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APPROXIMATION METHOD FOR SOLVING NON-MONOTONE EQUILIBRIUM PROBLEMS IN HILBERT SPACES

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Dedicated to Professor Tyrrell Rockafellar on the occasion of his 90th birthday

Abstract. In this paper, we introduce a proximal-type algorithm for solving equilibrium problems in Hilbert spaces, and prove the strong convergence of the generated sequence to a solution of the problem by assuming neither any monotonicity condition on the bifunction nor any weak continuity condition on its arguments. Moreover, assuming the boundedness of the generated sequence, we show that the solution set of the problem is nonempty and the sequence converges strongly to a solution of the problem.
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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let $K \subset H$ be a nonempty, closed and convex set. Suppose that $f : K \times K \to \mathbb{R}$ is a bifunction. The equilibrium problem EP(f,K) consists of finding $x^* \in K$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in K. \tag{1.1}$$

The set of solutions of EP(f,K) isdenoted by $S(f,K)$. The associated dual equilibrium problem is expressed as finding $x^* \in K$ such that $f(y, x^*) \leq 0$ for all $y \in K$. The solution set of the dual problem is denoted by $S_D(f,K)$.

Usually for approximating solutions of the equilibrium problems, some monotonicity assumptions on the bifunction f are assumed. We recall two such properties, although we will not use theses conditions in this paper. The bifunction f is said to be *monotone* if $f(x,y) + f(y,x) \leq 0$ for all $x, y \in K$, and *pseudo-monotone* if, for any pair $x, y \in K$, $f(x,y) \geq 0$ implies $f(y,x) \leq 0$.

The equilibrium problem covers in particular convex optimization problems, variational inequalities, Nash equilibrium problems, and other problems of interest in many applications. The

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study of equilibrium problems goes back to Ky Fan [10]. Subsequently, Brézis, Nirenberg and Stampacchia [5] studied the problem with a coercivity assumption on f. Blum and Oettli [3] proved the existence of solutions to $\mathrm{EP}(f,K)$ with monotonicity condition on f. Later the equilibrium problems were studied extensively for existence of solutions (see for example [12] and the references therein). Recently, much research was devoted to the approximation of solutions to equilibrium problems (see, e.g., [19, 20, 23] and the references therein). Equilibrium problems with monotone and pseudo-monotone bifunctions were studied extensively in Hilbert, Banach as well as in topological vector spaces by many authors (e.g., [2, 6, 7, 12, 13, 16, 17, 23]).

The proximal point algorithm is one of the most famous algorithms which has been applied originally for approximating a solution of a variational inequality associated with a maximal monotone operator, or finding a zero of a maximal monotone operator. This algorithm was first introduced by Martinet [18] and then systematically studied by Rockafellar [22] for maximal monotone operators. Let H be a real Hilbert space, and $A : D(A) \subset H \to H$ be a nonlinear, possibly multivalued maximal monotone operator. Assume that $A^{-1}(0) \neq \varnothing$. The proximal point algorithm is the scheme that generates the sequence $\{x_k\}$ by the following process:

$$x_{k-1} - x_k \in \lambda_k A x_k, \tag{1.2}$$

where $x_0 \in H$ and $\{\lambda_k\}$ is a positive sequence. Rockafellar [22] proved the weak convergence of the sequence $\{x_k\}$ to a zero of the maximal monotone operator A, with appropriate assumptions on $\{\lambda_k\}$. Brézis and Lions [4] extended his result with weaker assumptions on the parameter sequence $\{\lambda_k\}$. Moudafi [20] applied the following proximal point algorithm for solving monotone equilibrium problems:

$$f(x_k,y) + \lambda_{k-1}\langle y - x_k, x_k - x_{k-1}\rangle \geq 0, \quad \forall y \in K. \tag{1.3}$$

Recently, the authors of [9] proved the strong convergence of the sequence generated by the proximal point algorithm to a solution of a quasi-equilibrium problem in Banach spaces.

To approximate a solution of these problems, the extragradient method is one of the useful methods. This method was originally introduced by Korpelevich in 1976 for monotone variational inequalities (see [15]). An extragradient method for equilibrium problems in a Hilbert space H has been studied in [21]. It has the following form:

$$y_k \in \mathrm{Argmin}_{y\in K}\left\{\beta_k f(x_k,y) + \frac{1}{2}\|y - x_k\|^2\right\}, \tag{1.4}$$

$$x_{k+1} \in \mathrm{Argmin}_{y\in K}\left\{\beta_k f(y_k,y) + \frac{1}{2}\|y - x_k\|^2\right\}, \tag{1.5}$$

where β_k is a positive sequence. Weak convergence of the sequence generated by (1.4)–(1.5) to a solution of the equilibrium problem was established in [21]. Recently, the authors of [8] proved the strong convergence of the sequence generated by the extragradient method to a solution of a quasi-equilibrium problem in Banach spaces. Hoai et al. in [11] proposed the following proximal-type algorithm

$$x_k = (1 - \frac{1}{\varphi})y_k + \frac{1}{\varphi}x_{k-1}, \tag{1.6}$$

$$y_{k+1} = \mathrm{Argmin}_{y\in K}\left\{\lambda f(y_k,y) + \frac{1}{2}\|y - x_k\|^2\right\}, \tag{1.7}$$

and proved the weak convergence of the generated sequence to a solution of the equilibrium problem, where $\varphi = \frac{\sqrt{5}+1}{2}$, $x_0, y_1 \in K$ and $\lambda > 0$ is an appropriate number.

