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A BRIEF REVIEW OF OPTIMAL CONTROL THEORY FOR SYSTEMS GOVERNED BY EVOLUTION EQUATIONS ON BANACH SPACES

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Abstract. In this paper, we present a brief review of optimal control theory for evolution equations on Banach spaces. We consider classical controls, relaxed controls, feedback controls and operator valued measures as controls. We consider general systems containing (non Lipschitz) continuous as well as measurable vector fields which admit measure valued solutions extending the standard notions of classical, strong, mild and weak solutions. We present existence of optimal control policies and develop necessary conditions of optimality whereby one can determine the optimal controls. Based on the necessary conditions of optimality, we present an algorithm including a proof of its convergence whereby the optimal policies can be constructed. Further, we consider non-convex control problems as special cases where the relaxed controls specialize to switching controls, generalizing the bang-bang principle.

Keywords. Differential inclusions; Evolution equation; Optimal control; Structural control; Semilinear and quasilinear systems.

2020 Mathematics Subject Classification. 49J27, 49K27, 93C25, 93C27.

1. Introduction

The subject of control theory and its applications emerged from engineering sciences and technology more than a century ago. With the advancement of science and technology, the field gained widespread recognition and found applications not only in the fields of engineering but also in diverse areas such as defence, space technology, medicine, economics, management science, climate science, in fact almost all spheres of human activities. In all these fields, mathematics and physics play crucial roles. Many of the System dynamic models are based on fundamental principles of physics, mechanics, chemistry, biology, and social sciences requiring both applied and abstract mathematics. Optimal control theory for partial differential equations first appeared in Butkovskiy [43] and shortly after in Lions [53] followed by Ahmed and Teo [11] and Fattorini [49]. Inspired by these early works, more general theory of optimal control for infinite dimensional deterministic as well as stochastic systems continued to develop. With

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the phenomenal development and progress of control theory, the field has become multidisciplinary and highly mathematical and now has become an important area of mathematics. In this paper we present a very brief account of general control theory on infinite dimensional spaces covering systems governed by deterministic as well as stochastic partial differential equations as special cases. Also, we consider optimal structural control problems for semilinear evolution equations on Banach spaces determined by operator valued measures. Along with the presentation of theory, we have also included some interesting examples of practical importance and applications.

The rest of this paper is divided into four sections. (2): Deterministic Systems and Optimal Control (3): Structural Controls and Optimization (4): Stochastic Systems on Hilbert Spaces and Control (5): Measure-Valued Solutions and Optimal Control. We do not claim the review to be comprehensive in any sense. But we believe that the topics covered may inspire young readers to look into broader horizons of theory and applications of control sciences.

2. Deterministic Systems and Optimal Control

Let $\{X,E\}$ be a pair of real Banach spaces and U a Polish space (complete separable metric space) and $\mathscr{M}_1(U)$ the space of probability measures on U. In this section, we consider the following system on the Banach space X

$$dx = Axdt + \hat{F}(t,x,u_t)dt + G(t,x)\gamma(dt), x(0) = x_0, t \in I \equiv [0,T], \tag{2.1}$$

where A is the infinitesimal generator of a C_0-semigroup of operators $\{S(t), t \in I\} \subset \mathscr{L}(X)$, $F : I \times X \times U \longrightarrow X$, and $G : I \times X \longrightarrow \mathscr{L}(E,X)$ are Borel measurable maps. The system is driven by a pair of controls, denoted by $\{u,\gamma\}$, where $\hat{F}(t,x,u_t) \equiv \int_U F(t,x,\xi)u_t(d\xi)$ with u denoting the relaxed control, a measure valued function with values in $\mathscr{M}_1(U)$, and γ is an E valued vector measure containing impulsive controls as special cases. For details on vector measures, we refer to Diestel and Uhl Jr. [46] and Dunford and Schwartz [47].

Let $\mathscr{R}_{ad}$ denote the class of relaxed controls and $\mathscr{M}_{ad}$ the class of E-valued vector measures also considered as controls. In other words, the system is equipped with dual controls for increased reliability. The problem is to find a control $\nu = (u,\gamma) \in \mathscr{R}_{ad} \times \mathscr{M}_{ad}$ that minimizes the following cost functional

$$J(\nu) \equiv J(u,\gamma) = \int_I \ell(t,x(t))dt + \Phi(x(T)). \tag{2.2}$$

Since system (2.1) is subject to vector measures, the solutions may not be continuous. Thus $C(I,X)$ is not a suitable space for the solution trajectories. A suitable space is the space of (norm) bounded measurable functions on I with values in X denoted by $B_\infty(I,X)$ endowed with the norm topology

$$\| x \|_{B_\infty(I,X)} \equiv sup\{\| x(t) \|_X, t \in I\}.$$

With respect to this norm topology, it is a Banach space. Note that this space is a subspace of $L_\infty(I,X)$ and it contains the class of piecewise continuous X valued norm bounded functions which in turn contains $C(I,X)$.

Admissible Controls.

To solve the control problem, we need characterization of the set of admissible controls. Let U denote a compact Polish space, and $C(U)$ the Banach space of real valued continuous functions defined on U equipped with the standard sup-norm topology. The topological dual of this space is given by the space of regular Borel measures $\mathscr{M}(U)$. Let $L_1(I,C(U))$ denote the Lebesgue-Bochner space of integrable functions defined on I with values in the Banach space $C(U)$. The dual of this space is given by $L^w_\infty(I,\mathscr{M}(U))$, the class of weak star measurable functions with values in $\mathscr{M}(U)$. The set of relaxed controls is given by $\mathscr{R}_{ad} = L^w_\infty(I,\mathscr{M}_1(U)) \subset L^w_\infty(I,\mathscr{M}(U))$, where $\mathscr{M}_1(U)$ denotes the class of regular Borel probability measures on U. Clearly, by virtue of Alaoglu's theorem [47], $\mathscr{R}_{ad} \subset L^w_\infty(I,\mathscr{M}(U))$ is a weak star compact set. For the second class of controls, let E be a real Banach space and consider the space of E valued finitely additive vector measures of bounded variations denoted by $\mathscr{M}_{bfa}(\mathscr{A}_I,E)$, where $\mathscr{A}_I$ is the algebra of subsets of the set I. Endowed with the total variation norm, $\mathscr{M}_{bfa}(\mathscr{A}_I,E)$ is a Banach space. Here we consider the smaller class $\mathscr{M}_{bfa}(\Sigma_I,E)$, where Σ_I is the sigma algebra of subsets of the set I. Let $\mathscr{M}_{ad}$ be a weakly compact subset of $\mathscr{M}_{bfa}(\Sigma_I,E)$ denoting the second set of admissible controls. Thus system (2.1) is equipped with dual controls. In other words, for admissible controls, we choose the set $\mathscr{K}_{ad} \equiv \mathscr{R}_{ad} \times \mathscr{M}_{ad}$ endowed with the product topology $\tau_p = \tau_{w^*} \times \tau_w$, where τ_{w^*} and τ_w denote the weak star and weak topologies respectively. An element of $\mathscr{K}_{ad}$ is denoted by the pair $\nu = (u,\gamma)$. In the study of optimal controls, we use compactness of the admissible set $\mathscr{M}_{ad}$ in the τ_p topology as described above. For characterization of weakly compact sets in $\mathscr{M}_{bfa}(\mathscr{A}_I,E)$ and $\mathscr{M}_{bfa}(\Sigma_I,E)$; see the well-known result due to Brooks and Dinculeanu [Diestel, 45] which itself is a generalization of the celebrated theorem due to Bartle-Dundord-Schwartz Diestel [46]. For details on vector measures, we refer to [46, 47]. We need the following assumptions.

Basic Assumptions.

(A): The operator A is the infinitesimal generator of a C_0-semigroup [Ahmed,1] of bounded linear operators $\{S(t) \in \mathscr{L}(X), t \in I\}$ in the Banach space X and there exists a finite positive number M such that

$$\sup\{\| S(t) \|_{\mathscr{L}(X)}, t \in I\} \leq M.$$

(F): $F : I \times X \times U \longrightarrow X$ is a Borel measurable map in all the variables and continuous in the last two arguments satisfying the following growth and Lipschitz properties:

There exists a nonnegative constant K such that

$$(\mathbf{F1}):\ \| F(t,x,\xi) \|_X \leq K(1+ \| x \|_X)\ \forall\ (t,x) \in I \times X,$$

uniformly with respect to $\xi \in U$.

$$(\mathbf{F2}):\ \| F(t,x,\xi) - F(t,y,\xi) \|_X \leq K \| x-y \|_X\ \ \forall\ (t,x,y) \in I \times X \times X,$$

uniformly with respect to $\xi \in U$.

(G): $G : I \times X \longrightarrow \mathscr{L}(E,X)$ is a Borel measurable measurable map in all the variables and continuous in the last argument satisfying the following growth and Lipschitz properties:

There exists a nonnegative constant L such that

$$(\mathbf{G1}):\ \| G(t,x) \|_{\mathscr{L}(E,X)} \leq L(1+ \| x \|_X)\ \forall\ (t,x) \in I \times X$$

and

$$(\mathbf{G2}):\ \| G(t,x) - G(t,y) \|_{\mathscr{L}(E,X)} \leq L \| x-y \|_X\ \ \forall\ (t,x,y) \in I \times X \times X.$$

Using the above assumptions, we have the following result.

Theorem 2.1. Consider the system governed by the evolution equation (2.1) and suppose that assumptions (A),(F) and (G) hold. Then, for each initial state $x_0 \in X$ and control $\nu = (u,\gamma) \in \mathscr{K}_{ad}$, system (2.1) has a unique mild solution $x \in B_\infty(I,X)$.

Proof. See Ahmed [12, Theorem 4.1].

Based on the above theorem we, obtain the following result.

Theorem 2.2. Consider control system (2.1) and suppose taht the assumptions of Theorem 2.1 hold and the semigroup $S(t), t > 0$, is compact, and the set of admissible controls $\mathscr{K}_{ad} \equiv \mathscr{R}_{ad} \times \mathscr{M}_{ad}$ is endowed with the product topology τ_p. Then the control to solution map

$$\mathscr{K}_{ad} \ni \nu \longrightarrow x(\nu) \in B_\infty(I,X)$$

is continuous with respect to the topology τ_p on $\mathscr{K}_{ad}$ and the norm topology on $B_\infty(I,X)$.

Proof. See Ahmed [12, Theorem 4.3].

Theorem 2.3. Consider the system (2.1) with the cost functional (2.2) and admissible controls $\mathscr{K}_{ad}$, and suppose that the assumptions of Theorem 2.2 hold. Suppose both ℓ and Φ are Borel measurable in all their arguments and lower semi continuous in the state variable on X satisfying the following inequalities:

$$(i) : |\ell(t,x)| \leq \alpha_1(t) + \alpha_2 \parallel x \parallel_X^p,\ p \in [1,\infty), \\ \text{for} \quad \alpha_1 \in L_1^+(I) \text{ and } \alpha_2 > 0 \text{ finite}, \tag{2.3}$$

$$(ii) : |\Phi(x)| \leq \alpha_3 + \alpha_4 \parallel x \parallel_X^p, p \in [1,\infty), \alpha_3, \alpha_4 \geq 0 \text{ finite}. \tag{2.4}$$

Then there exists an optimal control $\nu^o \in \mathscr{K}_{ad}$ minimizing the cost functional (2.2).

Proof. See Ahmed [13, Theorem 4.4].

Differential Inclusions.

