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SUBDIFFERENTIALS OF A BILATERAL MINIMAL TIME FUNCTION IN NORMED SPACES

MI ZHOU[1], XIAO-LAN LIU[2,*]

[1]Institute of Polytechnic, Sanya College, Sanya, Hainan 572022, China

[2]School of Science, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, China

Abstract. In a general normed space setting, we study the bilateral minimal time function determined by a differential inclusion where the set-valued mapping involved has constant values of a bounded closed convex set U. In particular we show that proximal and Fréchet subdifferentials of a bilateral minimal time function are representable by virtue of corresponding normal cones of sublevel sets of the function and level or suplevel sets of the support function of U.

Keywords: bilateral minimal time function; subdifferentials; normal cone.

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1. Introduction

Let X be a normed vector space and U be a nonempty bounded closed convex subset of X with $0 \in intU$, where $intU$ denotes the set of interior points of the set U. Associated with U is the differential inclusion:

$$\dot{x}(t) \in U \;\; a.e. \;\; t \in [0,T], \;\; x(0) = \alpha. \tag{1.1}$$

*Corresponding author

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A solution to (1.1) is an absolutely continuous function $x(\cdot)$ defined on the time interval $[0,T]$ with initial value $x(0) = \alpha$, in which we say that $x(\cdot)$ is a trajectory that originates from α.

The minimal time control problem, associated to a nonempty closed subset S of X (called the target set), is a problem in which the goal is to steer an initial point α to S along some trajectory in minimal time. The minimal time value is denoted by $T_S(x)$, which could be $+\infty$ if no trajectory from α can reach S.

Various properties of the $T_S(x)$ have been studied in the literature; see [1,2,3,4,5,6]. Since this function is not necessarily convex, some researchers discussed its subdifferential in sense of nonsmooth analysis. Assuming that the origin is an interior point of U, Colombo and Wolenski [2,3] studied the proximal and Fr*é*chet subdifferentials of function $T_S(x)$ in a Hilbert space. He [7] studied the proximal and Fr*é*chet subdifferentials of function $T_S(x)$ in a Banach space. Recently, without requiring the condition $0 \in intU$, Jiang and He [8] obtained some analogous results based on [7] in general normed space setting.

Clarke and Nour [9] studied the Hamilton-Jacobi equation of the minimal time function. For the construction, Clarke and Nour considered the minimal time function as a function of two variables. This new function is called the bilateral minimal time function denoted by $T(\cdot,\cdot)$, and is defined as follows:

For $(\alpha,\beta) \in X \times X$, $T(\alpha,\beta)$ is the minimal time taken by a trajectory $x(\cdot)$ to go from α to β(when no such trajectory exist, $T(\alpha,\beta)$ is taken to $+\infty$), where F is a set-valued mapping, from points x in $\mathbb{R}^n$ to subsets $F(x)$ of $\mathbb{R}^n$. This kind of mapping is also quite useful in geometric optics and study of the eiknoal equation.

The purpose of this paper is to study the subdifferentials of $T(\cdot,\cdot)$, where the set-valued mapping involved has constant values of a bounded closed convex set U with $0 \in intU$. The following section contains a terse review of the required background plus some preliminary results. The main result in Section 3 and 4 is the formula of the proximal and the Fr*é*chet subdifferentials of $T(\cdot,\cdot)$ in terms of normal vectors to its level sets. A similar propagation formula for general nonlinear systems was proven by Nour[10] in Euclidean spaces. However, the normed space version given here is new.

2. Preliminaries

2.1. Background in nonsmooth analysis

Let X be a normed vector space with norm denoted by $\|\cdot\|$. Let X^* denote the topological dual of X. We use $B(x,r)$ to denote the open ball centered at x with radius $r>0$, and $\langle\cdot,\cdot\rangle$ to denote the pairing between X^* and X. Let $g: X\to\mathbb{R}$ be a lower semicontinuous function and $x\in X$. Let us review the following well-known classes of subdifferentials for g at x.

• The proximal subdifferential of g at x is the set

$$\partial^P g(x) := \{\xi\in X^* : \liminf_{\|v\|\to 0}\frac{g(x+v)-g(x)-\langle\xi,v\rangle}{\|v\|^2} > -\infty\}.$$

A user-friendly description of the proximal subdifferential is given by (refer to Theorem 2.5 of [11]):

$$\partial^P g(x) := \{\xi\in X^* : \exists\sigma>0, \delta>0 \; s.t. \; g(x+v)-g(x)\geq\langle\xi,v\rangle-\sigma\|v\|^2, \forall v\in B(0,\delta)\}.$$

If g is convex, then $\partial^P g(x)$ coincides the subdifferential of convex analysis and is simply denoted by $\partial g(x)$. In this case, the above description is equivalent with $\sigma=0$ and $\delta=\infty$, namely,

$$\partial^P g(x) := \partial g(x) = \{\xi\in X^* : g(x+v)-g(x)\geq\langle\xi,v\rangle, \forall v\in X\}.$$

• The Fr*é*chet subdifferential of g at x is the set

$$\partial^F g(x) := \{\xi\in X^* : \liminf_{\|v\|\to 0}\frac{g(x+v)-g(x)-\langle\xi,v\rangle}{\|v\|}\geq 0\}.$$

Let $S\subset X$ be a closed set and let $x\in S$. The proximal normal cone and Fr*é*chet normal cone of S at x are defined as the corresponding subdifferential of the indicator function of S at x and are denoted respectively as $N_S^P(x)$ and $N_S^F(x)$. That is , $\xi\in N_S^P(x)$ if and only if there exist $\sigma>0$ and $\delta>0$ such that $\langle\xi,y-x\rangle\leq\sigma\|y-x\|^2$ for all $y\in S\cap B(x,\delta)$, and $\xi\in N_S^F(x)$ if and only if for any $\sigma>0$, there exists $\delta>0$ such that $\langle\xi,y-x\rangle\leq\sigma\|y-x\|$ for all $y\in S\cap B(x,\delta)$.

For more information on nonsmooth analysis, refer to[11] and [12].

The support function of a set $U\subset X$ is defined by

$$\Im_U(\xi) := \sup_{x\in U}\langle\xi,x\rangle.$$

2.2. Gauge functions and polars

We are interested in a convex set $U \subset X$ which is also closed bounded and with $0 \in intU$. The Minkowski gauge function $g_U : X \to [0,\infty]$ associated to U is defined by

$$g_U(x) := \inf\{t > 0 : t^{-1}x \in U\}$$

and the polar U° of U is the set

$$U^\circ := \{\xi \in X^* : \langle \xi, u\rangle \leq 1, \forall u \in U\}.$$

We next further review some elementary properties of $g_U(\cdot)$, which of course hold for $g_{U^\circ}(\cdot)$, but are not explicitly stated. The gauge function $g_U(\cdot)$ is positively homogenous ($g_U(rx) = rg_U(x)$ for all $x \in X$ and $r \geq 0$), and subadditive ($g_U(x+y) \leq g_U(x) + g_U(y)$ for all $x, y \in X$), and therefore is also convex.

Some useful properties of Minkowski gauge functions are presented in the following proposition.

Proposition 2.1. *Let $U \subset X$ be a bounded closed convex set containing the origin as an interior point. Then*

(1) $v \in bdryU$ *iff* $g_U(v) = 1$.
(2) $U = \{x \in X : g_U(x) \leq 1\}$.
(3) $g_{U^\circ}(\xi) = \sup\limits_{u \in U} \langle \xi, u\rangle$ *and* $g_U(x) = \sup\limits_{\xi \in U^\circ} \langle \xi, x\rangle$.
(4) *Define* $\|U\| = \sup\{\|u\| : u \in U\}$ *and* $\|U^\circ\| = \sup\{\|\xi\| : \xi \in U^\circ\}$. *Then for every* $x \in X$, $\frac{\|x\|}{\|U\|} \leq g_U(x) \leq \|U^\circ\|\|x\|$.
(5) $g_U(\cdot)$ *is Lipschitz on* X *with modulus* $\|U^\circ\|$.