In this paper, by modifying the proximal-type algorithm, we are able to solve equilibrium problems in Hilbert spaces, and we prove the strong convergence of the generated sequence to a solution of the equilibrium problem without requiring either any monotonicity condition on the bifunction or any weak continuity condition on its arguments. Moreover, by assuming the boundedness of the generated sequence, we show that the generated sequence converges strongly to a solution of the problem, and hence the solution set of the problem is nonempty. Our convergence results hold without any pseudo-monotonicity and weak continuity assumptions on f. In particular, our results contain the case where the bifunction is the sum of two (or finitely many) pseudo-monotone bifunctions, or even the sum of a pseudo-monotone bifunction and a monotone bifunction. As we know, in this case, the sum may not be anymore pseudo-monotone, which is the case studied by previous authors.

The paper is organized as follows. In Section 2, we introduce some preliminary material. In Section 3, we introduce and analyze a proximal-type algorithm for solving equilibrium problems in Hilbert spaces, and we prove the strong convergence of the generated sequence to a solution of the equilibrium problem.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $C \subseteq H$ be nonempty, closed and convex. It is well known that for any $x \in H$ there exists a unique $\bar{x} \in C$ such that

$$\|\bar{x} - x\| = \inf\Big\{ \|y - x\| : y \in C \Big\}.$$

We define the mapping $P_C : H \to C$ by taking $P_C(x)$ to be this unique $\bar{x} \in C$. The mapping P_C is called the metric projection of H onto C (see [1]).

Lemma 2.1. ([1], Theorem 3.14) *Let $C \subset H$ be nonempty, closed and convex, $x \in H$ and $\bar{x} \in C$. Then $\bar{x} = P_C(x)$ if and only if $\langle y - \bar{x}, x - \bar{x} \rangle \leq 0$ for all $y \in C$.*

Lemma 2.2. ([1], Corollary 2.14) *Let $x, y \in H$ and $\alpha \in \mathbb{R}$. Then*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

Proposition 2.3. ([14], Kadec–Klee property) *Let X be a normed space. We say that X has the* Kadec–Klee property *if, for every sequence $\{x_k\} \subset X$, the following implication holds:*

The sequence $\{x_k\}$ converges weakly to some $x \in X$ and $\|x_k\| \to \|x\|$ imply that $\{x_k\}$ converges strongly to x.

We need the following elementary lemma to prove the main theorem in the next section.

Lemma 2.4. *Let $\{a_k\}$ be an unbounded nonnegative sequence. Then there exists a subsequence $\{a_{k_n}\}$ of $\{a_k\}$ such that $a_{k_n-2} \leq a_{k_n}$ for all n, and $a_{k_n} \to \infty$ as $n \to \infty$.*

Proof. Let $\{a_k\}$ be an unbounded nonnegative sequence. Define

$$k_1 = \min\{i \geq 3, a_i > M := \max\{a_1, a_2\}\}. \tag{2.1}$$

$$k_2 = \min\{i \geq k_1, a_i > a_{k_1}\}. \tag{2.2}$$

By continuing this process, we define

$$k_n = \min\{i \geq k_{n-1}, a_i > a_{k_{n-1}}\}. \tag{2.3}$$

Note that we have the following statements:
(2.1) shows that for all i, $3 \leq i < k_1$, we have $a_i \leq M < a_{k_1}$.
(2.2) shows that for all i, $k_1 \leq i < k_2$, we have $a_i \leq a_{k_1} < a_{k_2}$.
By continuing this process, (2.3) shows that for all i, $k_{n-1} \leq i < k_n$, we have $a_i \leq a_{k_{n-1}} < a_{k_n}$. It is clear that for all i, $i < k_n$, we have $a_i < a_{k_n}$. Therefore $a_{k_n-2} \leq a_{k_n}$ for all n. Now since $\{a_k\}$ is unbounded, we conclude that $a_{k_n} \to \infty$ as $n \to \infty$. $\square$

For a sequence $\{x_k\}$ in H, we denote the strong convergence of $\{x_k\}$ to $x \in H$ by $x_k \to x$, and the weak convergence by $x_k \rightharpoonup x$. Let H be a real Hilbert space and $K \subset H$ be a nonempty, closed and convex set. We now introduce the conditions on the bifunction $f : K \times K \to \mathbb{R}$ that we will need for studying the convergence analysis of the generated sequence in the next section.

A_1: $f(\cdot, x) : K \to \mathbb{R}$ is upper semicontinuous for all $x \in K$.
A_2: $f(x, \cdot) : K \to \mathbb{R}$ is convex and lower semicontinuous for all $x \in K$
A_3: f is Lipschitz-type continuous, i.e. there exist two positive constants c_1 and c_2 such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2, \quad \forall x,y,z \in K.$$

Remark 2.5. Note that A_3 implies that $f(x,x) \geq 0$ for all $x \in K$. Also, it can be easily shown that if $f(x,x) \geq 0$ for all $x \in K$ and f satisfies A_1 and A_2, then $S_D(f,K) \subseteq S(f,K)$.