There are many non-standard dynamic systems which can be formulated as differential inclusions. We present here a few simple examples. (i): A differential equation, like $\dot{x} = g(\alpha,t,x)$, with α representing a number of basic parameters whose exact values are unknown except their range Λ, can be formulated as a differential inclusion. Defining $G(t,x) = \{g(\alpha,t,x), \alpha \in \Lambda\}$, we have a differential inclusion $\dot{x} \in G(t,x)$. (ii): Differential inequalities can be reformulated as differential inclusion. Consider the differential inequality on Hilbert space X given by

$$< \dot{x} - Ax - F(x), y - x >_X \leq \phi(y) - \phi(x)\ \forall y \in X,$$

where A is the generator of a C_0-semigroup in X and F is a nonlinear map in X, and ϕ is a real valued convex function on X having sub-differential $\partial\phi(x)$ for each $x \in X$. Then the above inequality is equivalent to the following differential inclusion

$$\dot{x} \in Ax + F(x) + \partial\phi(x).$$

One can easily verify that the sub-differential $\partial\phi(x)$ is a closed convex set valued function. (iii): Another important example is given by a system of coupled differential-algebraic equations

$$\dot{x} = g(t,x,y), 0 = f(t,x,y), x, y \in X.$$

Introducing the multifunction $\Gamma(t,x) \equiv \{y \in X : f(t,x,y) = 0\}$ and assuming that it is nonempty, we introduce the multifunction $G(t,x) \equiv g(t,x,\Gamma(t,x))$ to obtain the differential inclusion $\dot{x} \in$

$G(t,x)$. (iv): A similar example is given by the differential equation $\Phi(t,x,\dot{x})=0$. Introducing the multifunction

$$G(t,x) \equiv \{y \in X : \Phi(t,x,y)=0\},$$

we obtain the differential inclusion $\dot{x} \in G(t,x)$. It is clear from these simple examples that by including differential inclusions in the study of stability and optimal control, we are able to cover a larger class of dynamic systems.

Now we are prepared to consider control problems involving differential inclusions. Let $\{X,Y\}$ be a pair of real Banach spaces, the former denoting the state space and the later the space where the controls take their values from and E is another real Banach space. We consider a control system governed by the following differential inclusion

$$\begin{aligned} &dx(t) \in Ax(t)dt + F(t,x(t))\nu(dt) + G(t,x(t),u(t))dt, t \in I=[0,T], \\ &x(0)=x_0, \end{aligned} \tag{2.5}$$

on the Banach space X where the operator A is the infinitesimal generator of C_0-semigroup $\{S(t), t \in I\}$ in X, and ν is a fixed countably additive E valued vector measure defined on the sigma algebra Σ_I of subsets of the interval I, and $F : I \times X$ to $\mathscr{L}(E,X)$, and G is a multifunction mapping $I \times X \times Y$ to $2^X \setminus \emptyset$. In many applications, the controls are generally limited and hence it is natural to consider a closed bounded set $U \subset Y$ as the range of controls. In order to admit non convex control constraint set $U \subset Y$, we replace regular controls (measurable functions) by relaxed controls taking values in the space of probability measures $\mathscr{M}_1(U)$. This requires replacing $u \in L_\infty(I,U)$ by $\mu \in L^w_\infty(I,\mathscr{M}_1(U))$ giving the following relaxed version of the system (2.5)

$$\begin{aligned} &dx(t) \in Ax(t)dt + F(t,x(t))\nu(dt) + \hat{G}(t,x(t),\mu_t)dt, t \in I=[0,T], \\ &x(0)=x_0, \end{aligned} \tag{2.6}$$

where $\hat{G}(t,x(t),\mu_t) \equiv \int_U G(t,x(t),\xi)\mu_t(d\xi)$. Let $\mathscr{U}_{ad} \equiv L^w_\infty(I,\mathscr{M}_1(U))$ denote the class of admissible relaxed controls. Endowed with the weak star topology, this is a compact topological space, a subset of the linear topological space $L^w_\infty(I,\mathscr{M}(U))$. Let Σ_I denote the sigma algebra of subsets of the interval I and $M_{ca}(\Sigma_I,E)$ denote the space of countably additive vector measures with values in a Banach space E.

Existence and regularity properties of solutions of the differential inclusion (2.6) can be studied using the existence and uniqueness of (mild) solution of the following evolution equation

$$dx = Axdt + F(t,x)\nu(dt) + f(t)dt, x(0)=\xi, t \in I. \tag{2.7}$$

Lemma 2.4. Consider the evolution equation (2.7) on Banach space X and suppose the following assumptions hold

(A1) A is the infinitesimal generator of a C_0-semigroup $S(t), t \geq 0$, in the Banach space X,

(A2) $\nu \in M_{ca}(\Sigma_I,E)$, having bounded variation (with $t=0$ not an atom), and there exists an $L \in L_1^+(I,|\nu|)$ with $|\nu|$ being a positive measure induced by the variation of the vector measure ν such that

$$|F(t,x)|_{\mathscr{L}(E,X)} \leq L(t)(1+|x|_X), \ |F(t,x)-F(t,y)|_{\mathscr{L}(E,X)} \leq L(t)(|x-y|_X), \ \forall\, x,y \in X.$$

Then, (i): for every $x_0 \in X$ and $f \in L_1(I,X)$, equation (2.7) has a unique mild solution $x \in B_\infty(I,X)$. (ii): the solution map Υ, $f \longrightarrow x \equiv \Upsilon(f)$, is Lipschitz continuous from $L_1(I,X)$ to $B_\infty(I,X)$.

Proof. See Ahmed [14, Theorem 3.3]

We are now prepared to consider the question of existence of solutions of the differential inclusion given by (2.6) (relaxed version of (2.5)). For any fixed $\mu \in \mathscr{U}_{ad} \equiv L^w_\infty(I,\mathscr{M}_1(U))$, define the set valued function $\hat{G}_\mu(t,x) \equiv \hat{G}(t,x,\mu_t)$ mapping $I \times X$ to $2^X \setminus \emptyset$. Using the solution operator Υ as seen in Lemma 2.4, we introduce the multifunction defined on $L_1(I,X)$ and taking values in the power set $2^{L_1(I,X)} \setminus \emptyset$ as follows

$$f \longrightarrow \mathscr{G}_\mu(f) \equiv \{g \in L_1(I,X) : g(t) \in \hat{G}_\mu(t,\Upsilon(f)(t)), a.e\; t \in I\}. \tag{2.8}$$

It is clear from the above expression and Lemma 2.4 that if an element $f \in L_1(I,X)$ is a fixed point of the multifunction $\mathscr{G}_\mu(\cdot)$ then $\Upsilon(f) = x \in B_\infty(I,X)$ is a solution of the differential inclusion (2.6). Let $Fix(\mathscr{G}_\mu)$ denote the set of fixed points of the multifunction $\mathscr{G}_\mu$. We prove that this set is nonempty.

With this goal in mind, let $cc(X)$ denote the class of nonempty closed convex subsets of X and let d_H denote the Hausdorff metric on $cc(X)$. It is known that with respect to this metric $(cc(X),d_H)$ is a complete metric space. We use the following assumptions:

(G1): for every $x \in X$ and $\mu \in \mathscr{U}_{ad}$, $I \ni t \to \hat{G}_\mu(t,x)$ is a measurable set valued function with values in $cc(X)$, the space of closed convex (possibly bounded) subsets of X.

(G2): for almost all $t \in I$, $x \to \hat{G}_\mu(t,x)$ is continuous in the Hausdorff metric and there exists a $K \in L_1^+(I)$ so that

$$d_H(\hat{G}_\mu(t,x),\hat{G}_\mu(t,y) \le K(t)|x-y|_X, \forall\, x,y \in X.$$

(G3): for the same K, $\sup\{|z|_X, z \in \hat{G}_\mu(t,x)\} \le K(t)(1+|x|_X)\ \forall\, x \in X$.

(G4): there exists an $h \in L_1^+(I)$ such that $\inf\{|z|_X, z \in \hat{G}_\mu(t,x)\} \le h(t)(1+|x|_X)\ \forall\, x \in X$.

Theorem 2.5. Consider the system governed by the differential inclusion (2.6) and suppose the assumptions of Lemma 2.4 hold and, for each $\mu \in \mathscr{U}_{ad}$, the multifunction $\hat{G}_\mu$ satisfies the properties $(\mathbf{G1}),(\mathbf{G2}),(\mathbf{G3}),(\mathbf{G4})$. Then for every $x_0 \in X$, the system (2.6) has a nonempty set of bounded and right continuous solutions $\mathscr{X}(\mu) \subset B_\infty(I,X)$.

Proof. See Ahmed [13, Theorem 3.5].

Next, we consider a min-max control problem. Consider the control system (2.6) corresponding to any control $\mu \in \mathscr{U}_{ad}$ and let $x \in \mathscr{X}(\mu)$. The cost functional corresponding to the pair $\{\mu,x\}$ is given by

$$C(\mu,x) \equiv \int_I \hat{\ell}(t,x(t),\mu_t)dt + \Phi(x(T)), \tag{2.9}$$

where $\hat{\ell}(t,x,\mu_t) \equiv \int_U \ell(t,x,\xi)\mu_t(d\xi)$. The objective is to find a control that minimizes the maximum risk, leading to the following min-max problem

$$\inf_{\mu \in \mathscr{U}_{ad}} \sup_{x \in \mathscr{X}(\mu)} C(\mu,x). \tag{2.10}$$

Problem (2.10) can be reformulated as follows. Find a control $\mu \in \mathscr{U}_{ad}$ that minimizes the following functional

$$J^o(\mu) \equiv \sup\{C(\mu,x), x \in \mathscr{X}(\mu)\}. \tag{2.11}$$

Clearly, this is equivalent to minimizing the maximum risk (maximum potential cost).

Consider the cost functional(2.9) satisfying the following conditions.

(C1:) The integrand ℓ is measurable in the first argument and continuous on $X \times Y$. There exists a $g \in L_1^+(I)$ and $1 \leq p < \infty$ such that

$$|\ell(t,x,y)| \leq g(t)(1+|x|_X^p + |y|_Y^p), t \in I, x,y \in X \times Y.$$

(C2:) The function Φ is a real valued continuous function on X and bounded on bounded sets.

Under the assumptions that X is a reflexive Banach space and that (C1) and (C2) hold, it follows from Theorem 2.5 that there exists an $\tilde{x} \in \mathscr{X}(\mu)$ such that $J^o(\mu) = C(\mu,\tilde{x})$. For detailed proof see [Ahmed, 14, Lemma 4.2]. Denoting this maximizing element $\tilde{x}$ by x_μ and using it in the expression (2.11) we obtain $J^o(\mu) = C(\mu, x_\mu)$.

To prove existence of an optimal control, it suffices to prove that there exists a $\mu^o \in \mathscr{U}_{ad}$ that minimizes the functional $J^o(\mu)$. This is presented in the following theorem after brief preparation. Define the multifunction

$$Q(t,x) \equiv \{(\zeta,\eta) \in R \times X : \zeta = \hat{\ell}(t,x,\gamma), \eta \in \hat{G}(t,x,\gamma), \text{ for some } \gamma \in \mathscr{M}_1(U)\}. \tag{2.12}$$

Let $clcoQ$ denote the closed convex hull of the set Q and $N_\varepsilon(x)$ denote the closed $\varepsilon(>0)$ neighbourhood of $x \in X$. A multifunction is said to satisfy the weak Cesari property if for each $t \in I$,

$$\bigcap_{\varepsilon>0} clcoQ(t, N_\varepsilon(x)) \subseteq Q(t,x)$$

for every $x \in X$.

Theorem 2.6. Consider the system governed by the differential inclusion (2.6) on the space X, a reflexive Banach space, with the admissible controls $\mathscr{U}_{ad}$, and suppose the assumptions (C1)-(C2) and those of Theorem 2.5 hold. Then there exists an optimal control $\mu^o \in \mathscr{U}_{ad}$ that solves the min-max problem (2.10) or equivalently (2.11).

Proof. See Ahmed [13, Theorem 4.4].

Based on the existence results as presented in Theorem 2.3 and Theorem 2.6, one can develop necessary conditions of optimality. We consider here the system given by equation (2.1) and the cost functional given by the expression (2.2). We leave the min-max control problem related to the differential inclusion (2.6) for interested readers. Another interesting min-max problem involving uncertain system is given by equation (4.12) (to be seen in Section 4).