Proof. (1) The ”if” part holds for any convex U with $0 \in U$, since $g_U(v) = 1$ implies $(1+\varepsilon)v \notin U$ for all $\varepsilon > 0$. For ”only if” direction, we prove the contrapositive, and assume $g_U(v) \leq g_0 < 1$. Let $\varepsilon < \frac{1-g_0}{max\{g_U(b'), b' \in \overline{B(0,1)}\}}$. Then for all $b \in \overline{B(0,1)}$, we have

$$g_U(v+\varepsilon b) \leq g_U(v) + \varepsilon g_U(b) \leq g_0 + \frac{1-g_0}{max\{g_U(b'), b' \in \overline{B(0,1)}\}} g_U(b) \leq 1.$$

Therefor $v + \varepsilon B \subseteq U$, and so $v \notin bdryU$, and then (i) holds. The next two properties (2) and (3) can be found in [12] in the more general context of topological vector space.

Now we prove (4) and (5). By the definition of $g_U(x)$ and $\|U^\circ\|$, we have that

$$\begin{aligned} g_U(x) &= \inf\{t > 0 : t^{-1}x \in U\} \\ &= \inf\{t > 0 : \langle \xi, t^{-1}x\rangle \le 1, \forall \xi \in U^\circ\} \\ &= \sup\{\langle \xi, x\rangle, \forall \xi \in U^\circ\} \\ &\le \|U^\circ\|\|x\|. \end{aligned}$$

Moreover, for any positive integer n, there exists $t_n > 0$ such that $t_n^{-1}x \in U$ and $t_n < g_U(x) + \frac{1}{n}$. By the definition of $\|U\|$, we have that $\|t_n^{-1}x\| \le \|U\|$, i.e., $\|t_n^{-1}\|\|x\| \le \|U\|$. It follows that $\|t_n\| \ge \frac{\|x\|}{\|U\|}$. Therefore,

$$g_U(x) > t_n - \frac{1}{n} \ge \frac{\|x\|}{\|U\|} - \frac{1}{n}.$$

Let $n \to \infty$. One can have that $g_U(x) \ge \frac{\|x\|}{\|U\|}$. Then (4) holds.

Now we prove (5). Since $g_U(\cdot)$ is a subadditive function and hence

$$g_U(x) - g_U(y) \le g_U(x-y), \forall x, y \in X. \tag{2.1}$$

In view of (4), $g_U(x-y) \le \|x-y\|\|U^\circ\|$, and it follows that

$$g_U(x) - g_U(y) \le \|x-y\|\|U^\circ\|, \forall x, y \in X.$$

Similarly, we can also obtain that

$$g_U(y) - g_U(x) \le \|y-x\|\|U^\circ\|, \forall x, y \in X.$$

Combining the above two inequalities, we have that

$$\|g_U(y) - g_U(x)\| \le \|y-x\|\|U^\circ\|, \forall x, y \in X.$$

Then the conclusion holds. This completes the proof.

2.3. The bilateral minimal time function

In this paper, the bilateral minimal time function $T(\cdot,\cdot) : X \times X \to [0,\infty)$ is defined as follows:

$$T(\alpha, \beta) = \inf\{T \ge 0 : \text{some trajectory } x(\cdot) \text{ has } x(0) = \alpha, x(T) = \beta\}.$$

If no trajectory between α and β exists, then $T(\alpha,\beta)=+\infty$. Clearly, we have $T(\alpha,\alpha)=0$ for all $\alpha\in X$. The bilateral minimal time function to be discussed which satisfies the differential inclusion (1.1) has the equivalent description as follows:

$$T(\alpha,\beta)=g_U(\beta-\alpha),\forall\alpha,\beta\in X. \tag{2.2}$$

We define

$$\Re(t):=\{(\alpha,\beta)\in X\times X: T(\alpha,\beta)<t, t>0\},$$

$$\Re:=\bigcup_{t>0}\Re(t)=\{(\alpha,\beta)\in X\times X: T(\alpha,\beta)<+\infty\}.$$

The assumption $0\in intU$ implies that $T(\alpha,\beta)<+\infty$ for all $(\alpha,\beta)\in X\times X$, but the next proposition says that $T(\cdot,\cdot)$ is globally Lipschitz.

Proposition 2.2. *The bilateral minimal time function $T(\cdot,\cdot)$ defined in (2.2) is globally Lipschitz on $X\times X$ with modulus $\|U^\circ\|$.*

Proof. Let$(\alpha,\beta),(\alpha',\beta')\in X\times X$. Then

$$\begin{aligned}
T(\alpha',\beta')-T(\alpha,\beta)&=g_U(\beta'-\alpha')-g_U(\beta-\alpha)\\
&\le g_U((\beta'-\alpha')-(\beta-\alpha))\\
&=g_U((\beta'-\beta)+(\alpha-\alpha'))\\
&\le g_U(\beta'-\beta)+g_U(\alpha-\alpha')\\
&\le \|U^\circ\|\|\beta'-\beta\|+\|U^\circ\|\|\alpha-\alpha'\|\\
&=\|U^\circ\|\|(\alpha'-\alpha,\beta'-\beta)\|.
\end{aligned}$$

where the first inequality holds by virtue of (2.1) , and the second inequality follows from the subadditive property of g_U. This completes the proof.

It is easy to check that

- $T(\cdot,\cdot)$ and $T(\cdot,\beta)$ are lower semi-continuous.
- For all $\alpha,\beta,\gamma\in X$, we have the following triangle inequality:

$$T(\alpha,\beta)\le T(\alpha,\gamma)+T(\gamma,\beta).$$

- If U is compact, and $T(\alpha,\beta)<\infty$, then the infimum defining $T(\alpha,\beta)$ is attained.

3. Proximal subdifferential of a bilateral minimal time function

Theorem 3.1. *For all $\alpha \in X$, then*

$$\partial^P T(\alpha,\alpha) = \{(\xi,-\xi) \in X^* \times X^* : \Im_U(-\xi) \le 1\}.$$

Proof. "$\subset$": Suppose $\alpha \in X$ and $(\xi,\theta) \in \partial^P T(\alpha,\alpha)$. Then there exist $\sigma > 0$ and $\delta > 0$ such that

$$T(\alpha',\beta') - T(\alpha,\alpha) - \left\langle (\xi,\theta),(\alpha'-\alpha,\beta'-\alpha)\right\rangle \ge -\sigma\|(\alpha'-\alpha,\beta'-\alpha)\|^2, \forall(\alpha',\beta') \in B((\alpha,\alpha),\delta). \tag{3.1}$$

Setting $\alpha' = \beta'$, together with $T(\alpha,\alpha) = 0$ for all $\alpha \in X$, we get that

$$0 \ge -\sigma\|(\alpha'-\alpha,\alpha'-\alpha)\|^2 + \left\langle (\xi,\theta),(\alpha'-\alpha,\alpha'-\alpha)\right\rangle, \forall \alpha' \in B(\alpha,\delta).$$

Let $v \in X$ and $\alpha' = \alpha_n = \alpha + \frac{v}{n}$ for all $n \in \mathbb{N}$. Then there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$, we have

$$0 \ge -\frac{\sigma}{n^2}\|(v,v)\|^2 + \frac{1}{n}\left\langle (\xi,\theta),(v,v)\right\rangle.$$

Multiplying n in the above inequality, we have that

$$0 \ge -\frac{\sigma}{n}\|(v,v)\|^2 + \left\langle (\xi,\theta),(v,v)\right\rangle.$$

Letting $n \to \infty$, we can obtain that $\langle (\xi,\theta),(v,v)\rangle \le 0$, and by the arbitrariness of v, then $-\xi = \theta$.