3. MAIN RESULTS

Let H be a real Hilbert space, $K \subset H$ be a nonempty, closed and convex set, and $f : K \times K \to \mathbb{R}$ be a bifunction. In this section, we prove the strong convergence of the sequence generated by the following proximal-type algorithm to a solution of the equilibrium problem by assuming $A_1 - A_3$.

Algorithm 1 The Proximal-type Algorithm

Initialization: Let $x_0, x_{-1}, y_{-1} \in K$, $B_0 = K$, $\lambda_0, \lambda_{-1} > 0$ and $\mu \in (0, \frac{1}{2})$.
Iterative Steps: Given x_k and y_k, compute

$$y_{k+1} = \operatorname{Argmin}\Big\{ f(y_k, y) + \frac{1}{2\lambda_k}\|y - x_k\|^2, \quad y \in K \Big\}, \tag{3.1}$$

$$x_{k+1} = P_{B_{k+1}}(x_0), \tag{3.2}$$

where

$$\begin{aligned} B_{k+1} = \Big\{ x \in B_k : \frac{1}{\lambda_{k-1}}\langle y_{k+1} - y_k, x_{k-1} - y_k\rangle + \frac{1}{\lambda_k}\langle x - y_{k+1}, x_k - y_{k+1}\rangle \\ \leq \frac{\mu}{2\lambda_{k+1}}(\|y_k - y_{k-1}\|^2 + \|y_{k+1} - y_k\|^2) \Big\} \end{aligned}$$

and

$$\lambda_{k+1} := \begin{cases} \lambda_k, & \text{if } f(y_{k-1}, y_{k+1}) - f(y_{k-1}, y_k) - f(y_k, y_{k+1}) \leq 0, \\ \min\Big\{ \frac{\mu(\|y_k - y_{k-1}\|^2 + \|y_{k+1} - y_k\|^2)}{2(f(y_{k-1}, y_{k+1}) - f(y_{k-1}, y_k) - f(y_k, y_{k+1}))}, \lambda_k \Big\}, & \text{otherwise.} \end{cases} \tag{3.3}$$

We divide the proof of the main theorem into several lemmas and propositions. In particular, we will show that the sequences $\{x_k\}$ and $\{y_k\}$ are well defined.

Lemma 3.1. *Suppose that the stepsize $\{\lambda_k\}$ is generated by Algorithm 1. Then $\lim_{k\to\infty}\lambda_k$ exists and $\min\left\{\lambda_0, \frac{\mu}{2\max\{c_1,c_2\}}\right\} \le \lambda_k \le \lambda_0$.*

Proof. Since the bifunction f is Lipschitz-type continuous by A_3, we have

$$\begin{aligned} f(y_{k-1},y_{k+1}) - f(y_{k-1},y_k) - f(y_k,y_{k+1}) &\le c_1\|y_k-y_{k-1}\|^2 + c_2\|y_{k+1}-y_k\|^2 \\ &\le \max\{c_1,c_2\}(\|y_k-y_{k-1}\|^2 + \|y_{k+1}-y_k\|^2). \end{aligned}$$

Therefore if $f(y_{k-1},y_{k+1}) - f(y_{k-1},y_k) - f(y_k,y_{k+1}) > 0$, we have

$$\frac{\|y_k-y_{k-1}\|^2 + \|y_{k+1}-y_k\|^2}{f(y_{k-1},y_{k+1}) - f(y_{k-1},y_k) - f(y_k,y_{k+1})} \ge \frac{1}{\max\{c_1,c_2\}}. \tag{3.4}$$

Using (3.3), we obtain that the lower bound of the sequence $\{\lambda_k\}$ is $\min\{\lambda_0, \frac{\mu}{2\max\{c_1,c_2\}}\}$, and its upper bound is λ_0. Since $\lambda_{k+1}\le\lambda_k$, then $\lim_{k\to\infty}\lambda_k$ exists and is different from zero. □

Proposition 3.2. *Suppose that f is a bifunction satisfying A_2. Then the sequence $\{y_k\}$ is well defined.*

Proof. Define $g : H \to \mathbb{R}\cup\{+\infty\}$ as

$$g(y) = \begin{cases} f(y_k,y) + \frac{1}{2\lambda_k}\|y-x_k\|^2 & y\in K, \\ +\infty & y\notin K. \end{cases} \tag{3.5}$$

By A_2, it is clear that the function g is proper, convex and lower semicontinuous. Also, g is coercive. Hence g has a minimizer y_{k+1}. Therefore the sequence $\{y_k\}$ is well defined. □

Proposition 3.3. *Assume that f is a bifunction satisfying A_2. Let $x,u\in K$ and $\lambda>0$. If*

$$z = \operatorname{Argmin}\left\{f(u,y) + \frac{1}{2\lambda}\|y-x\|^2, \quad y\in K\right\}, \tag{3.6}$$

then

$$\langle y-z, x-z\rangle \le \lambda\Big(f(u,y)-f(u,z)\Big), \qquad \forall y\in K.$$