Theorem 2.7. Consider the system (2.1) with the cost functional (2.2). Suppose that the pair $\{F,G\}$ is continuously Gâteaux differentiable on X, and the pair $\{\ell,\Phi\}$ is once differentiable in the state variable. Then, for $\nu^o = (u^o,\gamma^o) \in \mathscr{K}_{ad} \equiv \mathscr{R}_{ad} \times \mathscr{M}_{ad}$, with the corresponding solution $x^o \in B_\infty(I,X)$, to be optimal, it is necessary that there exists a $\psi \in B_\infty(I,X^*)$ satisfying the following inequality and differential equations:

$$dJ(\nu^o;\nu-\nu^o) \equiv \int_I <\psi(t), F(t,x^o,u_t-u_t^o>_{X^*,X} dt$$
$$+\int_I <G(t,x^o)^*\psi(t),\gamma(dt)-\gamma^o(dt)>_{E^*,E}\geq 0,\ \forall\ \nu=(u,\gamma)\in\mathscr{K}_{ad}, \quad (2.13)$$

$$-d\psi = A^*\psi dt + DF^*(t,x^o,u_t^o)\psi dt + DG^*(t,x^o;\psi)\gamma^o(dt) + \ell_x(t,x^o)dt, t\in I, \quad (2.14)$$
$$\psi(T) = D\Phi(x^o(T));$$

$$dx^o = Ax^o dt + F(t,x^o,u_t^o)dt + G(t,x^o)\gamma^o(dt), t\in I, \quad (2.15)$$
$$x^o(0) = x_o,$$

where $\{DF^*, DG^*\}$ denote the adjoints of the Gateaux differentials of $\{F,G\}$ respectively.

There are many interesting applications of deterministic control theory. We present here two examples.

(1) Immunotherapy. Optimal control theory for nonlinear reaction diffusion equations have been studied in [Ahmed, 18] dealing with immunotherapy to fight cancer. This therapy empowers the body's natural immune system to fight and kill the cancer cells without any side effects. There are several techniques; one of which involves synthesis of antigen-specific antibodies in laboratories which are then administered to cancer patients to boost up their immune system. We discuss this in details in Section 5.

(2) Climate Crisis Mitigation. It is known that green house gasses, like CO_2 and Methane, emitted by industries using fossil fuels, build up in the atmosphere forming a blanket around the planet and thereby raising its temperature, (in particular ocean surface temperature), leading to destructive climate events. Climate crisis is an existential threat to life on the planet. Timely intervention through necessary and bold actions against this crisis is urgently needed before it reaches a level beyond human capacity. In Ahmed & Biswas [33], the authors considered a dynamic (PDE) model and presented the optimal strategies for removal of greenhouse gas in the atmosphere to avert the climate crisis.

3. Structural Controls and Optimization

In this section, we consider dynamic systems using operator valued measures as structural controls. There are examples like mechanical systems and molecular systems which require reorientation and manipulation of their physical and molecular structures in order to optimize performance. A popular example is a skater in motion on ice frequently spreading out hands, body and legs to perform and prevent imbalance and fall. Similarly, an aircraft in motion through dense clouds use ailerons, elevators, and rudder to control its movement in pitch, roll, and yaw for safe flight. Similarly, large business corporations may occasionally require restructuring its administrative system and retraining its employees for efficiency. We will present some examples to conclude this section.

A general model that includes systems as described above is given by an evolution equation on infinite dimensional Banach spaces as follows:

$$dx = Axdt + M(dt)x + f(t,x)dt + g(t,x)\nu(dt), x(0) = x_0, t\in I\equiv[0,T], \quad (3.1)$$

where A is an unbounded operator with domain and range in E, $M(\cdot)$ is an operator valued measure with values in $\mathscr{L}(E)$, $f : I\times E \to E$, ν is an F valued vector measure with $g : I\times E \to \mathscr{L}(F,E)$. Let $B(I,E)$ denote the linear space of (norm) bounded measurable functions with values in the Banach space E. With respect to sup-norm topology it is a Banach space. Let $\mathscr{L}_s(E)$ denote the space of bounded linear operators in E endowed with the strong operator topology and $\mathscr{M}_{ca}(\Sigma_I,\mathscr{L}_s(E))$ the space of countably additive (in strong operator topology) operator valued measures taking values in $\mathscr{L}_s(E)$. The integral of an element $f\in B(I,E)$ with respect to an operator valued measure $M\in\mathscr{M}_{ca}(\Sigma_I,\mathscr{L}_s(E)$ is defined in the sense of Dobrakov Ahmed [15, Definition 2.3]. The semi-variation of any $M\in\mathscr{M}_{ca}(\Sigma_I,\mathscr{L}_s(E))$ on $D\in\Sigma_I$ is defined by the extended real valued sub-additive set function $\hat{M}(\cdot)$ given by

$$\hat{M}(D)\equiv\sup\Big\{\parallel\int_D M(ds)f(s)\parallel_E : f\in S(I,E), \parallel f\parallel_{B(I,E)}\leq 1\Big\},$$

where $S(I,E)(\subset B(I,E))$ denotes the class of sigma measurable simple functions. The semi-variation of M on I is then given by $|M|_s\equiv\hat{M}(I)\equiv\sup\{\hat{M}(D) : D\in\Sigma_I\}$.

We consider optimal structural control problems for the system governed by the evolution equation (3.1). For this, we need the following theorem asserting the existence and uniqueness of (mild) solutions as stated below. It is well known that, by definition, the mild solution of the system (3.1) is given by the solution (if one exists) of the following integral equation on Banach space E,

$$\begin{aligned} x(t) = S(t)x_0 + \int_0^t S(t-s)M(ds)x(s) + \int_0^t S(t-s)f(s,x(s))ds \\ + \int_0^t S(t-s)g(s,x(s))\nu(ds), t\in I. \end{aligned} \tag{3.2}$$

Theorem 3.1. Suppose that operator A is the infinitesimal generator of a C_0-semigroup $\{S(t), t\geq 0\}$ on E and $M\in\mathscr{M}_{ca}(\Sigma_I,\mathscr{L}_s(E))$ having finite semi-variation and admitting a dominating measure $\mu_o\in\mathscr{M}_{ca}^+(\Sigma_I)$. Let F be another Banach space and $\nu\in\mathscr{M}_{ca}(\Sigma_I,F)$ having bounded total variation. The vector fields $\{f,g\}$ are Borel measurable in all their arguments and locally Lipschitz on E with $f : I\times E\to E$ and $g : I\times E\to\mathscr{L}(F,E)$ satisfying the following conditions: there exists $K\in L_1^+(I)$ and $L\in L_1^+(I,|\nu|)$ so that

$$|f(t,x)|_E \leq K(t)\big(1+|x|_E\big), \parallel g(t,x)\parallel_{\mathscr{L}(F,E)}\leq L(t)\big(1+|x|_E\big) \tag{3.3}$$

and for each positive number r, there exist $K_r\in L_1^+(I)$, $L_r\in L_1^+(I,|\nu|)$ such that

$$\begin{aligned} &|f(t,x)-f(t,y)|_E \leq K_r(t)\big(|x-y|_E\big), \\ &\qquad \parallel g(t,x)-g(t,y)\parallel_{\mathscr{L}(F,E)}\leq L_r(t)\big(|x-y|_E\big), \ \forall\, x,y\in B_r(E). \end{aligned} \tag{3.4}$$

Then, for each initial state $x_0\in E$, the system (3.1) has a unique (mild) solution $x\in B(I,E)$.

Proof. See Ahmed [14, Theorem 3.5].

We consider the following structural optimal control problem. We assume that the space $\mathscr{M}_{ca}(\Sigma_I,\mathscr{L}_s(E))$ is endowed with the topology of set-wise convergence in the strong operator topology giving the topological space $(\mathscr{M}_{ca}(\Sigma_I,\mathscr{L}_s(E)),\tau_{sw})$ and let $\mathscr{M}_{ad}$ be a compact subset of this space considered as the set of admissible (structural) controls. Consider the system given by equation (3.1) with control $M \in \mathscr{M}_{ad}$ and the cost functional given by

$$J(M) \equiv \int_I \ell(t,x(t))dt + \Phi(x(T)), \tag{3.5}$$

where $x \in B(I,E)$ is the mild solution of equation (3.1) corresponding to the control $M \in \mathscr{M}_{ad}$. The problem is to find a control $M \in \mathscr{M}_{ad}$ that minimizes the cost functional (3.5).

Theorem 3.2 Consider the system given by equation (3.1) with the cost functional (3.5) and the set of admissible controls $\mathscr{M}_{ad}$ uniformly dominated by a $\mu \in \mathscr{M}_{ca}^+(\Sigma)$. Suppose the assumptions of Theorem 3.1 hold and let $\{\ell, \Phi\}$ be Borel measurable real valued functions on $I \times E$ and E respectively and lower semicontinuous on E, and there exist an $h \in L_1(I)$ and a finite real number $c > -\infty$ such that

$$\ell(t,x) \geq h(t), \Phi(x) \geq c, \; \forall \, x \in E.$$

Then there exists an optimal control $M_o \in \mathscr{M}_{ad}$ at which J attains its minimum.

Proof. Using integral equation (3.2), one can prove that the control to solution map $M \longrightarrow x(M)$ is continuous with respect to the τ_{sw} topology on $\mathscr{M}_{ca}(\Sigma_I, \mathscr{L}_s(E))$ and norm topology on $B_\infty(I,E)$. Existence of optimal control then follows from lower semicontinuity of the cost functional $M \longrightarrow J(M)$ with respect to the relative τ_{sw} topology on $\mathscr{M}_{ad}$ and its compactness in the same topology. Compactness of the set $\mathscr{M}_{ad}$ follows from Ahmed [2, Theorem 5].. Detailed arguments are similar to those given in the proof of [14, Theorem 4.3] and [15, Theorem 4.4 and Theorem 4.5].

Remark 3.3. For compactness of the admissible set of structural controls $\mathscr{M}_{ad}$, we have used the topology of set-wise convergence in the strong operator topology on $\mathscr{M}_{ca}(\Sigma_I, \mathscr{L}_s(E))$. This can be relaxed as in [3, Theorem 3.6] and [2, Theorem 1] provided the semigroup $\{S(t), t > 0\}$ generated by the unbounded operator A appearing in the system equation (3.1) is compact. This prevents its application to some important examples as presented below. In both the examples presented below, the operator $\mathscr{A}$ is the generator of C_0-unitary group of operators $\{U(t), t \in R\}$ with values $\mathscr{L}(E)$ and hence noncompact.

Examples: We mention two examples from theoretical mechanics having interesting practical applications. Here, lack of compactness of the unitary groups is compensated by assuming a stronger topology on the space of operator valued measures [2, Theorem 5] giving $(\mathscr{M}_{ca}(\Sigma_I, \mathscr{L}_s(E)), \tau_{sw})$.