We claim that if $\xi \in \partial^P T(\cdot,\alpha)(\alpha)$, we have $\Im_U(-\xi) \le 1$.

Indeed, let $v \in U$ and $t_\lambda := T(\alpha - \lambda v,\alpha)$, where $\lambda > 0$, i.e. $t_\lambda = g_U(\lambda v)$. Since $g_U(\cdot)$ is positively homogenous, $g_U(\lambda v) = \lambda g_U(v)$. According to (2) of Proposition 2.2., $g_U(v) \le 1$. Therefore, $t_\lambda \le \lambda$. Hence, $0 < t_\lambda \le \lambda < +\infty$. Take $\alpha' = \alpha - \lambda v$, $\beta' = \alpha$ in (3.1), then it follows from (3.1) that for sufficient small $\lambda > 0$

$$\lambda \ge t_\lambda \ge -\lambda\langle \xi, v\rangle - \lambda^2\sigma\|v\|^2,$$

which implies that $\langle -\xi, v\rangle \le 1$. Therefore $\Im_U(-\xi) \le 1$.

We also can obtain that if $\theta \in \partial^P T(\alpha,\cdot)(\alpha)$, then $\Im_U(\theta) \le 1$, by the similar method mentioned above.

Since $\partial^P T(\alpha,\alpha) \subset \partial^P T(\cdot,\alpha)(\alpha) \times \partial^P T(\alpha,\cdot)(\alpha)$, then, $\partial^P T(\alpha,\alpha) \subset \{(\xi,-\xi) \in X^* \times X^* : \Im_U(-\xi) \leq 1\}$.

"$\supset$": Suppose now that $\xi \in X^*$ and $\Im_U(-\xi) \leq 1$. We will show that $(\xi,-\xi) \in \partial^P T(\alpha,\alpha)$. Suppose to the contrary, then there exists a sequence $(\alpha_n,\beta_n) \in X \times X$ such that $(\alpha_n,\beta_n) \neq (\alpha,\alpha), (\alpha_n,\beta_n) \to (\alpha,\alpha)$ and

$$T_n = T(\alpha_n,\beta_n) < -n\|(\alpha_n-\alpha,\beta_n-\alpha)\|^2 + \langle(\xi,-\xi),(\alpha_n-\alpha,\beta_n-\alpha)\rangle, \forall n \in \mathbb{N}. \tag{3.2}$$

Then we have

$$\begin{aligned} T_n &\leq \|\xi\|\|(\alpha_n-\alpha,\beta_n-\alpha)\| - n\|(\alpha_n-\alpha,\beta_n-\alpha)\|^2 \\ &\leq \|\xi\|\|(\alpha_n-\alpha,\beta_n-\alpha)\| - \|(\alpha_n-\alpha,\beta_n-\alpha)\|^2, \end{aligned} \tag{3.3}$$

and $T_n \to 0$ as $n \to \infty$. Since $T_n < +\infty$, there exists a trajectory x_n on $[0,+\infty)$ such that $x_n(0) = \alpha_n$, and $x_n(T_n) = \beta_n$. Therefore,

$$\beta_n - \alpha_n = \int_0^{T_n} \dot{x_n}(t)dt.$$

Moreover, for $\forall n, \forall t \in [0,T_n]$, we have

$$\begin{aligned} \|x_n(t)-\alpha\| &\leq \|x_n(t)-\alpha_n\| + \|\alpha_n-\alpha\| \\ &= \|\int_0^t \dot{x_n}(t')dt'\| + \|\alpha_n-\alpha\| \\ &\leq \int_0^t \|\dot{x_n}(t')\|dt' + \|\alpha_n-\alpha\| \\ &\leq M\cdot T_n + \|\alpha_n-\alpha\|, (by\ the\ boundedness\ of\ U\ for\ some\ scalar\ M) \end{aligned}$$

and then

$$\|x_n(t)-\alpha\| \leq M\cdot T_n + \|(\alpha_n-\alpha,\beta_n-\alpha)\|. \tag{3.4}$$

Since,

$$\begin{aligned}
\langle \xi, \alpha_n - \beta_n \rangle &= \langle \xi, \alpha_n - \alpha \rangle + \langle \xi, \alpha - x_n(t) \rangle + \langle \xi, x_n(t) - \beta_n \rangle \\
&\leq \|\xi\| \|\alpha_n - \alpha\| + \|\xi\| \|x_n(t) - \alpha\| + \left\langle \xi, \int_{T_n}^{t} \dot{x_n}(t')dt' \right\rangle \\
&= \|\xi\| \|\alpha_n - \alpha\| + \|\xi\| \|x_n(t) - \alpha\| + \left\langle -\xi, \int_{t}^{T_n} \dot{x_n}(t')dt' \right\rangle \\
&\leq \|\xi\| \|\alpha_n - \alpha\| + \|\xi\| (MT_n + \|(\alpha_n - \alpha, \beta_n - \alpha)\|) + \|\xi\| MT_n \\
&\leq 2\|\xi\| \|(\alpha_n - \alpha, \beta_n - \alpha)\| + (2\|\xi\|M - 1)T_n + T_n \\
&\leq \|\xi\| \|(\alpha_n - \alpha, \beta_n - \alpha)\| + (2\|\xi\|M - 1)(\|\xi\| \|(\alpha_n - \alpha, \beta_n - \alpha)\| \\
&\quad - \|(\alpha_n - \alpha, \beta_n - \alpha)\|^2) + T_n \\
&= 2M\|\xi\|^2 \|(\alpha_n - \alpha, \beta_n - \alpha)\| - (2\|\xi\|M - 1)(\|(\alpha_n - \alpha, \beta_n - \alpha)\|^2) + T_n,
\end{aligned}$$

where the second inequality holds by virtue of (3.4) and by the boundedness of U for some scalar M and the forth inequality follows from (3.3). Then, there exists $K(n) > 0$ such that

$$T_n - \langle \xi, \alpha_n - \beta_n \rangle \geq -K(n) \|(\alpha_n - \alpha, \beta_n - \alpha)\|,$$

and this contradicts (3.2) since $\langle (\xi, -\xi), (\alpha_n - \alpha, \beta_n - \alpha) \rangle = \langle \xi, \alpha_n - \beta_n \rangle$.
This completes the proof.

Theorem 3.2. *For all* $(\alpha, \beta) \in X$ *with* $\alpha \neq \beta$*, then*

$$\partial^P T(\alpha, \beta) = N^P_{A(r)}(\alpha, \beta) \cap \{(\xi, \theta) \in X^* \times X^* : \Im_U(-\xi) = \Im_U(\theta) = 1\},$$

where $T(\alpha, \beta) = r$ *and* $A(r) = \{(x, y) \in X \times X : T(x, y) \leq r\}$.