Proof. Let $y\in K$ be arbitrary and $w = tz+(1-t)y$ where $t\in[0,1)$. Since z solves the minimization problem (3.6), we have

$$\begin{aligned} f(u,z) + \frac{1}{2\lambda}\|z-x\|^2 &\le f(u,w) + \frac{1}{2\lambda}\|w-x\|^2 \\ &= f(u,tz+(1-t)y) + \frac{1}{2\lambda}\|tz+(1-t)y-x\|^2 \\ &\le tf(u,z) + (1-t)f(u,y) \\ &+ \frac{1}{2\lambda}\Big(t\|x-z\|^2 + (1-t)\|x-y\|^2 - t(1-t)\|y-z\|^2\Big), \end{aligned}$$

which implies that

$$\begin{aligned} &\frac{1}{2\lambda}\Big((1-t)\|x-z\|^2 - (1-t)\|x-y\|^2 + t(1-t)\|y-z\|^2\Big) \\ &\le (1-t)f(u,y) - (1-t)f(u,z). \end{aligned}$$

Dividing this inequality by $(1-t)$ and taking the limit as $t \to 1^-$, we get

$$\frac{1}{2\lambda}\left(\|x-z\|^2 - \|x-y\|^2 + \|y-z\|^2\right) \le f(u,y) - f(u,z), \tag{3.7}$$

which shows that

$$\langle y-z, x-z\rangle \le \lambda\left(f(u,y) - f(u,z)\right). \tag{3.8}$$

□

Corollary 3.4. *Assume that f is a bifunction satisfying A_2. Let $x_k, y_k \in K$ and $\lambda_k > 0$. If*

$$y_{k+1} = \operatorname{Argmin}\left\{f(y_k,y) + \frac{1}{2\lambda_k}\|y-x_k\|^2, \quad y \in K\right\},$$

then

$$\langle y - y_{k+1}, x_k - y_{k+1}\rangle \le \lambda_k(f(y_k,y) - f(y_k,y_{k+1})), \quad \forall y \in K. \tag{3.9}$$

Proof. It is a direct consequence of Proposition 3.3. □

Lemma 3.5. *If $S_D(f,K) \neq \emptyset$ and the sequences $\{x_k\}$, $\{y_k\}$ and $\{\lambda_k\}$ are generated by Algorithm 1, then*

$$\frac{1}{\lambda_{k-1}}\langle y_{k+1}-y_k, x_{k-1}-y_k\rangle + \frac{1}{\lambda_k}\langle x^* - y_{k+1}, x_k - y_{k+1}\rangle \le \frac{\mu}{2\lambda_{k+1}}(\|y_k - y_{k-1}\|^2 + \|y_{k+1}-y_k\|^2) \tag{3.10}$$

for every $x^* \in S_D(f,K)$.

Proof. By Corollary 3.4, we have

$$\frac{1}{\lambda_k}\langle y - y_{k+1}, x_k - y_{k+1}\rangle \le f(y_k,y) - f(y_k,y_{k+1}) \tag{3.11}$$

and

$$\frac{1}{\lambda_{k-1}}\langle y - y_k, x_{k-1} - y_k\rangle \le f(y_{k-1},y) - f(y_{k-1},y_k), \tag{3.12}$$

for every $y \in K$. Let $x^* \in S_D(f,K)$. Then we have $f(y_k,x^*) \le 0$. Taking $y = x^*$ in (3.11) and $y = y_{k+1}$ in (3.12), we obtain respectively

$$\frac{1}{\lambda_k}\langle x^* - y_{k+1}, x_k - y_{k+1}\rangle \le -f(y_k,y_{k+1}), \tag{3.13}$$

and

$$\frac{1}{\lambda_{k-1}}\langle y_{k+1} - y_k, x_{k-1} - y_k\rangle \le f(y_{k-1},y_{k+1}) - f(y_{k-1},y_k). \tag{3.14}$$

By adding (3.13) and (3.14), we get

$$\begin{aligned}&\frac{1}{\lambda_{k-1}}\langle y_{k+1}-y_k, x_{k-1}-y_k\rangle + \frac{1}{\lambda_k}\langle x^* - y_{k+1}, x_k - y_{k+1}\rangle \\ &\le f(y_{k-1},y_{k+1}) - f(y_{k-1},y_k) - f(y_k,y_{k+1}).\end{aligned} \tag{3.15}$$

Now if $f(y_{k-1},y_{k+1}) - f(y_{k-1},y_k) - f(y_k,y_{k+1}) > 0$, by using (3.3), we have

$$\lambda_{k+1} \le \frac{\mu(\|y_k - y_{k-1}\|^2 + \|y_{k+1}-y_k\|^2)}{2(f(y_{k-1},y_{k+1}) - f(y_{k-1},y_k) - f(y_k,y_{k+1}))}, \tag{3.16}$$

which implies that

$$f(y_{k-1},y_{k+1})-f(y_{k-1},y_k)-f(y_k,y_{k+1})\leq \frac{\mu}{2\lambda_{k+1}}(\|y_k-y_{k-1}\|^2+\|y_{k+1}-y_k\|^2). \tag{3.17}$$

From (3.15) and (3.17), it follows that

$$\begin{aligned}&\frac{1}{\lambda_{k-1}}\langle y_{k+1}-y_k,x_{k-1}-y_k\rangle+\frac{1}{\lambda_k}\langle x^*-y_{k+1},x_k-y_{k+1}\rangle\\&\leq \frac{\mu}{2\lambda_{k+1}}(\|y_k-y_{k-1}\|^2+\|y_{k+1}-y_k\|^2).\end{aligned} \tag{3.18}$$

Also, note that if

$$f(y_{k-1},y_{k+1})-f(y_{k-1},y_k)-f(y_k,y_{k+1})\leq 0,$$

then (3.15) shows that (3.18) still holds.