(E1:) Structural Dynamics: Smart materials such as piezoelectric and magnetostrictive composites are used in modern technology to build active actuators and sensors. They provide active damping and stiffness for high precision machines or devices thereby suppressing vibration and preventing potential structural damage. Short bursts of currents can change the flexural (mechanical) properties of these materials providing structural damping and stiffness as and when required. We consider vector and operator valued measures as the appropriate mathematical models for such controls. A large family of structural dynamics can be described by the following second order evolution equation

$$dy_t + Aydt + D(dt)y_t + K(dt)y = h(t, Ly, y_t), y(0) = y_0, y_t(0) = y_1, t \geq 0, \tag{3.6}$$

where $y(t) \equiv \{y(t,\xi), \xi \in \Omega\}$ represents the spatial configuration of the body $\Omega \subset R^3$ at time t, such as displacement from the rest state (of a membrane, a road-bed of a suspension bridge etc.) In general, the operator A is a positive selfadjoint unbounded operator in a Hilbert space $H = L_2(\Omega)$ with domain and range $D(A), R(A) \subset H$. The operators D and K are respectively the damping and stiffness operators. In general these are operator valued measures, $D : \Sigma_I \to \mathscr{L}(H)$ and $K : \Sigma_I \to \mathscr{L}(\mathscr{D}(\sqrt{A}), H)$ where $\sqrt{A}$ denotes the positive square root of the operator A and $\mathscr{D}(\sqrt{A})$ the domain of $\sqrt{A}$. Typically, the operator L appearing in the function h is given by $L = \sqrt{A}$. Defining $y = x_1, y_t = x_2$, one can rewrite equation (3.6) as a first order evolution equation on the Hilbert space $E \equiv \mathscr{D}(\sqrt{A}) \times H$ as follows

$$dx = \mathscr{A} x dt + M(dt)x + f(t,x)dt, x(0) = x_0, t \in I, \tag{3.7}$$

where the operators $\{\mathscr{A}, f\}$, and the operator valued measure M taking values in $\mathscr{L}(E)$ for each $\sigma \in \Sigma_I$, are given by

$$\mathscr{A} = \begin{pmatrix} 0 & I_H \\ -A & 0 \end{pmatrix}, M(\sigma) = \begin{pmatrix} 0 & 0 \\ -K(\sigma) & -D(\sigma) \end{pmatrix}, f(t,x) = \begin{pmatrix} 0 \\ h(t, \sqrt{A}x_1, x_2) \end{pmatrix}.$$

The state space E, as introduced above, is a Hilbert space furnished with the norm $|x|_E$ where $|x|_E^2 = (|\sqrt{A}x_1|_H^2 + |x_2|_H^2)$ represents the sum of potential and kinetic energies. Under some additional technical conditions on the operator A, it is known from Ahmed and Wan [5, Theorem 5.5.1] that the operator $\mathscr{A}$ generates a unitary group of operator valued functions $\{U(t), t \in R\}$ in E. The mild solution of equation (3.7) is then given by the solution of the following integral equation in E

$$x(t) = U(t)x_0 + \int_0^t U(t-s)f(s,x(s))ds + \int_0^t U(t-s)M(ds)x(s), t \in I. \tag{3.8}$$

For suppression of vibration, an appropriate cost functional is given by

$$J(M) = (1/2)\int_I < Qx(t), x(t) >_E dt + \Phi(x(T)) + \Psi(M), \tag{3.9}$$

where $\Psi(M) = (1/2)\sum_i \parallel \left(\int_I M(dt)f_i(t) \parallel_E^2\right.$ with $\{f_i \in B_\infty(I,E), \parallel f_i \parallel_{B_\infty(I,E)} \leq 1\}$. The operator $Q \in \mathscr{L}^+(E)$ is bounded, positive and selfadjoint. Note that for $Q = I_E$ (the identity operator in E),

$$< Qx, x >= |x|_E^2 = (|\sqrt{A}x_1|_H^2 + |x_2|_H^2)$$

denotes the sum of the potential and kinetic energies. The problem is to find a structural control $M \in \mathscr{M}_{ad} \subset \mathscr{M}_{ca}(\Sigma_I, \mathscr{L}_s(E))$ that minimizes the functional (3.9).

(E2:) Molecular Dynamics: Study of molecular dynamics and their control has been an interesting subject in recent years because of the enormous prospect of developing smart materials with desirable structural properties (for example, nano-technology creating smart materials). The impact of such research on development of future engineering materials with application to biotechnology and medicine is expected to be highly significant. Changes in the molecular structure can be effected through control of the potential of the unperturbed Hamiltonian of the associated Schrödinger equation. This is done by laser induced electric field perturbing the original Hamiltonian. A simplified laser driven Schrödinger equation described in atomic units is given by the following partial differential equation

$$i\frac{\partial \psi}{\partial t} = H_o\psi + V(dt,\xi)\psi, \psi(0,\xi) = \psi_0(\xi), (t,\xi) \in I \times D, \tag{3.10}$$

where H_0 denotes the original Hamiltonian of the unperturbed system including the operators corresponding to kinetic and (Coulomb) potential energies. Let $D \subset R^3$ denote the body of the material subject to laser induced electric field V. Pulsed-Laser polymerization is known to be a leading technique to determine radical polymerization characteristics. The duration of Laser pulses is of the order of 10^{-18} seconds and hence it is reasonable to model this as a measure (including Dirac measures). As usual this equation is defined on a complex Hilbert space. Writing the real and imaginary parts of $\psi = \psi_1 + i\psi_2$ as a vector $x \equiv (x_1, x_2)' = (\psi_1, \psi_2)'$ we obtain a differential equation on the real Hilbert space $E \equiv L_2(D) \times L_2(D)$ as

$$dx = \mathscr{A} x dt + M(dt)x, x(0) = x_0, \tag{3.11}$$

where

$$\mathscr{A} = \begin{pmatrix} 0 & H_o \\ -H_o & 0 \end{pmatrix}, M(\sigma) = \begin{pmatrix} 0 & V(\sigma,\cdot) \\ -V(\sigma,\cdot) & 0 \end{pmatrix}, \text{ for } \sigma \in \Sigma_I.$$

It is interesting to note that V can be considered to be a simple multiplication operator such as $V(\sigma,\xi)\phi(\xi)$ or an integral-operator like $V(\sigma)\phi \equiv \int_D V(\sigma,\xi)\phi(\xi)d\xi, \phi \in L_2(D)$ for $\sigma \in \Sigma_I$. In this example the operator $\mathscr{A}$ is skew-adjoint and hence $i\mathscr{A}$ is self adjoint. Thus it follows from Stones theorem [1, Theorem 3.1.4] that $\mathscr{A}$ generates a unitary group of operators $\{U(t), t \in R\}$ in the Hilbert space E. Hence the mild solution of equation (3.11) is given by the solution of the following integral equation on the Hilbert space E

$$x(t) = U(t)x_0 + \int_0^t U(t-s)M(ds)x(s), t \in I. \tag{3.12}$$

The objective is to choose a control $M \in \mathscr{M}_{ad}$ so as to reach a specified target state $x_d \in E$ appropriate for polymer synthesis. For example, the cost functional may be taken as

$$J(M) = \Phi(x(T)) =< \mathbf{R}(x(T) - x_d), x(T) - x_d >_E, \tag{3.13}$$

where $\mathbf{R} \in \mathscr{L}^+(E)$ and symmetric. The objective is to choose a control M from the admissible set $\mathscr{M}_{ad}$ that minimizes the functional J given by the expression (3.13).

Theorem 3.4. Consider the system given by equation (3.7) ((3.11)) with the cost functional (3.9) ((3.13)) and the set of admissible controls $\mathscr{M}_{ad} \subset (\mathscr{M}_{ca}(\Sigma_I, \mathscr{L}_s(E)), \tau_{sw})$ compact in τ_{sw} topology and uniformly dominated by a scalar measure $\mu \in \mathscr{M}_{ca}^+(\Sigma)$. Then, there exists an optimal control $M^o \in \mathscr{M}_{ad}$ minimizing the cost functional J.

Proof. We present a brief outline. Using the integral equations (3.8) ((3.12)) one can verify that control to solution map $M \longrightarrow x$ is continuous with respect to τ_{sw} topology on $\mathscr{M}_{ca}(\Sigma_I, \mathscr{L}_s(E))$ and the norm topology on $B_\infty(I, E)$. Existence of an optimal control $M^o \in \mathscr{M}_{ad}$ then follows from lower semicontinuity of the functional $M \longrightarrow J(M)$ in the τ_{sw} topology and compactness of the set $\mathscr{M}_{ad}$ in the same topology. •

Next, we present the necessary conditions of optimality. To cover both the problems as presented above, we consider the following slightly more general system with the same set of admissible controls $\mathscr{M}_{ad}$ as follows:

$$dx = Axdt + M(dt)x + F(t,x)dt, x(0) = x_0, t \in I, \tag{3.14}$$

and the cost functional

$$J(M) = \int_I \ell(t,x)dt + \Phi(x(T)), \tag{3.15}$$

where $x \in B_\infty(I,E)$ is the (mild) solution of equation (3.14). The necessary conditions of optimality for this problem are given in the following theorem.

Theorem 3.5. Consider the system (3.14) with the cost functional (3.15) and suppose that F is locally Lipschitz having uniformly bounded Gâteaux derivative, and $\{\ell, \Phi\}$ are Borel measurable in their arguments and lower semicontinuous in the state variable on E, and there exist $\alpha_1 \in L_1^+(I)$ and $\alpha_2, \alpha_3, \alpha_4 > 0$ such that

$$|\ell(t,x)| \leq \alpha_1(t) + \alpha_2 |x|_E^p; \ |\Phi(x)| \leq \alpha_3 + \alpha_4 |x|^p, \text{ for some } p \geq 1.$$

The set of admissible structural controls $\mathscr{M}_{ad}$ satisfies the hypothesis as in Theorem 3.4. Then, for $M^o \in \mathscr{M}_{ad}$ to be the optimal control policy with the corresponding solution x^o, it is necessary that there exists a $\varphi \in B_\infty(I,E)$ satisfying the following inequality and differential equations:

$$dJ(M^o; M - M^o) = \int_I < \varphi(t), [M(dt) - M^o(dt)]x^o(t) >_E \geq 0, \ \forall\, M \in \mathscr{M}_{ad}, \tag{3.16}$$

$$-d\varphi = A^*\varphi dt + (M^o(dt))^*\varphi + F_x^*(t,x^o)\varphi dt, t \in I, \varphi(T) = \Phi_x(x^o(T)), \tag{3.17}$$

$$dx^o = Ax^o dt + M^o(dt)x^o + F(t,x^o)dt, x^o(0) = x_0, t \in I. \tag{3.18}$$

Using the necessary conditions of optimality given by (3.16)-(3.18), we develop a numerical algorithm to determine the optimal policy. Here we use a recent result on the Radon-Nikodym theorem for operator valued measures [3, Theorem 3.6 and Corollary 3.4]. According to this result an operator valued measure M can be expressed in terms of an operator valued function $B \in B_\infty(I, \mathscr{L}_s(E))$ and scalar measure $\mu \in \mathscr{M}_{cabv}^+(\Sigma_I)$ giving $M(dt) = B(t)\mu(dt)$. With this transition from the space of operator valued measures to operator valued functions and an isometric isomorphism as stated in [Ahmed, 4, Corollary 3.4], we can replace the set $\mathscr{M}_{ad}$ by an equivalent set $\mathscr{K}_\mu$, a subset of the space of operator valued functions which are μ integrable in the strong operator topology. Based on this result, the necessary conditions given by (3.16)-(3.18) can be restated as follows:

Corollary 3.6. Consider the system (3.14) with the cost functional (3.15) and suppose that the assumptions of Theorem 3.5 hold. Then, for $B^o \in \mathscr{K}_\mu$ to be the optimal control policy with the corresponding solution x^o, it is necessary that there exists a $\varphi \in B_\infty(I,E)$ satisfying the following inequality and differential equations:

$$dJ(B^o; B - B^o) = \int_I < \varphi(t), [B(t) - B^o(t)]x^o(t) >_E \mu(dt) \geq 0, \ \forall\, B \in \mathscr{K}_\mu, \tag{3.19}$$

$$-d\varphi = A^*\varphi dt + (B^o(t))^*\varphi(t)\mu(dt) + (F_x(t,x^o))^*\varphi dt, \varphi(T) = \Phi_x(x^o(T)), t \in I, \tag{3.20}$$

$$dx^o = Ax^o dt + B^o(t)x^o(t)\mu(dt) + F(t,x^o)dt, x^o(0) = x_0, t \in I. \tag{3.21}$$

Using the above corollary, we present an algorithm, along with the proof of its convergence, whereby one can compute the optimal policy.

Theorem 3.7. Consider the system (3.14) with the cost functional (3.15) and the necessary conditions of optimality given by Corollary 3.6. Then, there exists a sequence $\{B_n\} \in \mathscr{K}_\mu$ (as constructed here) along which the cost functional $J(B_n)$ monotonically decreases and converges to a (possibly local) minimum.

Proof. We present the necessary steps leading to the proof.

Step 1: Choose $B_1 \in \mathscr{K}_\mu$ and solve the state equation (3.21) by replacing B^o by B_1 giving the solution $x_1 \in B_\infty(I,E)$.

Step 2: Use the pair $\{B_1, x_1\}$ in the adjoint system (3.20) by replacing the pair $\{B^o, x^o\}$ and solve for φ_1 giving the triple $\{B_1, x_1, \varphi_1\}$.

