(x,y)) &= (\eta(y,x), (f(x))^2) + (\eta(x,y), (f(y))^2) \\ &= (y^2 + x^2, \sin x^2) + (x^2 + y^2, \sin y^2) \\ &= (2(x^2 + y^2), \sin x^2 + \sin y^2) \in -C \end{aligned}$$

i.e., $g(f(x), \eta(y,x)) + g(f(y), \eta(x,y)) \in -C$. Hence, $g$ is $\eta$-$f$-monotone with respect to $C$.

**Example 2.2** Let $X = R$, $K = R_+$, $Y = R^2$, and $C = \{(x,y) : x \geq 0, y \leq 0\}$.
Let $g : K \times X \to Y$, $f : K \to K$ and $\eta : K \times K \to X$ be the mappings such that

$$g(x,y) = (y, x^2), f(x) = \sin x,$$

and

$$\eta(x,y) = x - y, \forall \; x, y \in K.$$

Then,

$$\begin{aligned} g(f(x), \eta(x,w)) &= (\eta(x,w), (f(x))^2) \\ &= (x - w, \sin x^2) \notin -intC, \end{aligned}$$

implies $x \geq w$, and

$$\begin{aligned} g(f(x),\eta(y,w)) &= (\eta(y,w),(f(x))^2) \\ &= (y-w,\sin x^2) \notin -intC, \end{aligned}$$

implies that $y \geq w$, so it follows that

$$\begin{aligned} g(f(x),\eta(\lambda x+(1-\lambda)y,w)) &= (\eta(\lambda x+(1-\lambda)y,w),(f(x))^2) \\ &= (\lambda x+(1-\lambda)y-w,\sin x^2) \\ &= (\lambda x+(1-\lambda)y-w+\lambda w-\lambda w,\sin x^2) \\ &= (\lambda(x-w)+(1-\lambda)(y-w),\sin x^2) \notin -intC \end{aligned}$$

implies $g(f(x),\eta(\lambda x+(1-\lambda)y,w)) \notin -intC$. Hence, $g$ is $\eta$-$f$-generally convex.

## 3. Main results

We proof the following equivalence lemma which we need for the proof of our main results.

**Lemma 3.1.** *Let $X$ be Hausdorff topological vector space, $K$ be a closed convex subset of $X$ and $(Y,C)$ be an ordered Hausdorff topological vector space with $intC \neq \phi$. Let $g : K \times X \to Y$ be a vector valued mapping which is $\eta$-$f$-monotone with respect to $C$, homogeneous in the second argument and let $f : K \to K$ be an $\eta$-hemicontinuous mapping. Let $\eta : K \times K \to X$ be a continuous and affine mapping in the first argument such that $\eta(x,x) = 0$ and $\eta(x,y) = -\eta(y,x)$, for all $x,y \in K$. Then the following statements are equivalent.*
*Find $x_o \in K$ such that*

*(i)* $g(f(x_o),\eta(y,x_o)) \notin -intC, \forall\ y \in K$;

*(ii)* $g(f(y),\eta(y,x_o)) \notin -intC, \forall\ y \in K$.

**Proof.** (i)$\Rightarrow$ (ii). Let $x_o$ be a solution of (i), then we have

$$g(f(x_o),\eta(y,x_o)) \notin -intC, \forall\ y \in K.$$

Since $g$ is $\eta$-$f$-monotone with respect to $C$, we have

$$g(f(x_o),\eta(y,x_0)) + g(f(y),\eta(x_o,y)) \in -C,$$

as $\eta(x,y) = -\eta(y,x)$ and $g$ is homogeneous in the second argument, we have

$$g(f(x_o),\eta(y,x_0)) \in -C + g(f(y),\eta(y,x_o)). \tag{3.1}$$

Suppose to the contrary that (ii) is false. Then there exists $y \in K$ such that

$$g(f(y),\eta(y,x_o)) \in -intC.$$

It follows from (3.1) that

$$g(f(x_o),\eta(y,x_o)) \in -C - intC \subset -intC,$$

which contradicts (i). Thus $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$. Conversely, suppose that (ii) holds. That is,

$$g(f(y),\eta(y,x_o)) \notin -intC, \forall\ y \in K.$$

For each $y \in K$, $t \in [0,1]$, we let $y_t = ty + (1-t)x_o$. Since $K$ is convex, $y_t \in K$. Then we have

$$g(f(y_t),\eta(y_t,x_o)) \notin -intC.$$

Since $\eta$ is affine in the first argument and $\eta(x_o,x_o) = 0$, we have

$$g(f(ty + (1-t)x_o), t\eta(y,x_o)) \notin -intC.$$

By homogeneity of $g$ in the second argument and dividing by $t$, we obtain

$$g(f(ty + (1-t)x_o),\eta(y,x_o)) \notin -intC.$$

As $f$ is $\eta$-hemicontinuous, let $t \longrightarrow 0^+$, we have

$$g(f(x_o),\eta(y,x_o)) \notin -intC, \forall\ y \in K.$$

Thus $(ii) \Rightarrow (i)$. This completes the proof.