Proof. Let $(\alpha, \beta) \in X \times X$ with $\alpha \neq \beta$, and $T(\alpha, \beta) = r$. Let $(\xi, \theta) \in \partial^P T(\alpha, \beta)$.
Then there exist $\sigma > 0$ and $\delta > 0$ such that

$$T(\alpha', \beta') \geq r - \sigma \|(\alpha' - \alpha, \beta' - \beta)\|^2 + \left\langle (\xi, \theta), (\alpha' - \alpha, \beta' - \beta) \right\rangle. \tag{3.5}$$

$\forall (\alpha', \beta') \in B((\alpha, \beta), \delta)$. If we take $(\alpha', \beta') \in A(r) \cap B((\alpha, \beta), \delta)$, we get

$$0 \geq -\sigma \|(\alpha' - \alpha, \beta' - \beta)\|^2 + \left\langle (\xi, \theta), (\alpha' - \alpha, \beta' - \beta) \right\rangle.$$

Hence, $(\xi,\theta)\in N^P_{A(r)}(\alpha,\beta)$. We claim that if any $\xi\in\partial^P T(\cdot,\beta)(\alpha)$ with $\alpha\neq\beta$, then $\Im_U(-\xi)=1$. Indeed, let $v\in U$ and $t_\lambda=T(\alpha-\lambda v,\beta)$, where $\lambda>0$, i.e.,

$$g_U(\beta-\alpha+\lambda v)=t_\lambda\le g_U(\beta-\alpha)+g_U(\lambda v)=r+\lambda g_U(v)\le r+\lambda.$$

It follows from (3.5) that for $\lambda\in(0,\frac{\delta}{1+\|v\|})$,

$$\lambda\ge t_\lambda-r\ge\lambda\langle-\xi,v\rangle-\lambda^2\sigma\|v\|^2.$$

Dividing the above inequality by λ, we obtain that

$$1\ge\langle-\xi,v\rangle-\lambda\sigma\|v\|^2,\forall\lambda\in(0,\frac{\delta}{1+\|v\|}).$$

Letting $\lambda\to0^+$, we have that $\langle-\xi,v\rangle\le1$. Therefore, $\Im_U(-\xi)\le1$.

For any $\eta>0$, let $0<\varepsilon<min\{\frac{\eta^2}{(M^2\sigma+1)^2},r^2\}$, where $M=\sup\limits_{v\in U}\|v\|$ and σ is the constant in (3.5). Take $u=\frac{\beta-\alpha}{r_1}\in U$, where $r_1\in(r,r+\varepsilon)$. Then, we have that

$$\begin{aligned}T(\alpha+\sqrt{\varepsilon}u,\beta)&=g_U(\beta-\alpha-\sqrt{\varepsilon}u)\\&=g_U(\frac{(r_1-\sqrt{\varepsilon})(\beta-\alpha)}{r_1})\\&\le r_1-\sqrt{\varepsilon}\\&<r+\varepsilon-\sqrt{\varepsilon}.\end{aligned}$$

Hence, we have

$$\varepsilon-\sqrt{\varepsilon}>T(\alpha+\sqrt{\varepsilon}u,\beta)-r\ge\sqrt{\varepsilon}\langle\xi,u\rangle-\sigma\varepsilon\|u\|^2\ge\sqrt{\varepsilon}\langle\xi,u\rangle-\sigma\varepsilon M^2.$$

Therefore,

$$\inf_{w\in U}\langle\xi,w\rangle\le\langle\xi,u\rangle\le(1+\sigma M^2)\sqrt{\varepsilon}-1<\eta-1.$$

This verifies that $\inf\limits_{w\in U}\langle\xi,w\rangle\le-1$, i.e.,

$$-\inf_{w\in U}\langle\xi,w\rangle\ge1.$$

It is equivalent to

$$\sup_{w\in U}\langle-\xi,w\rangle\ge1.$$

That is,

$$\Im_U(-\xi)\ge1.$$

Since $\Im_U(-\xi) \leq 1$, we can directly get $\Im_U(-\xi) = 1$. Using the similar method mentioned above, we also can obtain that if $\theta \in \partial^P T(\alpha, \cdot)(\beta)$ with $\alpha \neq \beta$, then $\Im_U(\theta) = 1$.

Since

$$\partial^P T(\alpha, \beta) \subset \partial^P T(\cdot, \beta)(\alpha) \times \partial^P T(\alpha, \cdot)(\beta),$$

then, $\Im_U(-\xi) = \Im_U(\theta) = 1$.

Conversely, let $(\xi, \theta) \in N^P_{A(r)}(\alpha, \beta)$ be such that $\Im_U(-\xi) = \Im_U(\theta) = 1$. Then there exist $\sigma > 0$ and $\delta > 0$ such that

$$\big\langle (\xi, \theta), (\alpha' - \alpha, \beta' - \beta) \big\rangle \leq \sigma \|(\alpha' - \alpha, \beta' - \beta)\|^2, \forall (\alpha', \beta') \in B((\alpha, \beta), \delta) \cap A(r). \tag{3.6}$$

Let $M = \sup\limits_{u \in U} \|u\|, \sigma_1 = \sigma(1 + \frac{1}{2} M^2 K^2), \delta_1 = \frac{\delta}{2(1+KM)}$, where $K = \|U^\circ\|$ is the Lipschitz modulus of $T(\cdot, \cdot)$ mentioned in Proposition 2.2..

Now we prove that for all $(\alpha', \beta') \in B((\alpha, \beta), \delta)$,

$$T(\alpha', \beta') - T(\alpha, \beta) - \big\langle (\xi, \theta), (\alpha' - \alpha, \beta' - \beta) \big\rangle \geq -\sigma_1 \|(\alpha' - \alpha, \beta' - \beta)\|^2.$$

If $T(\alpha', \beta') = \infty$, it is trivial.

Now we assume that $t := T(\alpha', \beta') < \infty$. We divide the proof into two cases:

(1) If $(\alpha', \beta') \in B((\alpha, \beta), \delta_1)$ and $T(\alpha', \beta') = t > r$, for any $\varepsilon \in (0, \frac{\delta}{2(M+1)})$ and $t_1 \in (t, t + \varepsilon)$, take $z = \alpha' - \frac{(t_1 - r)(\alpha' - \beta')}{2t}$, and $z' = \beta' + \frac{(t_1 - r)(\alpha' - \beta')}{2t}$. Then

$$\begin{aligned}
T(z, z') &= g_U(\beta' - \alpha' + \frac{(t_1 - r)(\alpha' - \beta')}{t}) \\
&= g_U(\beta' - \alpha' - \frac{(t_1 - r)(\beta' - \alpha')}{t}) \\
&= g_U(\frac{(t - t_1 + r)(\beta' - \alpha')}{t}) \\
&= \frac{t - t_1 + r}{t} g_U(\beta' - \alpha') \\
&= t - t_1 + r \\
&\leq r + \varepsilon.
\end{aligned}$$

Let $\varepsilon \to 0^+$. We can obtain that $T(z,z') \le r$. Moreover,

$$\begin{aligned}
\|(z,z')-(\alpha,\beta)\| &\le \|(\alpha'-\alpha,\beta'-\beta)\| + (t_1-r)\|\frac{\beta'-\alpha'}{t}\| \\
&\le \|(\alpha'-\alpha,\beta'-\beta)\| + (t_1-r)M \\
&\le \|(\alpha'-\alpha,\beta'-\beta)\| + (t+\varepsilon-r)M \\
&\le (1+MK)\|(\alpha'-\alpha,\beta'-\beta)\| + \varepsilon M \\
&\le (1+MK)\delta_1 + \varepsilon M \\
&< (1+MK)\frac{\delta}{2(1+KM)} + \frac{\delta}{2(M+1)}M \\
&< \delta.
\end{aligned}$$