Step 3: Use the triple $\{B_1, x_1, \varphi_1\}$ in the expression (3.19) giving

$$dJ(B_1; B - B_1) = \int_I < \varphi_1(t), [B(t) - B_1(t)]x_1(t) >_E \mu(dt) \geq 0, \; \forall\, B \in \mathscr{K}_\mu. \tag{3.22}$$

If the above inequality holds, B_1 is optimal. It is rarely possible to reach the optimal feed back policy (operator valued function) in the first run, and so can be ignored. So to proceed from here, we need the tensor product space $E \otimes E$. Given a tensor $Z \equiv \sum \eta_i \otimes \zeta_i \in E \otimes E$, its cross-norm is defined as

$$\| Z \|_\otimes = \inf \left\{ \sum |\eta_i|_E |\zeta_i|_E : Z = \sum \eta_i \otimes \zeta_i \right\}.$$

Completion of the tensor product space $E \otimes E$ with respect to the above cross norm is a Banach space which we denote by V. It is known that the topological dual of the space V is given by the space of bounded linear operators $\mathscr{L}(E) = V^*$. For details, see Hàjek & Smith [52, Proposition 1.1]. In view of this, we can define a scalar (duality) product of an element $Z \in V$ with an element $B \in V^* = \mathscr{L}(E)$ satisfying

$$| < B, Z >_{V^*,V} | \leq \| B \|_{V^*} \| Z \|_V .$$

For an elementary (rank one) tensor like $Z \equiv \eta \otimes \zeta$, it is clear that $< B, Z >_{V^*,V} = < B\eta, \zeta >_E$ and that $| < B, Z >_{V^*,V} | \leq \| B \|_{V^*} |\eta|_E |\zeta|_E$. Let $\mathscr{D}$ denote the duality map with reference to the dual pair of Banach spaces $\{V, V^*\}$ as follows. For any $Z \in V$, the image of the duality map is given by

$$\mathscr{D}(Z) = \{ B \in V^* : < B, Z >_{V^*,V} = \| Z \|_V^2 = \| B \|_{V^*}^2 \} \subset V^*.$$

It follows from Hahn-Banach theorem that $\mathscr{D}(Z)$ is a nonempty convex set valued function and upper semicontinuous on V.

Step 4: Define the tensor $Z_1(t) = x_1(t) \otimes \varphi_1(t), t \in I$, and choose $\Gamma_1(t) \in \mathscr{D}(Z_1(t)), t \in I$. Using the tensor notation, we rewrite the expression on the right side of the inequality (3.19) as follows

$$dJ(B_1; B - B_1) = \int_I < B(t) - B_1(t), Z_1(t) >_{V^*,V} \mu(dt), \; B \in \mathscr{K}_\mu. \tag{3.23}$$

Take $B \equiv B_2 = B_1 - \varepsilon \Gamma_1$ with $\varepsilon > 0$ sufficiently small so that $B_2 \in \mathscr{K}_\mu$. Substituting this in the above expression, we obtain

$$\begin{aligned} dJ(B_1; B_2 - B_1) &= -\varepsilon \int_I < \Gamma_1(t), Z_1(t) >_{V^*,V} \mu(dt) \\ &= -\varepsilon \int_I \| \Gamma_1(t) \|_{V^*}^2 \mu(dt) = -\varepsilon \int_I \| Z_1(t) \|_V^2 \mu(dt). \end{aligned}$$

Using Lagrange formula for computing $J(B_2)$, we find that

$$\begin{aligned} J(B_2) &= J(B_1) + dJ(B_1; B_2 - B_1) + o(\| B_2 - B_1 \|) \\ &= J(B_1) - \varepsilon \int_I \| \Gamma_1(t) \|_{V^*}^2 \mu(dt) + o(\varepsilon). \end{aligned} \tag{3.24}$$

Clearly, for $\varepsilon > 0$ sufficiently small, we have $J(B_2) < J(B_1)$. Using B_2, as constructed above, and repeating the steps 1- 4, we find B_3 satisfying $J(B_3) < J(B_2)$. Repeating this procedure adinfinitum, we arrive at the following conclusion,

$$J(B_1) \geq J(B_2) \geq J(B_3) \geq \cdots \geq J(B_n) \geq \cdots .$$

It follows from the (absolute) growth properties of $\{\ell, \Phi\}$ that $J(B) > -\infty$ for all $B \in \mathscr{K}_\mu$. Thus the sequence $\{J(B_n)\}$ monotonically converges (possibly) to its local minimum. •

Remark 3.8. Using the algorithm given by Theorem 3.7, one can construct the optimal control policies for both the structural problems considered above.

Partially Observed Control: The class of measure valued functions as controls for partially observed nonlinear evolution equations on Banach spaces is closely related to dynamic structural control. Let U be a compact Polish space (a topological space that is metrizable with a metric turning it into a complete separable metric space) with Σ_U denoting the sigma algebra of subsets of the set U and $M_B(U)$ the space of Borel measures on Σ_U and $M_1(U) \subset M_B(U)$ the class of probability measures. Here we use the theory of "lifting" due to [54, Tulcea & Tulcea]. By virtue of the theory of "lifting", [Ahmed, 20], the topological dual of $L_1(I, C(U))$ is given by $L_\infty^w(I, M_B(U))$ which denotes the space of weak-star measurable functions defined on I and taking values in $M_B(U)$. Clearly, it follows from Alaoglu's theorem that the set $\mathscr{M}_{ad} \equiv L_\infty^w(I, M_1(U)) \subset L_\infty^w(I, M_B(U))$ is weak-star compact. We choose this as the set of admissible controls. Let $\{X, Y\}$ be a pair of real Banach spaces denoting the state space and the space of observations (measurement space). The system is governed by the following pair of evolution equations

$$dx = Axdt + F(t,x)dt + \int_U G(t,y,\xi)\mu_t(d\xi)dt, x(0) = x_0, t \in I; \tag{3.25}$$

$$dy = A_0 y dt + F_0(t,y)dt + G_0(t,x)\gamma(dt), t \in I, y(0) = y_0. \tag{3.26}$$

Here the first equation represents the main system to be controlled whose state is inaccessible and the second equation stands for the observer (monitor) whose state is fully accessible. System (3.25) is to be controlled on the basis of the observation y through the feedback operator G which in turn is controlled by the measure valued function $\mu_t, t \in I$. The observer receives state information through the operator G_0 which may be occasionally cut off by the measure γ whenever the null set $N(\gamma) \equiv \{\triangle \in \Sigma_I : \gamma(\triangle) = 0\} \neq \emptyset$. During these time periods (blank periods) the system (3.25) operates on the basis of information generated by the system (3.26) in the absence of G_0. These periods are considered as the periods of partial blackout (system failure) and the process $\{y\}$, used to control the system (3.25), does not carry current state $\{x\}$ information though it does carry the past information till the cut off time. In any case, whatever information is available, the measure $\mu \in \mathscr{M}_{ad}$ is required to adjust the controller (actuator) G so as to achieve the objective as closely as possible. The objective functional given by

$$J(\mu) = \int_{I \times U} \ell(t,x,y,\xi)\mu_t(d\xi)dt + \Phi(x(T), y(T)) \tag{3.27}$$

is considered as the cost functional. The objective is to choose a control measure $\mu \in \mathscr{M}_{ad}$ that minimizes the cost functional (3.27). We prove existence of optimal controls in the following theorem. We leave it to the interested reader to develop necessary conditions of optimality.

Theorem 3.9. Suppose the pair $\{A, A_0\}$ generate C_0-semigroups $\{S(t), S_0(t), t \in I\}$ on X and Y respectively and the vector fields $\{F, G, F_0, G_0\}$ satisfy standard assumptions (Local Lipschitz and linear growth). Then, for every $x_0 \in X, y_0 \in Y$, $\mu \in \mathscr{M}_{ad}$ and $\gamma \in M_{ca}(\Sigma_I, R)$, the system (3.25)-(3.26) has a unique mild solution $(x, y) \in B_\infty(I, X \times Y)$.

Using the above result, one can prove existence of optimal control $\mu^o \in \mathscr{M}_{ad}$ minimizing the functional (3.27).

Proof. For detailed proof, see [20, Theorem 4.1 and Theorem 4.2] where the first theorem proves weak-star continuity of the control to solution map, and weak star continuity of the map $\mu \longrightarrow J(\mu)$, on $\mathscr{M}_{ad} \subset L_\infty^w(I, M_B(U))$. The second theorem uses this to prove existence of optimal control. •

Remark 3.10. **(A):** Note that the set of ordinary controls given by $L_\infty(I, U)$ is a proper subset of the set $\mathscr{M}_{ad}$.

(B): It is interesting to note that by virtue of Krein-Milman theorem [47], the weak-star closed convex hull of the set of extreme points of the set $\mathscr{M}_{ad}$ equals $\mathscr{M}_{ad}$ itself, symbolically, $\overline{co}^{w^*}(Ext.(\mathscr{M}_{ad})) = \mathscr{M}_{ad}$. Clearly, if $\{\delta_{e_i}, i \in \mathbb{N}\}$ is the set of extreme points, any $\mu \in \mathscr{M}_{ad}$ can be approximated as closely as desired by the sum $\sum_i \alpha_i \delta_{e_i}$, with $\alpha_i \geq 0$ satisfying $\sum \alpha_i = 1$, [20, Remark 4.4].

(C): In case $U = \{v_i, i = 1, 2, \cdots, n\}$, we have the simplest set of controls given by $\mu_t(d\xi) = \sum_i \zeta_i(t)\delta_{v_i}(d\xi)$ with $\zeta_i(t) \geq 0$ satisfying $\sum \zeta_i(t) = 1$ for all $t \in I$.

For more on structural control, we refer to [6, 12, 14, 15, 16, 17, 30, 31, 40, 41].

4. Stochastic Systems on Hilbert Spaces and Control

In this section, we consider briefly optimal control problems of stochastic systems on Hilbert spaces. Readers, interested in the fundamentals of stochastic differential equations on infinite dimensional spaces, are referred to the popular monograph due to Da Prato and Zabczyk [43]. Let X be a real Hilbert space denoting the state space, and H a separable Hilbert space where the Wiener process $W \equiv \{W(t), t \in I\}$ takes its values from. Let $(\Omega, \mathscr{F} \supset \mathscr{F}_{t\geq 0}, P)$ be a complete filtered probability space where $\mathscr{F}_{t\geq 0}$ represents the smallest class of increasing right continuous (with left limits) sub-sigma algebras of the sigma algebra $\mathscr{F}$ with respect to which the Wiener process is progressively measurable. Throughout this section, we assume that the incremental covariance of this process is given by $Q \in \mathscr{L}_1^+(H)$ (the space of positive nuclear operators in H). Let U be a compact Polish space and $M_1(U)$ the space of regular probability measures on U, a subset of the space of signed measures $M(U)$. The set of relaxed controls are weak-star measurable $\mathscr{F}_t$-adapted processes defined on I and taking values in $M_1(U)$. This is denoted by the set $\mathscr{U}_{ad} = L_\infty^a(I, M_1(U)) \subset L_\infty^a(I, M(U))$. Note that this contains the class of all standard Borel measurable $\mathscr{F}_t$-adapted random processes with values in U denoted by $L_\infty^a(I, U)$. We consider the system governed by the following stochastic differential equation on the Hilbert space X with control $u \in \mathscr{U}_{ad}$,

$$dx = Axdt + \hat{b}(t,x,u_t)dt + \hat{\sigma}(t,x,u_t)dW(t), x(0) = x_0, t \in I = [0,T], T < \infty, \quad (4.1)$$

where A is the infinitesimal generator of a C_0-semigroup in X, $b : I \times X \times U \longrightarrow X$ and $\sigma : I \times X \times U \longrightarrow \mathscr{L}(H,X)$ are Borel measurable maps, and

$$\hat{b}(t,x,u_t) \equiv \int_U b(t,x,\xi)u_t(d\xi), \text{ and } \hat{\sigma}(t,x,u_t) \equiv \int_U \sigma(t,x,\xi)u_t(d\xi).$$

The problem is to find a control from the admissible set $\mathscr{U}_{ad}$ that minimizes the following cost functional

$$J(u) = \mathbb{E}\left\{ \int_I \hat{\ell}(t,x(t),u_t)dt + \Phi(x(T)) \right\}, \quad (4.2)$$

where $\ell : I \times X \times U \to R$ is Borel measurable in all the arguments and continuous in the second and third, with $\hat{\ell}(t,x,u_t) \equiv \int_U \ell(t,x,\xi)u_t(d\xi)$, and $\Phi : X \longrightarrow R$ is continuous.