□

Lemma 3.6. *Assume that f is a bifunction satisfying A_1-A_3 and the sequence $\{x_k\}$ is generated by Algorithm 1. Then we have*
(i) $S_D(f,K)\subseteq\bigcap_{k=0}^{\infty}B_k$.
(ii) *If $\bigcap_{k=0}^{\infty}B_k\neq\emptyset$, then the sequence $\{x_k\}$ is bounded.*
(iii) *If the sequence $\{x_k\}$ is bounded, then each weak limit point of $\{x_k\}$ belongs to $\cap_{k=0}^{\infty}B_k$.*

Proof. (i) If $\bar{x}\in S_D(f,K)$, then Lemma 3.5 shows that $\bar{x}\in B_k$ for all $k\geq 0$. Therefore $S_D(f,K)\subseteq\bigcap_{k=0}^{\infty}B_k$.

(ii) Let $\bigcap_{k=0}^{\infty}B_k\neq\emptyset$. It is clear that B_k is closed and convex for all $k\geq 0$, hence $\bigcap_{k=0}^{\infty}B_k$ is closed and convex. Let $x^*=P_{\cap_{k=0}^{\infty}B_k}(x_0)$. Since $x^*\in\bigcap_{k=0}^{\infty}B_k$, using (3.2), we get

$$\langle x^*-x_{k+1},x_0-x_{k+1}\rangle\leq 0,$$

which implies that

$$\|x^*-x_{k+1}\|^2+\|x_{k+1}-x_0\|^2\leq\|x^*-x_0\|^2,\ \text{for all } k\geq 0. \tag{3.19}$$

This shows that the sequence $\{x_k\}$ is bounded.

(iii) Assume that the sequence $\{x_k\}$ is bounded. Let $n\geq 0$. Since $x_k\in B_n$ for all $k\geq n$, and B_n is closed and convex, then any weak limit point p of the sequence $\{x_k\}$ belongs to B_n. Since $n\geq 0$ is arbitrary, $p\in\cap_{k=0}^{\infty}B_k$.

□

Proposition 3.7. *Assume that f is a bifunction satisfying A_1-A_3 and the sequence $\{x_k\}$ is generated by Algorithm 1. If $S_D(f,K)\neq\emptyset$ or the sequence $\{x_k\}$ is bounded, then $\{x_k\}$ converges strongly to $x^*=P_{\cap_{k=0}^{\infty}B_k}(x_0)$.*

Proof. Lemma 3.6 shows that $\{x_k\}$ is bounded and each weak limit point of $\{x_k\}$ belongs to $\cap_{k=0}^{\infty}B_k$. We first show that $x_k\rightharpoonup x^*$. Since $x^*\in\cap_{k=0}^{\infty}B_k$ and $x_k=P_{B_k}(x_0)$, we have

$$\langle x^*-x_k,x_0-x_k\rangle\leq 0,$$

which implies that

$$\|x^*-x_k\|^2+\|x_k-x_0\|^2\leq\|x^*-x_0\|^2,\qquad \text{for all } k\geq 0. \tag{3.20}$$

Therefore we have

$$\|x_k-x_0\|^2\leq\|x^*-x_0\|^2,\qquad \text{for all } k\geq 0. \tag{3.21}$$

Let $\{x_{k_j}\}$ be any weakly convergent subsequence of $\{x_k\}$ to some point p. Replacing k by k_j, it follows from (3.21) that

$$\|p-x_0\|^2 \le \liminf_{j\to\infty}\|x_{k_j}-x_0\|^2 \le \limsup_{j\to\infty}\|x_{k_j}-x_0\|^2 \le \|x^*-x_0\|^2. \tag{3.22}$$

Since $x^* = P_{\cap_{k=0}^{\infty}B_k}(x_0)$ and by Lemma 3.6, $p \in \cap_{k=0}^{\infty}B_k$, we have $\|p-x_0\|^2 = \|x^*-x_0\|^2$. Therefore $p = x^*$. This shows that $x_k \rightharpoonup x^*$. Now we prove $x_k \to x^*$. Using (3.21), we get

$$\|x^*-x_0\|^2 \le \liminf_{k\to\infty}\|x_k-x_0\|^2 \le \limsup_{k\to\infty}\|x_k-x_0\|^2 \le \|x^*-x_0\|^2, \tag{3.23}$$

which implies that $\lim_{k\to\infty}\|x_k-x_0\|^2 = \|x^*-x_0\|^2$, and hence $\lim_{k\to\infty}\|x_k\|^2 = \|x^*\|^2$. Therefore by Kadec-Klee property, $x_k \to x^* = P_{\cap_{k=0}^{\infty}B_k}(x_0)$.