This verifies that $(z,z') \in A(r) \cap B((\alpha,\beta),\delta)$. It follows from (3.6) that

$$\begin{aligned}
&T(\alpha',\beta') - T(\alpha,\beta) - \big\langle (\xi,\theta),(\alpha'-\alpha,\beta'-\beta)\big\rangle \\
&= t-r-\big\langle (\xi,\theta),(\alpha'-z,\beta'-z')\big\rangle - \big\langle (\xi,\theta),(z-\alpha,z'-\beta)\big\rangle \\
&\ge t-r-\big\langle (\xi,\theta),(\alpha'-z,\beta'-z')\big\rangle - \sigma\|(z-\alpha,z'-\beta)\|^2 \\
&= t-r+\frac{t_1-r}{2}\left\langle \xi,\frac{\beta'-\alpha'}{t}\right\rangle - \frac{t_1-r}{2}\left\langle \theta,\frac{\beta'-\alpha'}{t}\right\rangle - \sigma\|(z-\alpha,z'-\beta)\|^2 \\
&\ge t-t_1-\sigma\|(z-\alpha,z'-\beta)\|^2 \\
&\ge -\varepsilon-\sigma\|(z-\alpha'+\alpha'-\alpha,z'-\beta'+\beta'-\beta)\|^2 \\
&= -\varepsilon-\sigma\|(z-\alpha',z'-\beta')+(\alpha'-\alpha,\beta'-\beta)\|^2 \\
&\ge -\varepsilon-\sigma\|(z-\alpha',z'-\beta')\|^2-\sigma\|(\alpha'-\alpha,\beta'-\beta)\|^2 \\
&= -\varepsilon-\sigma\|(-\frac{(t_1-r)(\alpha'-\beta')}{2t},\frac{(t_1-r)(\alpha'-\beta')}{2t})\|^2-\sigma\|(\alpha'-\alpha,\beta'-\beta)\|^2 \\
&\ge -\varepsilon-\frac{\sigma}{2}M^2(t_1-r)^2-\sigma\|(\alpha'-\alpha,\beta'-\beta)\|^2 \\
&\ge -\varepsilon-\frac{\sigma}{2}M^2(t+\varepsilon-r)^2-\sigma\|(\alpha'-\alpha,\beta'-\beta)\|^2.
\end{aligned}$$

Let $\varepsilon \to 0^+$. We obtain that

$$\begin{aligned}
&T(\alpha',\beta') - T(\alpha,\beta) - \big\langle(\xi,\theta),(\alpha'-\alpha,\beta'-\beta)\big\rangle \\
&\geq -\frac{\sigma}{2}M^2(t-r)^2 - \sigma\|(\alpha'-\alpha,\beta'-\beta)\|^2 \\
&\geq -\frac{\sigma}{2}M^2(K\|(\alpha'-\alpha,\beta'-\beta)\|)^2 - \sigma\|(\alpha'-\alpha,\beta'-\beta)\|^2 \\
&\geq -\sigma(1+\frac{1}{2}M^2K^2)\|(\alpha'-\alpha,\beta'-\beta)\|^2 \\
&= -\sigma_1\|(\alpha'-\alpha,\beta'-\beta)\|^2,
\end{aligned}$$

where the second inequality follows from Lipschitz condition.

(2) If $(\alpha',\beta') \in B((\alpha,\beta),\delta_1)$ and $T(\alpha',\beta') = t < r$, for any $\varepsilon \in (0, r-t)$ and $t_\varepsilon \in (t, t+\varepsilon)$, letting $u \in U$ be such that $\langle -\xi, u\rangle > \Im_U(-\xi) - \varepsilon = 1-\varepsilon$, and $\langle \theta, u\rangle > \Im_U(\theta) - \varepsilon = 1-\varepsilon$, and letting $z = \alpha' - \frac{r-t_\varepsilon}{2}u$, $z' = \beta' + \frac{r-t_\varepsilon}{2}u$, we find that

$$\begin{aligned}
T(z,z') &= g_U(\beta'-\alpha'+(r-t_\varepsilon)u) \\
&\leq g_U(\beta'-\alpha') + (r-t_\varepsilon)g_U(u) \\
&\leq t + r - t_\varepsilon \\
&< r.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|(z,z')-(\alpha,\beta)\| &\leq \|(\alpha'-\alpha,\beta'-\beta)\| + (r-t_\varepsilon)\|u\| \\
&\leq \|(\alpha'-\alpha,\beta'-\beta)\| + (r-t)M \\
&\leq (1+MK)\|(\alpha'-\alpha,\beta'-\beta)\| \\
&\leq (1+MK)\delta_1 \\
&< (1+MK)\frac{\delta}{2(1+KM)} \\
&< \delta.
\end{aligned}$$

This verifies that $(z,z') \in A(r) \cap B((\alpha,\beta),\delta)$. It follows from (3.6) that

$$\begin{aligned}
& T(\alpha',\beta') - T(\alpha,\beta) - \big\langle (\xi,\theta),(\alpha'-\alpha,\beta'-\beta)\big\rangle \\
&= t - r - \big\langle (\xi,\theta),(\alpha'-z,\beta'-z')\big\rangle - \big\langle (\xi,\theta),(z-\alpha,z'-\beta)\big\rangle \\
&\geq t - r + \frac{r-t_\varepsilon}{2}\langle -\xi,u\rangle + \frac{r-t_\varepsilon}{2}\langle \theta,u\rangle - \sigma\|(z-\alpha,z'-\beta)\|^2 \\
&> t - r + (r-t_\varepsilon)(1-\varepsilon) - \sigma\|(z-\alpha,z'-\beta)\|^2 \\
&\geq t - r + (r-t-\varepsilon)(1-\varepsilon) - \sigma\|(z-\alpha,z'-\beta)\|^2 \\
&= -\varepsilon(1+r-t-\varepsilon) - \sigma\|(z-\alpha'+\alpha'-\alpha,z'-\beta'+\beta'-\beta)\|^2 \\
&\geq -\varepsilon(1+r-t-\varepsilon) - \sigma\|(z-\alpha',z'-\beta')\|^2 - \sigma\|(\alpha'-\alpha,\beta'-\beta)\|^2 \\
&= -\varepsilon(1+r-t-\varepsilon) - \sigma\|(-\frac{r-t_\varepsilon}{2}u,\frac{r-t_\varepsilon}{2}u)\|^2 - \sigma\|(\alpha'-\alpha,\beta'-\beta)\|^2 \\
&= -\varepsilon(1+r-t-\varepsilon) - \frac{\sigma}{2}\|u\|^2(t_\varepsilon-r)^2 - \sigma\|(\alpha'-\alpha,\beta'-\beta)\|^2 \\
&\geq -\varepsilon(1+r-t-\varepsilon) - \frac{\sigma}{2}\|M\|^2(t_\varepsilon-r)^2 - \sigma\|(\alpha'-\alpha,\beta'-\beta)\|^2 \\
&\geq -\varepsilon(1+r-t-\varepsilon) - \frac{\sigma}{2}\|M\|^2(t+\varepsilon-r)^2 - \sigma\|(\alpha'-\alpha,\beta'-\beta)\|^2.
\end{aligned}$$

Let $\varepsilon \to 0^+$. Applying Lipschitz condition, we obtain that

$$\begin{aligned}
& T(\alpha',\beta') - T(\alpha,\beta) - \big\langle (\xi,\theta),(\alpha'-\alpha,\beta'-\beta)\big\rangle \\
&\geq -\frac{\sigma}{2}M^2(t-r)^2 - \sigma\|(\alpha'-\alpha,\beta'-\beta)\|^2 \\
&\geq -\frac{\sigma}{2}M^2(K\|(\alpha'-\alpha,\beta'-\beta)\|)^2 - \sigma\|(\alpha'-\alpha,\beta'-\beta)\|^2 \\
&= -\sigma(1+\frac{1}{2}M^2K^2)\|(\alpha'-\alpha,\beta'-\beta)\|^2 \\
&= -\sigma_1\|(\alpha'-\alpha,\beta'-\beta)\|^2.
\end{aligned}$$

This completes the proof.