Let $B^a_\infty(I, L_2(\Omega, X))$ denote the Banach space of $\mathscr{F}_t$-adapted X-valued random processes having finite second moments furnished with the norm topology given by

$$\| x \|^2 \equiv \sqrt{\{\sup\{\mathbb{E}|x(t)|^2_X, t \in I\}\}}.$$

We present the following theorem.

Theorem 4.1. Consider the system (4.1) with admissible controls $\mathscr{U}_{ad}$ (a weak-star compact subset of $L^w_\infty(I, M(U))$), and the cost functional (4.2). Suppose the operator A is the infinitesimal generator of a C_0-semigroup on the Hilbert space X, the drift vector $b : I \times X \times U \longrightarrow X$ and the diffusion operator $\sigma : I \times X \times U \longrightarrow \mathscr{L}(H,X)$ are all Borel measurable maps. There exist positive constants $\{K, \alpha\}$ such that, for $(t,x,y) \in I \times X \times X$, the following conditions are satisfied:

$$\begin{aligned}
&(\mathbf{A1}) : |b(t,x,\xi)|^2_X \leq K[1 + |x|^2_X], \ |b(t,x,\xi) - b(t,y,\xi|^2_X \leq K[|x-y|^2_X], \\
&(\mathbf{A2}) : Tr(\sigma(t,x,\xi)Q\sigma^*(t,x,\xi)) \leq K[1 + |x|^2_X], \\
&\qquad\quad Tr(\sigma(t,x,\xi) - \sigma(t,y,\xi))Q(\sigma^*(t,x,\xi) - \sigma(t,y,\xi)) \leq K[|x-y|^2_X]
\end{aligned}$$

uniformly in $\xi \in U$.

(A3): The cost integrands $\{\ell, \Phi\}$ are Borel measurable in all their arguments and lower semicontinuous on X satisfying

$$|\ell(t,x,\xi)| \leq \alpha(1 + |x|^2_X), \ |\Phi(x)| \leq \alpha(1 + |x|^2_X), \text{ for } (t,x) \in X$$

uniformly with respect to $\xi \in U$. Then, there exists an optimal control in $\mathscr{U}_{ad}$ minimizing the cost functional (4.2).

Proof. We present a brief outline. Under the assumptions (A1) and (A2), the system given by equation (4.1) with the ($\mathscr{F}_0$-measurable) initial sate $x_0 \in L_2(\Omega, X)$ and $u \in \mathscr{U}_{ad}$, has a unique mild solution $x \in B^a_\infty(I, L_2(\Omega, X))$ having continuous modification. Existence of continuous modification of x is proved following similar approach as given by Da Prato-Kwapien-Zabczyk [45]. The control to solution map $u \longrightarrow x$ is continuous with respect to the relative weak-star topology on $\mathscr{U}_{ad} \subset L^a_\infty(I, M(U))$ and the norm topology on $B^a_\infty(I, L_2(\Omega, X))$. Under the assumption (**A3**), both ℓ and Φ are lower semicontinuous on X and hence, by virtue of Fatou's lemma, this leads to lower semicontinuity of the map $u \longrightarrow J(u)$ in the weak-star topology on

$\mathscr{U}_{ad}$. Existence of optimal control then follows from Weak-star compactness of the set $\mathscr{U}_{ad}$. For details; see [19, Theorem 3.1, Theorem 4.3 & Theorem 5.1]. •

Assured of existence of optimal control, one looks for necessary (and sufficient) conditions of optimality whereby one can determine the optimal control. This demands more regularity of the drift and diffusion operators in the state variable. Here, we present the necessary conditions of optimality.

Theorem 4.2. Suppose that $\{A, b, \sigma, \ell, \Phi, \mathscr{U}_{ad}\}$ satisfy the basic assumptions of Theorem 4.1, and further, the maps $\{b, \sigma, \ell, \Phi\}$ are continuously Gâteaux differentiable on X with the Gâteaux differentials b_x and σ_x uniformly bounded on X, and there exist $\beta_1 \in L_1^+(I)$ and $\{\beta_2, \beta_3, \beta_4 > 0\}$ such that the following inequalities hold:

$$|\ell_x(t,x,\xi)|_X \leq \beta_1(t) + \beta_2|x|_X,\ |\Phi_x(x)|_X \leq \beta_3 + \beta_4|x|_X, t \in I, x \in X, \xi \in U. \tag{4.3}$$

Then, for a control $u^o \in \mathscr{U}_{ad}$ to be optimal with $x^o \in B^a_\infty(I, L_2(\Omega, X))$ being the corresponding (mild) solution of equation (3.26), it is necessary that there exists a $\varphi \in B^a_\infty(I, L_2(\Omega, X))$ satisfying the following inequalities and differential equations,

$$\begin{aligned} dJ(u^o; u-u^o) = \mathbb{E}\Big\{\int_I\Big(&< \varphi(t), \hat{b}(t, x^o(t), u_t - u^o_t) >_X \\ &-Tr\big(\hat{\sigma}_x(t, x^o(t), u^o_t; \varphi)Q\hat{\sigma}^*(t, x^o(t), u_t - u^o_t)\big) \\ &+\hat{\ell}(t, x^o(t), u_t - u^o_t)\Big)dt\Big\} \geq 0,\ \forall\, u \in \mathscr{U}_{ad}; \end{aligned} \tag{4.4}$$

$$dx^o = Ax^o(t)dt + \hat{b}(t, x^o(t), u^o_t)dt + \hat{\sigma}(t, x^o(t), u^o_t)dW,\ x^o(0) = x_0, t \in I; \tag{4.5}$$

$$\begin{aligned} -d\varphi = A^*\varphi(t)dt + \big(\hat{b}^*_x(t, x^o(t), u^o_t) - \hat{\Sigma}(t, x^o(t), u^o_t)\big)\varphi(t)dt + \hat{\ell}_x(t, x^o(t), u^o_t)dt \\ +\hat{\sigma}_x(t, x^o(t), u^o_t; \varphi(t))dW,\ \varphi(T) = \Phi_x(x^o(T)), t \in I; \end{aligned} \tag{4.6}$$

where the operator $\hat{\Sigma} \in B^a_\infty(I, \mathscr{L}(X))$ follows from the following expression

$$Tr(\hat{\sigma}_x(t, x^o(t), u^o_t; \varphi)Q(\hat{\sigma}_x(t, x^o(t), u^o_t; z))^* \equiv < \hat{\Sigma}(t, x^o(t), u^o_t)\varphi, z >_X .$$

Proof. For proof, see Ahmed [19, Theorem 6.2 & Theorem 6.3].

Remark 4.3. Several interesting special cases follow from the above result giving simpler versions of the necessary conditions of optimality: (i) σ independent of control,(ii) σ independent of state, (iii) σ independent of both control and state. Further, by setting $\sigma \equiv 0$, one obtains the necessary conditions of optimality for deterministic systems.

Stochastic PDE with Noisy Boundary. A large class of stochastic partial differential equations subject to both distributed and boundary noise and (distributed) control have been studied by the author in [24]. For lack of space, we present only a very brief summary. Using semigroup theory and appropriate Sobolev spaces for both the boundary and distributed data, the system is transformed into an abstract integral equation containing the unbounded semigroup generator. The Sobolev spaces used are related to the fractional powers of the semigroup generator. Using relaxed controls, several optimal control problems are considered proving existence of optimal control policies [24, Theorem 4.3] and presenting necessary conditions of optimality [24, Theorem 5.1 & Theorem 5.2]. Some physical examples (involving special cases of the general theory) are presented dealing with (1): Quadratic Gaussian linear regulator problem and (2): Microwave distributed heating problem. For details, we refer [24].

Partially Observed Stochastic Systems and Control: In section 3, we considered partially observed deterministic systems controlled by measure valued functions (covering Dirac measures) with reference to Ahmed [20]. Here we consider partially observed stochastic systems and their feedback controls. In recent years, there has been substantial development of optimal feedback control theory for partially observed stochastic infinite dimensional systems. For details on optimal linear and nonlinear feedback control laws, see, e.g., [16, 17, 18, 19, 20, 21, 22, 23, 40, 41]. Here, we mention [22] which considers uncertain partially observed system giving optimal linear feedback law solving a min-max control problem. In [21], questions of existence of optimal Lipschitz feedback control laws are studied. We present only a brief outline based on the results given in Ahmed [21]. Let $\{X,Y\}$ be a pair of real Banach spaces and $Z \equiv X \times Y$ their product space furnished with the standard norm topology turning it into a Banach space. Consider the system governed by the following pair of coupled stochastic evolution equations on the product space Z as follows:

$$dx = Axdt + F(t,x)dt + C(y(t))dt + G(t,x)dW, x(0) = x_0, t \in I \equiv [0,T], \tag{4.7}$$

$$dy = A_0 ydt + F_0(t,x)dt + G_0(t,y)dW_0, y(0) = y_0, t \in I, \tag{4.8}$$

with C denoting a non-linear feedback control law. The space X denotes the state space of the system (4.7) to be controlled. The process x is not accessible and so not observable. The space Y denotes the state space of the system described by equation (4.8) called the observer, whose state is fully accessible and so observable. The noisy observable (or output) process y is used to control the main system given by (4.7). Both $W \equiv \{W(t), t \geq 0\}$ and $W_0 \equiv \{W_0(t), t \geq 0\}$ are stochastically independent H and H_0-cylindrical Brownian motions respectively, defined on a complete probability space with a filtration $(\Omega, \mathscr{F}, \mathscr{F}_{t\geq 0}, P)$, where H and H_0 are real separable Hilbert spaces. Clearly, both the systems are noisy. The problem is to find a feedback operator $C \in \mathscr{C}_{ad}$ that minimizes the cost functional

$$J(C) = \mathbb{E}\left\{ \int_I \ell(t,x(t),y(t))dt + \Phi(x(T),y(T)) \right\}, \tag{4.9}$$

where $\mathscr{C}_{ad}$ denotes the set of admissible output feedback operators or controllers contained in X^Y, the space of all maps from Y to X. There are many possible topologies that can be placed on subspaces of X^Y which are fundamentally based on the Tychonoff product topology. For details see [Willard, 54]. Here we consider a special class of maps contained in X^Y. For any pair of finite positive numbers $K > 0, c > 0$, let

$$\begin{aligned} \mathscr{C}_{ad} = Lip_{K,c}(Y,X) \equiv \{ & f : Y \longrightarrow X : \\ & |f(0)|_X \leq c, |f(y_1) - f(y_2)|_X \leq K|y_1 - y_2|_Y \ \forall\, y_1, y_2 \in Y\}, \end{aligned}$$

denote the class of Lipschitz maps from $Y \to X$ with Lipschitz constants not exceeding K and their norms at $0 \in Y$ not exceeding c. Assuming X to be a reflexive Banach space, we introduce the topology τ_{pw} of point wise convergence on Y in the weak topology of X. It is known that the set $\mathscr{C}_{ad}$, endowed with this topology giving $(\mathscr{C}_{ad}, \tau_{pw})$, is a compact (Hausdorff) topological space [Ahmed, 21, Proposition 4.1]. Let $B^a_\infty(I, L_2(\Omega, Z))$ denote the Banach space of $\mathscr{F}_{t\geq 0}$-adapted Z valued random processes having finite second moments. It follows from [21, Theorem 4.4, & Theorem 4.5] that there exists an optimal feedback policy $C^o \in \mathscr{C}_{ad}$ that minimizes the cost functional J given by (4.9).

There are other interesting non-standard control problems considered in [21]. For example, (i): maximizing the probability of reaching a target set in the state space X; (ii): minimizing the probability of hitting a danger zone given by an open set in X; (iii): maximizing the exit time from a given set in X; (iv) mass transfer from one given set to another in the state space X. For details, the interested reader is referred to [21].