□

Theorem 3.8. *Assume that f is a bifunction satisfying $A_1 - A_3$ and the sequence $\{x_k\}$ is generated by Algorithm 1. If $S_D(f,K) \neq \emptyset$ or the sequence $\{x_k\}$ is bounded, then $\{x_k\}$ converges strongly to $x^* \in S(f,K)$.*

Proof. If $S_D(f,K) \neq \emptyset$ or the sequence $\{x_k\}$ is bounded, then Proposition 3.7 shows that $\{x_k\}$ converges strongly to $x^* = P_{\cap_{k=0}^{\infty}B_k}(x_0)$. Therefore we only need to show that $x^* \in S(f,K)$. Since $x^* \in \bigcap_{k=0}^{\infty}B_k$, we obtain from the definition of B_k that

$$\begin{aligned}&\frac{1}{\lambda_{k-1}}\langle y_{k+1}-y_k, x_{k-1}-y_k\rangle + \frac{1}{\lambda_k}\langle x^*-y_{k+1}, x_k-y_{k+1}\rangle\\ &\le \frac{\mu}{2\lambda_{k+1}}(\|y_k-y_{k-1}\|^2+\|y_{k+1}-y_k\|^2),\end{aligned} \tag{3.24}$$

which implies that

$$\begin{aligned}&\frac{1}{2\lambda_{k-1}}(\|y_k-x_{k-1}\|^2+\|y_{k+1}-y_k\|^2-\|y_{k+1}-x_{k-1}\|^2)\\ &+\frac{1}{2\lambda_k}(\|x^*-y_{k+1}\|^2+\|y_{k+1}-x_k\|^2-\|x^*-x_k\|^2)\\ &\le \frac{\mu}{2\lambda_{k+1}}(\|y_k-y_{k-1}\|^2+\|y_{k+1}-y_k\|^2).\end{aligned}$$

Rearranging the terms, we get

$$\begin{aligned}&\Big(\frac{1}{2\lambda_{k-1}}-\frac{\mu}{2\lambda_{k+1}}\Big)\|y_{k+1}-y_k\|^2+\Big(\frac{1}{2\lambda_k}\|y_{k+1}-x_k\|^2-\frac{1}{2\lambda_{k-1}}\|y_{k+1}-x_{k-1}\|^2\Big)\\ &+\frac{1}{2\lambda_{k-1}}\|y_k-x_{k-1}\|^2+\frac{1}{2\lambda_k}(\|x^*-y_{k+1}\|^2-\|x^*-x_k\|^2)\\ &\le \frac{\mu}{2\lambda_{k+1}}\|y_k-y_{k-1}\|^2.\end{aligned} \tag{3.25}$$

From (3.25) we get

$$\begin{aligned}
&\Big(\frac{1}{2\lambda_{k-1}}-\frac{\mu}{2\lambda_{k+1}}\Big)\|y_{k+1}-y_k\|^2+\Big(\frac{1}{2\lambda_k}(\|y_{k+1}\|-\|x_k\|)^2-\frac{1}{2\lambda_{k-1}}(\|y_{k+1}\|+\|x_{k-1}\|)^2\Big)\\
&+\frac{1}{2\lambda_{k-1}}(\|y_k\|-\|x_{k-1}\|)^2+\frac{1}{2\lambda_k}((\|x^*\|-\|y_{k+1}\|)^2-\|x^*-x_k\|^2)\\
&\le\frac{\mu}{2\lambda_{k+1}}(\|y_k\|+\|y_{k-1}\|)^2. \qquad (3.26)
\end{aligned}$$

Lemma 3.1 shows that $\lim_{k\to\infty}\lambda_k=\lambda>0$. Since $\mu\in(0,\frac{1}{2})$, for large enough k, we have $\Big(\frac{1}{2\lambda_{k-1}}-\frac{\mu}{2\lambda_{k+1}}\Big)\|y_{k+1}-y_k\|^2\ge 0$. Therefore rearranging the terms in (3.26), we get

$$\begin{aligned}
&(\frac{1}{2\lambda_k}-\frac{1}{2\lambda_{k-1}})\|y_{k+1}\|^2+\frac{1}{2\lambda_k}(-2\|y_{k+1}\|\|x_k\|+\|x_k\|^2)-\frac{1}{2\lambda_{k-1}}(2\|y_{k+1}\|\|x_{k-1}\|+\|x_{k-1}\|^2)\\
&+\frac{1}{2\lambda_{k-1}}(\|y_k\|-\|x_{k-1}\|)^2+\frac{1}{2\lambda_k}((\|x^*\|-\|y_{k+1}\|)^2-\|x^*-x_k\|^2)\\
&\le\frac{\mu}{2\lambda_{k+1}}(\|y_k\|+\|y_{k-1}\|)^2 \qquad (3.27)
\end{aligned}$$

We claim that the sequence $\{y_k\}$ is bounded. By contradiction, if $\{y_k\}$ is unbounded, then Lemma 2.4 shows that there exists a subsequence $\{y_{k_n}\}$ of $\{y_k\}$ such that $\|y_{k_n-1}\|\le\|y_{k_n+1}\|$ and $\|y_{k_n+1}\|\to\infty$ as $n\to\infty$. Replacing k by k_n in (3.27), since $\|y_{k_n-1}\|\le\|y_{k_n+1}\|$, we get for large enough n,

$$\begin{aligned}
&(\frac{1}{2\lambda_{k_n}}-\frac{1}{2\lambda_{k_n-1}})\|y_{k_n+1}\|^2+\frac{1}{2\lambda_{k_n}}(-2\|y_{k_n+1}\|\|x_{k_n}\|+\|x_{k_n}\|^2)\\
&-\frac{1}{2\lambda_{k_n-1}}(2\|y_{k_n+1}\|\|x_{k_n-1}\|+\|x_{k_n-1}\|^2)\\
&+\frac{1}{2\lambda_{k_n-1}}(\|y_{k_n}\|-\|x_{k_n-1}\|)^2+\frac{1}{2\lambda_{k_n}}((\|x^*\|-\|y_{k_n+1}\|)^2-\|x^*-x_{k_n}\|^2)\\
&\le\frac{\mu}{2\lambda_{k_n+1}}(\|y_{k_n}\|+\|y_{k_n+1}\|)^2\\
&\le\frac{\mu}{2\lambda_{k_n+1}}(2\|y_{k_n}\|^2+2\|y_{k_n+1}\|^2) \qquad (3.28)
\end{aligned}$$