4. Fr*é*chet subdifferential of a bilateral minimal time function

Theorem 4.1. *For all $\alpha \in X$, then*

$$\partial^F T(\alpha,\alpha) = \{(\xi,-\xi) \in X^* \times X^* : \Im_U(-\xi) \leq 1\}.$$

Proof. "$\subset$": Let $(\xi,\theta)\in\partial^F T(\alpha,\alpha)$. Then for any $\sigma>0$, there exists $\delta>0$ such that

$$T(\alpha',\beta')\geq -\sigma\|(\alpha'-\alpha,\beta'-\alpha)\|+\left\langle(\xi,\theta),(\alpha'-\alpha,\beta'-\alpha)\right\rangle,\forall(\alpha',\beta')\in B((\alpha,\alpha),\delta). \tag{4.1}$$

Take $\alpha'=\beta'$. Together with $T(\alpha,\alpha)=0$ for all $\alpha\in X$, we get that

$$0\geq -\sigma\|(\alpha'-\alpha,\alpha'-\alpha)\|+\left\langle(\xi,\theta),(\alpha'-\alpha,\alpha'-\alpha)\right\rangle,\forall\alpha'\in B(\alpha,\delta).$$

Let $v\in X$ and $\alpha'=\alpha_n=\alpha+\frac{v}{n}$ for all $n\in\mathbb{N}$. Then there exists $n_0\in\mathbb{N}$ such that for $n>n_0$, we have

$$0\geq -\frac{\sigma}{n}\|(v,v)\|+\frac{1}{n}\left\langle(\xi,\theta),(v,v)\right\rangle.$$

Hence, $\langle(\xi,\theta),(v,v)\rangle\leq 0$, and by the arbitrariness of σ and v, then $-\xi=\theta$.

We claim that if $\xi\in\partial^F T(\cdot,\alpha)(\alpha)$, we have $\Im_U(-\xi)\leq 1$. Indeed, let $v\in U$ and $t_\lambda:=T(\alpha-\lambda v,\alpha)$, where $\lambda>0$, i.e. $t_\lambda=g_U(\lambda v)$. Since $g_U(\cdot)$ is positively homogenous, $g_U(\lambda v)=\lambda g_U(v)$. According to (2) of Proposition 2.1., $\lambda g_U(v)\leq 1$. Therefore, $t_\lambda\leq\lambda$. hence, $0<t_\lambda\leq\lambda<+\infty$.Take $\alpha'=\alpha-\lambda v$, $\beta'=\alpha$ in(4.1), then it follows from (4.1) that for sufficient small $\lambda>0$

$$\lambda\geq t_\lambda\geq -\lambda\left\langle\xi,v\right\rangle-\lambda\sigma\|v\|,$$

which implies that $\langle-\xi,v\rangle\leq 1+\sigma\|v\|$. Since $\sigma>0$ and $v\in U$ are arbitrary, then $\Im_U(-\xi)\leq 1$. We also can obtain that if $\theta\in\partial^F T(\alpha,\cdot)(\alpha)$, then $\Im_U(\theta)\leq 1$, by the similar method mentioned above.

Since $\partial^F T(\alpha,\alpha)\subset\partial^F T(\cdot,\alpha)(\alpha)\times\partial^F T(\alpha,\cdot)(\alpha)$, we have $\partial^F T(\alpha,\alpha)\subset\{(\xi,-\xi)\in X^*\times X^*:\Im_U(-\xi)\leq 1\}$.

"$\supset$": Suppose that $\xi\in X^*$ and $\Im_U(-\xi)\leq 1$. We will show that $(\xi,-\xi)\in\partial^F T(\alpha,\alpha)$. Suppose,to the contrary, then there exists a sequence $(\alpha_n,\beta_n)\in X\times X$ such that $(\alpha_n,\beta_n)\neq(\alpha,\alpha),(\alpha_n,\beta_n)\to(\alpha,\alpha)$ and

$$T_n=T(\alpha_n,\beta_n)<-n\|(\alpha_n-\alpha,\beta_n-\alpha)\|+\langle(\xi,-\xi),(\alpha_n-\alpha,\beta_n-\alpha)\rangle,\forall n\in\mathbb{N}. \tag{4.2}$$

Then we have

$$T_n\leq\|\xi\|\|(\alpha_n-\alpha,\beta_n-\alpha)\|, \tag{4.3}$$

and $T_n \to 0$ as $n \to \infty$. Since $T_n < +\infty$, there exists a trajectory x_n on $[0,+\infty)$ such that $x_n(0) = \alpha_n$, and $x_n(T_n) = \beta_n$. Therefore

$$\beta_n - \alpha_n = \int_0^{T_n} \dot{x_n}(t)dt.$$

Moreover, for $\forall n, \forall t \in [0,T_n]$, we have

$$\begin{aligned}
\|x_n(t)-\alpha\| &\leq \|x_n(t)-\alpha_n\| + \|\alpha_n-\alpha\| \\
&= \|\int_0^t \dot{x_n}(t')dt'\| + \|\alpha_n - \alpha\| \\
&\leq \int_0^t \|\dot{x_n}(t')\|dt' + \|\alpha_n-\alpha\| \\
&\leq M\cdot T_n + \|\alpha_n-\alpha\|, (by\ the\ boundedness\ of\ U\ for\ some\ scalar\ M)
\end{aligned}$$

and then

$$\|x_n(t)-\alpha\| \leq M\cdot T_n + \|(\alpha_n-\alpha, \beta_n-\alpha)\|. \tag{4.4}$$

Since

$$\begin{aligned}
\langle \xi, \alpha_n - \beta_n\rangle &= \langle \xi, \alpha_n-\alpha\rangle + \langle \xi, \alpha - x_n(t)\rangle + \langle \xi, x_n(t) - \beta_n\rangle \\
&\leq \|\xi\|\|\alpha_n-\alpha\| + \|\xi\|\|x_n(t)-\alpha\| + \left\langle \xi, \int_{T_n}^t \dot{x_n}(t')dt' \right\rangle \\
&= \|\xi\|\|\alpha_n-\alpha\| + \|\xi\|\|x_n(t)-\alpha\| + \left\langle -\xi, \int_t^{T_n} \dot{x_n}(t')dt' \right\rangle \\
&\leq \|\xi\|\|\alpha_n-\alpha\| + \|\xi\|(MT_n + \|(\alpha_n-\alpha,\beta_n-\alpha)\|) + \|\xi\|MT_n \\
&\leq 2\|\xi\|\|(\alpha_n-\alpha,\beta_n-\alpha)\| + (2\|\xi\|M-1)T_n + T_n \\
&\leq 2\|\xi\|\|(\alpha_n-\alpha,\beta_n-\alpha)\| + 2(M\|\xi\|^2 - \|\xi\|)\|(\alpha_n-\alpha,\beta_n-\alpha)\| + T_n \\
&= 2M\|\xi\|^2\|(\alpha_n-\alpha,\beta_n-\alpha)\| + T_n,
\end{aligned}$$

where the second inequality holds by virtue of (4.4) and the fourth inequality follows from (4.3). Then, there exists $K > 0$ such that

$$T_n - \langle \xi, \alpha_n - \beta_n\rangle \geq -K\|(\alpha_n-\alpha,\beta_n-\alpha)\|,$$

and this contradicts (4.2). Note that $\langle (\xi,-\xi),(\alpha_n-\alpha,\beta_n-\alpha)\rangle = \langle \xi, \alpha_n-\beta_n\rangle$. This completes the proof.

Theorem 4.2. *For all* $(\alpha,\beta)\in X$ *with* $\alpha\neq\beta$*, then*

$$\partial^F T(\alpha,\beta)=N^F_{A(r)}(\alpha,\beta)\cap\{(\xi,\theta)\in X^*\times X^*:\Im_U(-\xi)=\Im_U(\theta)=1\},$$

where $T(\alpha,\beta)=r$ *and* $A(r)=\{(x,y)\in X\times X: T(x,y)\leq r\}$.