We present here two related examples. Let $\mathscr{B}(Z)$ denote the sigma algebra of Borel sets in Z. For any feedback control $C \in \mathscr{C}_{ad}$, let $z(C) \in B_\infty^a(I, L_2(\Omega, Z))$ denote the corresponding mild solution of the system of evolution equations (4.7)-(4.8), and for $t \in I$ and any $\Gamma \in \mathscr{B}(Z)$, let $m_t^C(\Gamma) = Prob.\{z(C)(t) \in \Gamma\}$, denote the probability law of the random element $z(C)(t)$. For any set $D \in \mathscr{B}(X)$ and $t \in I$, let $\mu_t^C(D) \equiv m_t^C(D \times Y)$ giving $Prob.\{x(C)(t) \in D\}$. The problem is to find a control law $C \in \mathscr{C}_{ad}$ that maximizes the hitting probability of a given target set $K \subset X$ at time T. The corresponding objective functional is given by

$$J(C) = \mu_T^C(K). \tag{4.10}$$

The problem is to find a $C \in \mathscr{C}_{ad}$ that maximizes the functional (4.10). Given that K is a closed subset of X, one can prove that the functional $C \longrightarrow J(C)$ is upper semi-continuous on $\mathscr{C}_{ad}$ in the τ_{pw} topology. Then, existence of a maximizing control $C^o \in \mathscr{C}_{ad}$ follows from compactness of the set $\mathscr{C}_{ad}$ in the same topology. A closely related problem is finding a stabilizing control that maximizes the probability of the state trajectory to reside inside a closed bounded set such as the closed ball B_r of radius $r > 0$ around the origin. Assuming λ to be any positive measure on I, and using Fatou's lemma and the fact that B_r is a closed set, one can show that the functional

$$J(C) = \int_0^T \mu_t^C(B_r)\lambda(dt) \tag{4.11}$$

is upper semi-continuous on $\mathscr{C}_{ad}$. Hence existence of a maximizing control law $C^o \in \mathscr{C}_{ad}$ follows from τ_{pw} compactness of the set $\mathscr{C}_{ad}$.

Min-Max Control: Min-Max control problems may arise in many different contexts, for example, battle against nature. Human race is always trying to minimize the maximum possible impact of natural disasters like pandemic, tornadoes, flood etc. by building infrastructure and other measures of appropriate response. Here, we consider a simpler min-max problem presented by the following uncertain system

$$\begin{aligned} dx &= Axdt + B(t)udt + \sigma(t)dW(t), x(0) = x_0 \\ y(t) &= L(t)x + \xi(t) \\ u(t) &= K(t)y(t), t \in I. \end{aligned} \tag{4.12}$$

Without going into details, we mention that the state space X and the observer space Y are real reflexive Banach spaces, the control space U is a Hilbert space, and the process $\{W\}$ is the cylindrical Brownian motion taking values in another Hilbert space H. The first equation defined on the Banach space X, represents the system to be controlled. The second equation defined on the Banach Y and perturbed by additive noise $\{\xi\}$, represents the observer (monitor) giving the noisy output $\{y\}$. The probability law of the uncertain process $\{\xi\}$ is unknown, but it is known that it has a bounded range $B_r(Y)$ (closed ball of radius $r > 0$ in Y). The third equation representing the controller, uses the noisy output $\{y\}$ to control the system through the operator $\{K\}$. The objective functional is given by

$$J(K,\xi) = (1/2)\mathbb{E}\int_I < Q(t)x, x >_{X^*,X} dt + < Mx(T), x(T) >_{X^*,X},$$

where $< \cdot, \cdot >_{X^*,X}$ denotes the standard duality product with $Q \in B_\infty(I, \mathscr{L}^+_{nu}(X,X^*))$, and $M \in \mathscr{L}^+_{nu}(X,X^*)$ being strongly measurable functions on I taking values in the set of positive nuclear operators $\mathscr{L}^+_{nu}(X,X^*) \subset \mathscr{L}_{nu}(X,X^*)$. The operator valued functions appearing in the system equations are given elements as indicated here; $B \in B_\infty(I, \mathscr{L}(U,X))$, $L \in B_\infty(I, \mathscr{L}(X,Y))$ and $K \in B_\infty(I, \mathscr{L}(Y,U))$ all measurable in strong operator topology. Let $\Lambda \subset \mathscr{L}(Y,U)$ be a closed bounded convex set and $\mathscr{B}_{ad} \equiv B_\infty(I,\Lambda)$ denote the set of admissible feedback control laws furnished with the Tychonoff product topology. The objective is to find a controller $K \in \mathscr{B}_{ad}$ that minimizes the maximum potential (risk) measured in terms of cost. This leads to the following Min-Max problem:

$$\inf\left\{ \sup\left\{J(K,\xi), \xi \in B_r(Y)\right\}, K \in \mathscr{B}_{ad} \right\}.$$

For detailed treatment of the above problem, the reader is referred to Ahmed [22]. Proof of existence of optimal feedback control law is presented in [22, Theorem 5.2] including the necessary conditions of optimality presented in [22, Theorem 5.4]. There has been substantial progress on optimal control of partially observed deterministic as well as stochastic systems on infinite dimensional spaces, see [19, 20, 21, 22, 23, 24, 30, 40].

Hamilton-Jacobi-Bellman (HJB) Equation: One well known technique for construction of optimal feedback control laws is based on Bellman's principle of optimality. Using this technique, one can construct optimal feedback control laws only for fully observed systems through the HJB equation which is a partial differential equation for the so called value function $V(t,x), (t,x) \in I \times X$. By replacing the relaxed controls by ordinary controls $\mathscr{U}_{ad} = L_\infty(I,U)$, where U is a compact Polish space, consider the system (4.1) with the cost functional given by equation (4.2). Let $\xi^u_{t,x}(s), s \in [t,T]$ denote the solution of equation (4.1) starting from state x at time t corresponding to control $u \in \mathscr{U}_{ad}$. Define the value function $V(t,x)$ on $I \times X$ as follows

$$V(t,x) \equiv \inf.\left\{ \mathbb{E}\left(\int_t^T \ell(s, \xi^u_{t,x}(s), u(s))ds + \Phi(\xi^u_{t,x}(T)) \right), u \in \mathscr{U}_{ad} \right\}. \tag{4.13}$$

Using the above expression and following Bellman's principle of optimality, and denoting by $\{DV, D^2V\}$ the first and second Frèchet derivatives of V on X, and using Ito-differential rule, one can easily derive the HJB equation as follows:

$$\begin{aligned} -\partial_t V &= < A^*DV(t,x), x >_{X^*,X} + H_o(t,x,DV,D^2V), (t,x) \in I \times X, \qquad (4.14)\\ V(T,x) &= \Phi(x), x \in X, \end{aligned}$$

where the Hamiltonian H_o is given by

$$\begin{aligned} H_o(t,x,DV,D^2V) \equiv \inf\Big\{ &< DV(t,x), F(t,x,u) > \\ &+ (1/2)Tr(\sigma(t,x,u)D^2V(t,x)\sigma^*(t,x,u)) + \ell(t,x,u), u \in U \Big\}, (t,x) \in I \times X. \qquad (4.15) \end{aligned}$$

The optimal (or minimum) cost is then given by $V(0,x_0)$. In case the initial state x_0 is an X-valued random variable with probability law $\nu_0 = \mathscr{L}(x_0)$, the minimum expected cost is given by $J_{op} = \int_X V(0,\zeta)\nu_0(d\zeta)$. Given the solution V of equation (4.14), the optimal feedback control law $u^o(t,x), (t,x) \in I \times X$ is then determined by minimizing the expression within the parenthesis of equation (4.15).

Note that equation (4.14) is a highly nonlinear second order partial differential equation on infinite dimensional Banach (or Hilbert) space. There has been some notable theoretical developments giving interesting results on the questions of existence of solutions for a limited class of such HJB equations. We refer the reader to Gozzi-Rouy [50], Gozzi-Rouy-Swiech [51], Chow-Menaldi [44], and Ahmed [25, 26]. It is very pleasing to note, that from the solution of the HJB equation one can directly construct the optimal state feedback control law. However, it is applicable only if the state is fully observable. This is a serious limitation of the HJB approach. Further, solving a nonlinear partial differential equation like (4.14) numerically on infinite dimensional space X is a formidable task if not impossible. Constructing the feedback law based on the numerical result is yet another forbidding task.

McKean-Vlasov Equation: Before we conclude this section, we present another important class of stochastic evolution equations known as McKean-Vlasov equation. Here, the vector field depends on the state as well as its probability law. A common example is found in plasma physics where ions and electrons interact through electro magnetic forces. The motion of individual particles is influenced by the collective behaviour of the plasma. Similarly, in any society, the behaviour of individuals is influenced by the collective behaviour (social norm) of the society. Let X be a real Hilbert space with $X_0 \subset X$ a nonempty set not containing 0 in its interior and $\mathscr{B}(X_0)$ the Borel sigma algebra of subsets of the set X_0. The dynamics of the McKean-Vlasov system is given by the following evolution equation on the Hilbert space X driven by the H-Brownian motion $\{W\}$ and the centred compensated Poisson random measure q_Λ with the intensity measure $\Lambda \in M^+(X_0)$,

$$\begin{aligned} dx &= Axdt + f(t,x,\mu_t)dt + \sigma(t,x,\mu_t)dW + \int_{X_0} g(t,x,\mu_t,v)q_\Lambda(dt \times dv) \\ x(0) &= x_0, \text{ and } \mu_t = \mathscr{L}(x(t)), \text{ the probability law of } x(t),\ t \in I = [0,T]. \end{aligned} \tag{4.16}$$

A special case of this general model, known as the mean-field stochastic evolution, is given by

$$\begin{aligned} dx &= Axdt + f(t,x(t),\mathbb{E}(x(t)))dt + \sigma(t,x(t),\mathbb{E}(x(t)))dW \\ &\qquad + \int_{X_0} g(t,x(t),\mathbb{E}(x(t)),v)q_\Lambda(dt \times dv) \\ x(0) &= x_0, \text{ and } \mathbb{E}(x(t)) = \bar{x}(t) = \text{ mean of } x(t)\ ,\ t \in I = [0,T]. \end{aligned} \tag{4.17}$$

Without going into details we mention the following results. For detailed proof of existence of solutions of the evolution equation (4.16), see Ahmed [32, Theorem 4.2]. For continuous dependence of solution on the Lévy (intensity) measure Λ, that determines the compensated Poisson random measure q_Λ; see [32, Theorem 5.1& Corollary 5.2]. Optimal controls are considered in [32, Theorem 6.2, Theorem 6.3]. For the Mean-Field equation (4.17), existence of solution including its regularity properties are given in [37, Theorem 3.1]. Existence of optimal control is proved in [37, Theorem 4.2] and necessary conditions of optimality are given in [37, Theorem 5.3 & Theorem 5.4]. Following this, application to LQGR (linear quadratic

Gaussin regulator) problem and its solution are presented. In [36], nonlinear diffusion governed by McKean-Vlasov equation is presented including several physical examples dealing with standard and nonstandard optimal control problems.

Nonlinear Stochastic Schrödinger Equation: In Section 3, we considered linear Schrödinger equation driven by operator valued measure as control representing laser pulses. Nonlinear Schrödinger equation describes the propagation of optical pulses through nonlinear media like optical fibres. Optical fibres have wide range of applications including communication (internet,phone,cable TV) and medical imaging etc. Here we consider a general nonlinear stochastic Schrödinger equation with state feedback control. This is represented by the following evolution equation on the Hilbert space $E = L_2(D) \times L_2(D)$,

$$dx = \mathscr{A}xdt + B(t)xdt + F(x)dt + C(t)dW, x(0) = x_0, t \in I.$$

The operator $\mathscr{A}$ is the generator of a unitary group of operators $\{U(t), t \in I\} \subset \mathscr{L}(E)$ as seen in example (**E2**) of section3, F is a nonlinear map in E, and C is an operator valued function on I with values in $\mathscr{L}_2(H,E)$ (equivalently $C(t)^*C(t) \in \mathscr{L}_1(H)), t \in I$ and W is an H valued cylindrical Brownian motion and $B \in \mathscr{B}_{ad} \subset B_\infty(I, \mathscr{L}_s(E))$. The set $\mathscr{B}_{ad}$ denotes the class of admissible feedback operator valued functions. The cost functional is given by

$$J(B) = \mathbb{E}\Big\{ \int_I \ell(t, x(B)(t))dt + \Phi(x(B)(T)) \Big\},$$

where $x(B) \in B^a_\infty(I, L_2(\Omega, E))$ is the mild solution of the above equation corresponding to the choice of feedback operator $B \in \mathscr{B}_{ad}$. For detailed analysis of this control problem, we refer the reader to [16, 17], where existence of optimal feedback control policy, necessary conditions of optimality, and convergence of an algorithm for construction of the optimal policy, are presented with complete proof.