Rearranging the terms in (3.28), we get

$$\begin{aligned}
&(\frac{1}{\lambda_{k_n}}-\frac{1}{2\lambda_{k_n-1}}-\frac{\mu}{\lambda_{k_n+1}})\|y_{k_n+1}\|^2+\frac{1}{2\lambda_{k_n}}(-2\|y_{k_n+1}\|\|x_{k_n}\|+\|x_{k_n}\|^2)\\
&-\frac{1}{2\lambda_{k_n-1}}(2\|y_{k_n+1}\|\|x_{k_n-1}\|+\|x_{k_n-1}\|^2)+(\frac{1}{2\lambda_{k_n-1}}-\frac{\mu}{\lambda_{k_n+1}})\|y_{k_n}\|^2\\
&+\frac{1}{2\lambda_{k_n-1}}(-2\|y_{k_n}\|\|x_{k_n-1}\|+\|x_{k_n-1}\|^2)+\frac{1}{2\lambda_{k_n}}(\|x^*\|^2-2\|x^*\|\|y_{k_n+1}\|-\|x^*-x_{k_n}\|^2)\\
&\le 0. \qquad (3.29)
\end{aligned}$$

Since $\lim_{k\to\infty}\lambda_k=\lambda>0$, $\mu\in(0,\frac{1}{2})$, $x_k\to x^*$, and the sequence $\{y_{k_n+1}\}$ is unbounded, we reach a contradiction by taking limsup as $n\to\infty$ on the left hand side of (3.29). Therefore $\{y_k\}$ is bounded.

Suppose that $\limsup_{k\to\infty}\|y_{k+1}-y_k\|^2 = c \geq 0$. It is easily seen that

$$\lim_{k\to\infty}\Big(\frac{1}{2\lambda_k}\|y_{k+1}-x_k\|^2 - \frac{1}{2\lambda_{k-1}}\|y_{k+1}-x_{k-1}\|^2\Big) = 0. \tag{3.30}$$

Therefore by taking limsup as $k\to\infty$ in (3.25), we get

$$\Big(\frac{1-\mu}{2\lambda}\Big)c + \frac{1}{2\lambda}\liminf_{k\to\infty}\|y_k - x_{k-1}\|^2 + \frac{1}{2\lambda}\liminf_{k\to\infty}\|x^* - y_{k+1}\|^2 \leq \frac{\mu}{2\lambda}c. \tag{3.31}$$

Since $\mu\in(0,\frac{1}{2})$, (3.31) shows that $c=0$, hence

$$\lim_{k\to\infty}\|y_{k+1}-y_k\|^2 = 0 \tag{3.32}$$

and

$$\liminf_{k\to\infty}\|y_k - x_{k-1}\|^2 = \liminf_{k\to\infty}\|x^* - y_{k+1}\|^2 = 0. \tag{3.33}$$

Adding (3.11) and (3.14), we have

$$\begin{aligned}&\frac{1}{\lambda_{k-1}}\langle y_{k+1}-y_k, x_{k-1}-y_k\rangle + \frac{1}{\lambda_k}\langle y - y_{k+1}, x_k - y_{k+1}\rangle\\ &\leq f(y_k,y) + f(y_{k-1},y_{k+1}) - f(y_{k-1},y_k) - f(y_k,y_{k+1}).\end{aligned} \tag{3.34}$$

If $f(y_{k-1},y_{k+1}) - f(y_{k-1},y_k) - f(y_k,y_{k+1}) > 0$, by using (3.3), we have

$$\lambda_{k+1} \leq \frac{\mu(\|y_k - y_{k-1}\|^2 + \|y_{k+1}-y_k\|^2)}{2(f(y_{k-1},y_{k+1}) - f(y_{k-1},y_k) - f(y_k,y_{k+1}))} \tag{3.35}$$

which implies that

$$f(y_{k-1},y_{k+1}) - f(y_{k-1},y_k) - f(y_k,y_{k+1}) \leq \frac{\mu}{2\lambda_{k+1}}(\|y_k-y_{k-1}\|^2 + \|y_{k+1}-y_k\|^2) \tag{3.36}$$

Therefore by using (3.34) and (3.36), we obtain

$$\begin{aligned}&\frac{1}{\lambda_{k-1}}\langle y_{k+1}-y_k, x_{k-1}-y_k\rangle + \frac{1}{\lambda_k}\langle y - y_{k+1}, x_k - y_{k+1}\rangle\\ &\leq f(y_k,y) + \frac{\mu}{2\lambda_{k+1}}(\|y_k-y_{k-1}\|^2 + \|y_{k+1}-y_k\|^2).\end{aligned} \tag{3.37}$$