Proof. Let $(\alpha,\beta)\in X\times X$ with $\alpha\neq\beta$, and $T(\alpha,\beta)=r$. Let $(\xi,\theta)\in\partial^F T(\alpha,\beta)$. Then for any $\sigma>0$, there exists $\delta>0$ such that

$$T(\alpha',\beta')\geq r-\sigma\|(\alpha'-\alpha,\beta'-\beta)\|+\left\langle(\xi,\theta),(\alpha'-\alpha,\beta'-\beta)\right\rangle,\forall(\alpha',\beta')\in B((\alpha,\beta),\delta). \tag{4.5}$$

If we take $(\alpha',\beta')\in A(r)\cap B((\alpha,\beta),\delta)$, we get

$$0\geq-\sigma\|(\alpha'-\alpha,\beta'-\beta)\|+\left\langle(\xi,\theta),(\alpha'-\alpha,\beta'-\beta)\right\rangle.$$

Hence, $(\xi,\theta)\in N^F_{A(r)}(\alpha,\beta)$. We claim that if any $\xi\in\partial^F T(\cdot,\beta)(\alpha)$ with $\alpha\neq\beta$, then $\Im_U(-\xi)=1$. Indeed, let $v\in U$ and $t_\lambda=T(\alpha-\lambda v,\beta)$, where $\lambda>0$, i.e.

$$g_U(\beta-\alpha+\lambda v)=t_\lambda\leq g_U(\beta-\alpha)+g_U(\lambda v)\leq g_U(\beta-\alpha)+\lambda g_U(v)\leq r+\lambda.$$

By the definition of $\partial^F T(\cdot,\beta)(\alpha)$ and referring to (4.5), we have that

$$\lambda\geq t_\lambda-r\geq\lambda\langle-\xi,v\rangle-\lambda\sigma\|v\|.$$

Since $\sigma>0$ is arbitrary, $\langle-\xi,v\rangle\leq 1$. Therefore, $\Im_U(-\xi)\leq 1$. For any $\eta>0$, let $0<\varepsilon<min\{\frac{\eta^2}{4},r^2,\frac{\delta^2}{M^2}\}$ and $r_1\in(r,r+\varepsilon)$, where $M=\sup\limits_{v\in U}\|v\|$ and σ is the constant in (4.5). Then $0<\sqrt{\varepsilon}<r<r_1$. Take $u=\frac{\beta-\alpha}{r_1}\in U$. Then we have that

$$\begin{aligned}T(\alpha+\sqrt{\varepsilon}u,\beta)&=g_U(\beta-\alpha-\sqrt{\varepsilon}u)\\&=g_U(\frac{(r_1-\sqrt{\varepsilon})(\beta-\alpha)}{r_1})\\&\leq r_1-\sqrt{\varepsilon}\\&<r+\varepsilon-\sqrt{\varepsilon}.\end{aligned}$$

Hence, we get that

$$\varepsilon-\sqrt{\varepsilon}>T(\alpha+\sqrt{\varepsilon}u,\beta)-r\geq\sqrt{\varepsilon}\langle\xi,u\rangle-\sigma\sqrt{\varepsilon}\|u\|\geq\sqrt{\varepsilon}\langle\xi,u\rangle-\sigma\sqrt{\varepsilon}M.$$

That is

$$\langle \xi, u\rangle \le \sqrt{\varepsilon}+\sigma M-1<\eta-1.$$

Since $\eta>0$ is arbitrary, we have that $\inf_{w\in U}\langle \xi, w\rangle \le -1$, i.e.,

$$-\inf_{w\in U}\langle \xi, w\rangle \ge 1.$$

It is equivalent to

$$\sup_{w\in U}\langle -\xi, w\rangle \ge 1.$$

That is,

$$\Im_U(-\xi)\ge 1$$

Since $\Im_U(-\xi)\le 1$, we can get $\Im_U(-\xi)=1$. Using the similar method mentioned above, we also can obtain that if $\theta\in\partial^F T(\alpha,\cdot)(\beta)$ with $\alpha\neq\beta$, then $\Im_U(\theta)=1$. Since

$$\partial^F T(\alpha,\beta)\subset \partial^F T(\cdot,\beta)(\alpha)\times\partial^F T(\alpha,\cdot)(\beta),$$

then $\Im_U(-\xi)=\Im_U(\theta)=1$.

Conversely, let $(\xi,\theta)\in N^F_{A(r)}(\alpha,\beta)$ be such that $\Im_U(-\xi)=\Im_U(\theta)=1$. Since $T(\cdot,\cdot)$ is Lipschitz in X with the modulus $K=\|U^\circ\|$ mentioned above in Proposition 2.2..

For any $\sigma>0$, take $\sigma_0\in(0,\frac{\sigma}{1+KM})$. Since $(\xi,\theta)\in N^F_{A(r)}(\alpha,\beta)$, we may assume that $\delta>0$ satisfies that

$$\big\langle(\xi,\theta),(\alpha'-\alpha,\beta'-\beta)\big\rangle\le\sigma_0\|(\alpha'-\alpha,\beta'-\beta)\|,\forall(\alpha',\beta')\in B((\alpha,\beta),\delta)\cap A(r). \tag{4.6}$$

Put $\sigma_1=\sigma_0(1+\frac{1}{2}MK)$ and $\delta_1=\frac{\delta}{2(1+MK)}$.

Now we prove that for $(\alpha',\beta')\in B((\alpha,\beta),\delta)$,

$$T(\alpha',\beta')-T(\alpha,\beta)-\big\langle(\xi,\theta),(\alpha'-\alpha,\beta'-\beta)\big\rangle\ge-\sigma_1\|(\alpha'-\alpha,\beta'-\beta)\|.$$

If $T(\alpha',\beta')=\infty$, it is trivial. Now we assume that $t:=T(\alpha',\beta')<\infty$. We divide the proof into two cases.

(1) If $(\alpha',\beta') \in B((\alpha,\beta),\delta_1)$ and $T(\alpha',\beta') = t > r$, for any $\varepsilon \in (0, \frac{\delta}{2M})$ and $t_1 \in (t, t+\varepsilon)$, take $z = \alpha' - \frac{(t_1-r)(\alpha'-\beta')}{2t}$, and $z' = \beta' + \frac{(t_1-r)(\alpha'-\beta')}{2t}$. Then

$$\begin{aligned}
T(z,z') &= g_U(\beta' - \alpha' + \frac{(t_1-r)(\alpha'-\beta')}{t}) \\
&= g_U(\beta' - \alpha' - \frac{(t_1-r)(\beta'-\alpha')}{t}) \\
&= g_U(\frac{(t-t_1+r)(\beta'-\alpha')}{t}) \\
&= \frac{(t-t_1+r)}{t} g_U((\beta'-\alpha')) \\
&= t - t_1 + r \\
&\leq r + \varepsilon .
\end{aligned}$$

Let $\varepsilon \to 0^+$. This verifies that $T(z,z') \leq r$. Moreover, we have

$$\begin{aligned}
\|(z,z') - (\alpha,\beta)\| &\leq \|(\alpha'-\alpha, \beta'-\beta)\| + (t_1 - r)\|\frac{\beta'-\alpha'}{t}\| \\
&\leq \|(\alpha'-\alpha, \beta'-\beta)\| + (t_1 - r)M \\
&\leq \|(\alpha'-\alpha, \beta'-\beta)\| + (t + \varepsilon - r)M \\
&\leq (1+MK)\delta_1 + \varepsilon M \\
&< (1+MK)\frac{\delta}{2(1+KM)} + \frac{\delta}{2M}M \\
&< \delta.
\end{aligned}$$