**Theorem 3.1.** *Let $X$ be a Hausdorff topological vector space and $K$ be a compact and convex subset of $X$, and $(Y,C)$ be an ordered Hausdorff topological vector space with $intC \neq \phi$. Let $\eta : K \times K \to X$ be a continuous mapping and affine in the both arguments such that $\eta(x,x) = 0$ and $\eta(x,y) = -\eta(y,x)$, for all $x,y \in K$. Let $g : K \times X \to Y$ be a vector valued mapping which* *is $\eta$-$f$-monotone with respect to C, homogeneous in the second argument, $f : K \to K$ be an $\eta$-hemicontinuous mapping and let the mapping $x \to g(f(y),\eta(y,x))$ be continuous. Then,problem (2.1) admits a solution, that is, there exists $x_o \in K$ such that*

$$g(f(x_o),\eta(y,x_o)) \notin -intC, \forall\ \ y \in K.$$

**Proof.** For $y \in K$, we define

$$M(y) = \{x \in K : g(f(x),\eta(y,x)) \notin -intC\},$$

$$S(y) = \{x \in K : g(f(y),\eta(y,x)) \notin -intC\}.$$

Clearly $M(y) \neq \phi$, as $y \in M(y)$. We divide the proof into three steps.

**Step 1** We claim that $M : K \to 2^K$ is KKM-mapping. If $M$ is not a KKM-mapping, then there exists $x \in Co\{y_1,y_2,y_3,...y_n\}$ such that for all $t_i \in [0,1], i = 1,2,3,...,n$ with $\sum_{i=1}^{n} t_i = 1$, we have

$$x = \sum_{i=1}^{n} t_i y_i \notin \bigcup_{i=1}^{n} M(y_i).$$

Thus, we have

$$g(f(x),\eta(y_i,x)) \in -intC, i = 1,2,3,....,n.$$

Since $f$ is affine, $\eta$ is affine in the second argument and $\eta(y_i,y_i) = 0$, we have

$$g(f(\sum_{i=1}^{n} t_i y_i),\eta(y_i,\sum_{i=1}^{n} t_i y_i)) = \sum_{i=1}^{n}\sum_{i=1}^{n} t_i t_i g(f(y_i),\eta(y_i,y_i)) \in -intC.$$

It follows that $0 \in -intC$, which is a contradiction. Thus $M$ is KKM mapping.

**Step 2** $\bigcap_{y\in K} M(y) = \bigcap_{y\in K} S(y)$ and $S$ is also a KKM mapping.

If $x \in M(y)$, then $g(f(x),\eta(y,x)) \notin -intC$. By the $\eta$-$f$-monotonicity of $g$ with respect to $C$, homogeneity and using the fact that $\eta(x,y) = -\eta(y,x)$, we have

$$g(f(x),\eta(y,x)) \in g(f(y),\eta(y,x)) - C. \tag{3.2}$$

Suppose that $x \notin S(y)$. Then, we have

$$g(f(y),\eta(y,x)) \in -intC.$$

It follows from (3.2) that

$$g(f(x),\eta(y,x)) \in -intC - C \subset -intC,$$

which contradicts that $x \in M(y)$. Therefore $x \in S(y)$, that is, $M(y) \subset S(y)$. Then

$$\bigcap_{y\in K} M(y) \subset \bigcap_{y\in K} S(y).$$

On the other hand, suppose that $x \in \bigcap_{y\in K} S(y)$. We have

$$g(f(y),\eta(y,x)) \notin -intC, \forall \ y \in K.$$

By Lemma 3.1, we have

$$g(f(x),\eta(y,x)) \notin -intC, \forall \ y \in K.$$

Therefore, $x \in M(y)$, that is, $S(y) \subset M(y)$. Hence, $\bigcap_{y\in K} M(y) \supset \bigcap_{y\in K} S(y)$. So,

$$\bigcap_{y\in K} M(y) = \bigcap_{y\in K} S(y).$$

Also $\bigcap_{y\in K} S(y) \neq \phi$, since $y \in S(y)$. From above, we know that $M(y) \subset S(y)$ and by Step 1, we know that $M$ is a KKM-mapping. Thus $S$ is also KKM-mapping.