On the other hand, if $f(y_{k-1},y_{k+1}) - f(y_{k-1},y_k) - f(y_k,y_{k+1}) \leq 0$, then by using (3.34), it is clear that (3.37) still holds. Since $x_k\to x^*$, by using (3.32) and (3.33), we deduce that there exists a subsequence $\{y_{k_j}\}$ of $\{y_k\}$ such that $y_{k_j}\to x^*$ as $j\to\infty$, and

$$\lim_{k\to\infty}\|y_{k_j} - x_{k_j-1}\| = \lim_{k\to\infty}\|x_{k_j} - y_{k_j+1}\| = 0. \tag{3.38}$$

Since $\{y_k\}$ is bounded and $\lim_{k\to\infty}\|y_{k+1}-y_k\|^2 = 0$, replacing k by k_j and then taking limsup as $j\to\infty$ in (3.37), we deduce that $\limsup_{j\to\infty} f(y_{k_j},y)\geq 0$. Since $y_{k_j}\to x^*$, by using A_1, we get $f(x^*,y)\geq 0$ for all $y\in K$, i.e., $x^*\in S(f,K)$. □

We end this section by providing some examples of the equilibrium problems where our main result can be applied.

Example 3.1. Assume that $H=\mathbb{R}^n$ and $K=\Big\{(\xi_1,\xi_2,\cdots,\xi_n)\in\mathbb{R}^n : \xi_i\geq 0\Big\}$. Let $A:\mathbb{R}^n\to\mathbb{R}^n$

be a Lipschitz continuous operator with Lipschitz constant L. Let the bifunction $f : K \times K \to \mathbb{R}$ be defined by $f(x,y) = \langle A(x), y-x\rangle$. Note that

$$\begin{aligned} f(x,y)+f(y,z)-f(x,z) &= \langle A(x), y-x\rangle + \langle A(y), z-y\rangle - \langle A(x), z-x\rangle \\ &= \langle A(x), y-z\rangle + \langle A(y), z-y\rangle \\ &= \langle A(x)-A(y), y-z\rangle \\ &\geq -\|A(x)-A(y)\|\|y-z\| \\ &\geq -L\|x-y\|\|y-z\| \\ &\geq \frac{-L}{2}(\|x-y\|^2+\|y-z\|^2) \end{aligned} \tag{3.39}$$

which shows that A_3 is satisfied. Clearly, A_1 and A_2 are also satisfied. If $\langle A(x), x\rangle \geq 0$ for all $x \in K$ or the sequence $\{x_k\}$ generated by Algorithm 1 is bounded, then Theorem 3.8 implies that $\{x_k\}$ converges to a solution of the equilibrium problem. Moreover if we replace K by $\mathbb{R}^n$, it is clear that the sequence $\{x_k\}$ generated by Algorithm 1 converges to a zero of the operator A.

Example 3.2. Assume that $H = \ell^2 = \Big\{\zeta = (\zeta_1, \zeta_2, \zeta_3, \cdots) : \big(\sum_{i=1}^{\infty} |\zeta_i|^2\big)^{\frac{1}{2}} < \infty\Big\}$ and $K := \Big\{\zeta = (\zeta_1, \zeta_2, \zeta_3, \cdots) \in \ell^2 : \zeta_i \geq 0, \forall i \in \mathbb{N}\Big\}$. Let $f : K \times K \to \mathbb{R}$ be defined as $f(x,y) = \langle 2x, y-x\rangle + \|y\| - \|x\|$. Note that

$$\begin{aligned} f(x,y)+f(y,z)-f(x,z) &= \langle 2x, y-x\rangle + \|y\| - \|x\| + \langle 2y, z-y\rangle + \|z\| - \|y\| \\ &\quad - (\langle 2x, z-x\rangle + \|z\| - \|x\|) \\ &= \langle 2x-2y, y-z\rangle \\ &\geq -2\|x-y\|\|y-z\| \\ &\geq -\|x-y\|^2 - \|y-z\|^2 \end{aligned} \tag{3.40}$$

which shows that A_3 is satisfied. Clearly, A_1 and A_2 are also satisfied. Since $0 \in S_D(f,K)$, the conditions of Theorem 3.8 hold. If $\{x_k\}$ is the sequence generated by Algorithm 1, then $\{x_k\}$ converges strongly to a solution of the equilibrium problem.

Example 3.3. Let $H = L^2([0,1])$ with the inner product

$$\langle x, y\rangle := \int_0^1 x(t)y(t)dt, \qquad \forall\, x, y \in L^2([0,1])$$

and the induced norm

$$\|x\| := \Big(\int_0^1 |x(t)|^2 dt\Big)^{\frac{1}{2}}, \qquad \forall\, x \in L^2([0,1]).$$

Suppose that $K := \Big\{x(t) \in L^2([0,1]) : \|x(t)\| \leq \frac{1}{2}\Big\}$, and $\varphi : K \to \mathbb{R}$ is defined by $\varphi(x) = \cos(2\pi\|x\| + \frac{\pi}{2})$. It is clear that φ is convex and continuous, hence it is weakly lower semi-continuous. We define the bifunction $f : K \times K \to \mathbb{R}$ as $f(x,y) = \varphi(y) - \varphi(x)$. It is easy to see that the assumptions $A_1 - A_3$ are satisfied. Note that any function $x(t) \in K$ satisfying $\|x\| = \frac{1}{4}$ belongs to $S_D(f,K)$. Therefore the conditions of Theorem 3.8 hold. If $\{x_k\}$ is the sequence generated by Algorithm 1, then $\{x_k\}$ converges strongly to a solution of the problem.

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