This verifies that $(z,z')\in A(r)\cap B((\alpha,\beta),\delta)$. By virtue of (4.6), we see that

$$\begin{aligned}
&T(\alpha',\beta')-T(\alpha,\beta)-\big\langle(\xi,\theta),(\alpha'-\alpha,\beta'-\beta)\big\rangle\\
&=t-r-\big\langle(\xi,\theta),(\alpha'-z,\beta'-z')\big\rangle-\big\langle(\xi,\theta),(z-\alpha,z'-\beta)\big\rangle\\
&\geq t-r-\big\langle(\xi,\theta),(\alpha'-z,\beta'-z')\big\rangle-\sigma_0\|(z-\alpha,z'-\beta)\|\\
&=t-r+\frac{t_1-r}{2}\left\langle\xi,\frac{\beta'-\alpha'}{t}\right\rangle-\frac{t_1-r}{2}\left\langle\theta,\frac{\beta'-\alpha'}{t}\right\rangle-\sigma_0\|(z-\alpha,z'-\beta)\|\\
&\geq t-t_1-\sigma_0\|(z-\alpha,z'-\beta)\|\\
&=-\varepsilon-\sigma_0\|(z-\alpha',z'-\beta')+(\alpha'-\alpha,\beta'-\beta)\|\\
&\geq-\varepsilon-\sigma_0\|(z-\alpha',z'-\beta')\|-\sigma_0\|(\alpha'-\alpha,\beta'-\beta)\|\\
&=-\varepsilon-\sigma_0\|(-\frac{(t_1-r)(\alpha'-\beta')}{2t},\frac{(t_1-r)(\alpha'-\beta')}{2t})\|-\sigma_0\|(\alpha'-\alpha,\beta'-\beta)\|\\
&\geq-\varepsilon-\frac{\sigma_0}{2}M(t_1-r)-\sigma_0\|(\alpha'-\alpha,\beta'-\beta)\|\\
&\geq-\varepsilon-\frac{\sigma_0}{2}M(t+\varepsilon-r)-\sigma_0\|(\alpha'-\alpha,\beta'-\beta)\|.\\
&\geq-\varepsilon-\sigma_0(1+\frac{1}{2}MK)\|(\alpha'-\alpha,\beta'-\beta)\|-\frac{\sigma_0}{2}M\varepsilon\\
&\geq-(1+\frac{\sigma_0M}{2})\varepsilon-\sigma_1\|(\alpha'-\alpha,\beta'-\beta)\|.
\end{aligned}$$

Let $\varepsilon\to0^+$. We obtain that

$$T(\alpha',\beta')-T(\alpha,\beta)-\big\langle(\xi,\theta),(\alpha'-\alpha,\beta'-\beta)\big\rangle\geq-\sigma_1\|(\alpha'-\alpha,\beta'-\beta)\|.$$

(2) If $(\alpha',\beta')\in B((\alpha,\beta),\delta_1)$ and $T(\alpha',\beta')=t<r$, for any $\varepsilon\in(0,r-t)$ and $t_\varepsilon\in(t,t+\varepsilon)$, letting $u\in U$ such that $\langle-\xi,u\rangle>\Im_U(-\xi)-\varepsilon=1-\varepsilon$, $\langle\theta,u\rangle>\Im_U(\theta)-\varepsilon=1-\varepsilon$, and letting $z=\alpha'-\frac{r-t_\varepsilon}{2}u$, $z'=\beta'+\frac{r-t_\varepsilon}{2}u$, we find that

$$\begin{aligned}
T(z,z')&=g_U(\beta'-\alpha'+(r-t_\varepsilon)u)\\
&\leq g_U(\beta'-\alpha')+(r-t_\varepsilon)g_U(u)\\
&\leq t+r-t_\varepsilon\\
&<r.
\end{aligned}$$

Moreover,

$$\begin{aligned}\|(z,z')-(\alpha,\beta)\| &\leq \|(\alpha'-\alpha,\beta'-\beta)\|+(r-t_{\varepsilon})\|u\|\\ &\leq \|(\alpha'-\alpha,\beta'-\beta)\|+(r-t)M\\ &\leq (1+MK)\|(\alpha'-\alpha,\beta'-\beta)\|\\ &\leq (1+MK)\delta_1\\ &= (1+MK)\frac{\delta}{2(1+MK)}\\ &< \delta.\end{aligned}$$

This verifies that $(z,z')\in A(r)\cap B((\alpha,\beta),\delta)$. It follows from (4.6) that

$$\begin{aligned}&T(\alpha',\beta')-T(\alpha,\beta)-\big\langle(\xi,\theta),(\alpha'-\alpha,\beta'-\beta)\big\rangle\\ &=t-r-\big\langle(\xi,\theta),(\alpha'-z,\beta'-z')\big\rangle-\big\langle(\xi,\theta),(z-\alpha,z'-\beta)\big\rangle\\ &\geq t-r+\frac{r-t_{\varepsilon}}{2}\langle-\xi,u\rangle+\frac{r-t_{\varepsilon}}{2}\langle\theta,u\rangle-\sigma_0\|(z-\alpha,z'-\beta)\|\\ &> t-r+(r-t_{\varepsilon})(1-\varepsilon)-\sigma_0\|(z-\alpha,z'-\beta)\|\\ &\geq t-r+(r-t-\varepsilon)(1-\varepsilon)-\sigma_0\|(z-\alpha,z'-\beta)\|\\ &=-\varepsilon(1+r-t-\varepsilon)-\sigma_0\|(z-\alpha'+\alpha'-\alpha,z'-\beta'+\beta'-\beta)\|\\ &\geq -\varepsilon(1+r-t-\varepsilon)-\sigma_0\|(z-\alpha',z'-\beta')\|-\sigma_0\|(\alpha'-\alpha,\beta'-\beta)\|\\ &=-\varepsilon(1+r-t-\varepsilon)-\sigma_0\|(-\frac{r-t_{\varepsilon}}{2}u,\frac{r-t_{\varepsilon}}{2}u)\|-\sigma_0\|(\alpha'-\alpha,\beta'-\beta)\|\\ &=-\varepsilon(1+r-t-\varepsilon)-\frac{\sigma_0}{2}\|u\|(t_{\varepsilon}-r)-\sigma_0\|(\alpha'-\alpha,\beta'-\beta)\|\\ &\geq -\varepsilon(1+r-t-\varepsilon)-\frac{\sigma_0}{2}\|M\|(t_{\varepsilon}-r)-\sigma_0\|(\alpha'-\alpha,\beta'-\beta)\|\\ &\geq -\varepsilon(1+r-t-\varepsilon)-\frac{\sigma_0}{2}\|M\|(t+\varepsilon-r)-\sigma_0\|(\alpha'-\alpha,\beta'-\beta)\|.\end{aligned}$$

Letting $\varepsilon\to 0^+$, we have that

$$\begin{aligned}&T(\alpha',\beta')-T(\alpha,\beta)-\big\langle(\xi,\theta),(\alpha'-\alpha,\beta'-\beta)\big\rangle\\ &\geq -\sigma_0 M(t-r)-\sigma_0\|(\alpha'-\alpha,\beta'-\beta)\|\\ &\geq -\sigma_0(1+KM)\|(\alpha'-\alpha,\beta'-\beta)\|\\ &=-\sigma\|(\alpha'-\alpha,\beta'-\beta)\|\end{aligned}$$

and

$$
\begin{aligned}
& T(\alpha',\beta') - T(\alpha,\beta) - \left\langle (\xi,\theta),(\alpha'-\alpha,\beta'-\beta) \right\rangle \\
& \geq -\frac{\sigma_0}{2} M(t-r) - \sigma_0 \|(\alpha'-\alpha,\beta'-\beta)\| \\
& \geq -\frac{\sigma_0}{2} M(K\|(\alpha'-\alpha,\beta'-\beta)\|) - \sigma_0 \|(\alpha'-\alpha,\beta'-\beta)\| \\
& = -\sigma_0 (1+\frac{1}{2}MK)\|(\alpha'-\alpha,\beta'-\beta)\| \\
& = -\sigma_1 \|(\alpha'-\alpha,\beta'-\beta)\|.
\end{aligned}
$$

This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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