**Step 3** For all $y \in K, S(y)$ is closed.

Let $\{x_n\}$ be a sequence in $S(y)$ such that $\{x_n\}$ converges to $x \in K$. Then

$$g(f(y),\eta(y,x_n)) \notin -intC, \forall \ n.$$

Since the mapping $x \longrightarrow g(f(y),\eta(y,x))$ is continuous, we have

$$g(f(y),\eta(y,x_n)) \longrightarrow g(f(y),\eta(y,x)) \notin -intC.$$

We conclude that $x \in S(y)$, that is, $S(y)$ is a closed subset of a compact set $K$ and hence compact. By KKM Theorem 2.1, $\bigcap_{y\in K} S(y) \neq \phi$ and also $\bigcap_{y\in k} M(y) \neq \phi$. Hence there exists $x_o \in \bigcap_{y\in K} M(y) = \bigcap_{y\in K} S(y)$, that is, there exist $x_o \in K$ such that

$$g(f(x_o),\eta(y,x_o)) \notin -intC, \forall \ y \in K.$$

Thus $x_o$ is a solution of problem (2.1). This completes the proof.

If $g$ is $\eta$-$f$-pseudomonotone then we can be obtain the following result from Theorem 3.1.

**Corollary 3.1.** *Let $K$ be a compact convex subset of $X$ and $(Y,C)$ be an ordered topological vector space with $intC \neq \phi$. Let $g : K \times X \to Y$ be a vector-valued mapping which is $\eta$-$f$-pseudomonotone with respect to $C$ and $f : K \to K$ be an $\eta$-hemicontinuous. Let $\eta : K \times K \to X$*

*be a continuous mapping and affine in the both arguments such that $\eta(x,x) = 0$, for all $x \in K$. Let the mapping $x \longrightarrow g(f(x), \eta(y,x))$ be continuous. Then the problem (2.1) is solvable.*

**Proof.** By step 1 of theorem 3.1, it follows that $M$ is KKM-mapping. Also it follows from $\eta$-$f$-pseudomonotonicity of $g$ that $M(y) \subset S(y)$, thus $S$ is also KKM-mapping. By step 3 of Theorem 3.1, the conclusion follows.

**Theorem 3.2.** *Let $X$ be a reflexive Banach space, $(Y,C)$ be an ordered topological vector space with $intC \neq \phi$. Let $K$ be a non empty, bounded and convex subset of $X$. Let $g : K \times X \to Y$ be vector valued mapping which is $\eta$-$f$-monotone with respect to C, homogeneous in the second argument, $\eta$-$f$-generally convex on K and $f : K \to K$ be an $\eta$-hemicontinuous mapping. Let $\eta : K \times K \to X$ be a continuous and affine mapping in the both arguments such that $\eta(x,x) = 0$, and $\eta(x,y) = -\eta(y,x)$, for all $x,y \in K$. Then, problem (2.1) is solvable, that is, there exists $x_o \in K$ such that*

$$g(f(x_o), \eta(y,x_o)) \notin -intC, \forall \ y \in K.$$

**Proof.** For each $y \in K$, let

$$M(y) = \{x \in K : g(f(x), \eta(y,x)) \notin -intC\},$$

$$S(y) = \{x \in K : g(f(y), \eta(y,x)) \notin -intC\}.$$

From the proof of the Theorem 3.1, we know that $S(y)$ is closed and $S$ is a KKM-mapping. We also know that

$$\bigcap_{y \in K} M(y) = \bigcap_{y \in K} S(y).$$

Since $K$ is a bounded, closed and convex subset of reflexive Banach space $X$, therefore $K$ is weakly compact.

Now, we show that $S(y)$ is convex. Suppose that $x_1, x_2 \in S(y)$ and $t_1, t_2 \geq 0$ with $t_1 + t_2 = 1$. Then

$$g(f(y), \eta(y,x_i)) \notin -intC, i = 1,2.$$

Since $g$ is $\eta$-$f$-generally convex, we have

$$g(f(y), \eta(y, t_1x_1 + t_2x_2)) \notin -intC,$$

that is, $t_1x_1 + t_2x_2 \in S(y)$, which implies that S(y) is convex. Since $S(y)$ is closed and convex, $S(y)$ is weakly closed.

As $S$ is a KKM-mapping, $S(y)$ is weakly closed subset of $K$, therefore $S(y)$ is weakly compact. By KKM Theorem 2.1, there exists $x_o \in K$ such that

$$g(f(x_o), \eta(y, x_o)) \notin -intC, \forall \ \ y \in K.$$

Hence problem (2.1) is solvable. This completes the proof.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

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