5. Measure-Valued Solutions and Optimal Control

If f is merely continuous and even uniformly bounded on the state space X (a real Banach space), the Cauchy problem given by the following evolution equation

$$\dot{x} = Ax + f(x), x(0) = x_0, \tag{5.1}$$

may have no solution. On the other hand if X is finite dimensional, then just continuity is sufficient to prove the existence of a local solution which may blow up in finite time. But if $dim(X) = \infty$, continuity does not guaranty existence. This statement is clearly supported by the following simple example due to [5]. Let $X = c_0$, the sequence space $x = \{x_n, n \geq 1\}$ with norm $\parallel x \parallel \equiv sup\{|x_n|, n \in N\}$ satisfying $lim_{n\to\infty} x_n = 0$. It is well known that with respect to this norm it is a Banach space. Define the map

$$f(x) \equiv \{f_n(x_n) = \sqrt{|x_n|} + (1/1+n), x_n \in R, n = 1, 2, \cdots\}. \tag{5.2}$$

It is clear that f is a continuous map from X to X. Now consider the evolution equation on X given by

$$\dot{x} = f(x), x(0) = 0, t \geq 0. \tag{5.3}$$

For each $n \in N$, it is easy to verify that the scalar equation $\dot{x}_n = f_n(x_n), x_n(0) = 0$, has a continuous solution and that $|x_n(t)| \geq t^2/4$ for all $n \in N$ and therefore $x(t) = \{x_n(t), n \geq 1\} \notin X$ for any $t > 0$. In other words, equation (5.3) has no solution. Another example due to Godunov on a Hilbert space also demonstrates the nonexistence of a solution even though f is continuous.

Thus the standard notions of solutions, such as classical, strong, weak, and mild solutions, are not sufficient to cover much more general vector fields or equivalently much larger class of evolution equations on infinite dimensional spaces. This led to the notion of (generalized or) measure valued solutions which is comparable with the notion of (generalized) solutions in the sense of distribution for linear systems. For comprehensive study of measure-valued solutions, interested readers are referred to the recent book on the subject due to [Ahmed & Wang, 6], and many interesting papers on the subject due to [4, 7, 8, 9, 10, 13, 23, 27, 28, 29, 34, 38, 47]. The book [5] covers a wide variety of evolution equations and their stochastic versions on infinite dimensional spaces. It covers deterministic systems with continuous as well as measurable vector fields, impulsive systems, differential inclusions, stochastic evolution equations, deterministic as well as stochastic neutral differential equations. It is shown that these equations with non-standard vector fields have generalized solutions which are measure valued functions. Then a variety of classical and non-classical optimal control problems are presented proving existence of optimal control policies and giving necessary conditions of optimality. After a brief introduction of the theory of measure-valued solutions, here we present three physical examples, (1) Stochastic Navier-Stokes equation and optimal control of turbulence, (2) Optimization of Nonlinear Filtering (3) Bio-Medical system and Immunotherapy.

For study of measure-valued solutions we need the characterization of duals of certain function spaces. Let Z denote a normal topological space and $BC(Z)$ the space of bounded Borel measurable real valued functions equipped with the supnorm topology. The topological dual of this space is given by $M_{rba}(Z)$, the space of regular bounded finitely additive set functions defined on the Borel algebra of sets $\mathscr{B}(Z)$. With respect to these topologies, they are Banach spaces and the dual of $BC(Z)$ is $M_{rba}(Z)$ (see Dunford and Schwartz [47, Theorem 2]). Note that if Z is a compact Hausdorff space, $M_{rba}(Z) = M_{rca}(Z)$, the later denoting the space of regular countably additive measures. Let X be a Banach space and $L_1(I,X)$ denote the Banach space of Lebesgue-Bochner integrable X valued functions defined on $I = [0,T]$ furnished with standard norm topology. The dual of $L_1(I,X)$ is $L_\infty(I,X^*)$ if, and only if, X^* has the Radon-Nikodym property (RNP) Diestel [46, Theorem IV.1]. However, it follows form the theory of "lifting" [Tulcea & Tulcea, 53, Theorem 7 & its Corollary] that the dual of $L_1(I,X)$ is given by $L^w_\infty(I,X^*)$ known as the space of weak-star measurable functions on I taking values in X^*. The spaces $\{BC(Z), B(Z)\}$ and their duals $\{M_{rba}(Z), M_{ba}(Z)\}$ do not satisfy RNP. So, in view of the above result, the topological duals of $L_1(I,B(Z))$ and $L_1(I,BC(Z))$ are given by $L^w_\infty(I,M_{ba}(Z))$ and $L^w_\infty(I,M_{rba}(Z))$ respectively. Throughout the rest of the paper, we use $\{\Pi_{ba}(Z) \subset M_{ba}(Z)\}$ and $(\{\Pi_{rba}(Z) \subset M_{rba}(Z)\})$ to denote the class of finitely additive (regular finitely additive) probability measures defined on the algebra of Borel sets $\mathscr{B}(Z)$.

We consider the following Cauchy problem in Banach space E,

$$\dot{x} = Ax + f(t,x), x(0) = x_0 \in E, t \in I, \tag{5.4}$$

where A is the generator of a C_0-semigroup $S(t), t \geq 0$, in E, and $f : I \times E \longrightarrow E$ is measurable in t on I and continuous on E satisfying the following linear growth condition:

$$\parallel f(t,x) \parallel_E \leq K(t)(1+ \parallel x \parallel_E), x \in E, \text{ with } K \in L^+(I). \tag{5.5}$$

Using the growth property of f and the following integral equation

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s,x(s))ds, t \in I,$$

and Gronwall inequality one can easily verify that there exists a ball $B_r \subset E$ of finite radius $r > 0$ centred at the origin, such that $x(t) \in B_r$ for all $t \in I$.

Following Fattorini's notion of relaxed or generalized solutions, we have the following definition.

Definition 5.1. An element $\lambda \in L^w_\infty(I, M_{rba}(B_r))$ is a measure valued solution of equation (5.4) if for each $e^* \in E^*$, the following identity holds,

$$\int_{B_r} < e^*, \xi > \lambda_t(d\xi) = < S^*(t)e^*, x_0 > + \int_0^t ds \int_{B_r} < S^*(t-s)e^*, f(s,\xi) > \lambda_s(d\xi), t \in I. \quad (5.6)$$

Theorem 5.2. Let E be a Banach space with dual E^* and $B_r \subset E$ a ball of radius r centred at $0 \in E$. Let A be the generator of the C_0-semigroup $\{S(t), t \in I\}$ in E and $f : I \times E \longrightarrow E$ is Borel measurable and continuous in $x \in E$ satisfying the growth property (5.5). Then the system (5.4) has a measure valued solution $\lambda \in L^w_\infty(I, M_{rba}(E))$.

Proof. See Fattorini [48] and Ahmed [39].

Remark 5.3. The linear growth hypothesis (5.5) can be relaxed to admit vector fields f having polynomial growth and more. In fact, it can be simply replaced with much more relaxed assumptions.

It is known from general topology that every metric space is a Tychonoff space and that Stone-Cech compactification of a Tychonoff space is a compact Hausdorff space. Hence Stone-Cech compactification of E denoted by $E^+ \equiv \beta E$ is a compact Hausdorff space. Since homeomorphic spaces are topologically equivalent we can write $E \subset E^+$. We introduce a more general notion of solutions which is applicable to systems with vector fields $\{f\}$ admitting polynomial nonlinearities and more without requiring the growth property. Let $\mathscr{F}$ denote the class of test functions given by

$$\mathscr{F} \equiv \{\varphi \in BC(E) : D\varphi \text{ exists}, D\varphi \in BC(E, E^*)\}$$

and define the operator $\mathscr{A}$ with domain given by

$$\mathscr{D}(\mathscr{A}) \equiv \{\varphi \in \mathscr{F} : \mathscr{A}\varphi \in BC(E^+)\},$$

where

$$(\mathscr{A}\varphi)(\xi) = (A^*D\varphi(\xi), \xi >_{E^*,E} + < D\varphi(\xi), f(\xi) >_{E^*,E}, \xi \in E, \text{ for } \varphi \in D(\mathscr{A}).$$

Note that $\mathscr{D}(\mathscr{A}) \neq \emptyset$, for example, for $\psi \in \mathscr{F}$, the function φ given by $\varphi(\xi) \equiv \psi(\lambda R(\lambda, A)\xi)$, belongs to $\mathscr{D}(\mathscr{A})$ for each $\lambda \in \rho(A)$, the resolvent set of A.

Definition 5.4. The system given by the evolution equation (5.4) is said to have a measure valued solution $\lambda \in L^w_\infty(I, M_{rba}(E))$ if, for every $\varphi \in \mathscr{D}(\mathscr{A})$ having compact support in E, it satisfies the following equation

$$\lambda_t(\varphi) = \varphi(x_0) + \int_0^t \lambda_s(\mathscr{A}\varphi)ds, t \in I. \quad (5.7)$$

We present a result on the existence of measure solution of the above equation.

Theorem 5.5. Let A be the generator of a C_0-semigroup in E and suppose $f : I \times E \longrightarrow E$ is Borel measurable and for each $x \in E$, integrable on I and, for almost all $t \in I$, it is continuous and bounded in x on bounded subsets of E satisfying the following approximation properties:

(ai):there exists a sequence $\{f_n\}$ such that $f_n(t,x) \in D(A)$ for $x \in E$ and almost all $t \in I$;

(aii): for almost all $t \in I$, and $e^* \in E^*$, $< e^*, f_n(t,x) >_{E^*,E} \longrightarrow < e^*, f(t,x) >_{E^*,E}$ for each $x \in E$ and there exists a pair of sequences $\alpha_n, \beta_n \in L^+(I)$ possibly $\lim_{n\to\infty}\{\| \alpha_n \|_{L_1(I)}, \| \beta_n \| L_1(I)\} \longrightarrow \infty$ such that

$$\| f_n(t,x) - f_n(t,y) \|_E \leq \alpha_n(t) \| x - y \|_E \; \| f_n(t,x) \|_E \leq \beta_n(t)(1 + \| x \|_E).$$

Then, for each $x_0 \in E$, the evolution equation (5.4) has a measure valued solution $\lambda \in L_\infty^w(I, \Pi_{rba}(E^+))$ satisfying the identity (5.7); and further, $I \ni t \longrightarrow \lambda_t$ is weak star continuous.

Proof. See Ahmed [7, 39].

Remark 5.6. In theorem 5.5, we assumed that the initial state is a given point $x_0 \in E$. In fact there is no difficulty in admitting any element $\lambda_0 \in \Pi_{rba}(E)$ as the initial state.

In [39], one can find two examples. One is given by

$$\dot{x}(t) = Ax(t) + |x(t)|_E^{p-1} x(t), x(0) = x_0, t \in I$$

where A is the generator of a contraction semigroup in Banach space E. It is shown that for $p > 1$ the standard mild solution blows up in finite time $\tau = (1/(p-1) \| x_0 \|_E^{p-1})$. Another example is given by the nonlinear Klein-Gordon equation, describing the relativistic motion of particles, and having applications in solid state physics, cosmology and even suspension bridges. It contains vector fields having polynomial growth and its mild solution, if one exists, may blow up in finite time due to change of sign of a critical parameter. These are immaterial in case we consider measure valued solutions.

Theorem 5.5 is one of many earlier results on measure valued solutions for evolution equations on Banach spaces. In the literature one can find several results on existence of measure valued solutions for semilinear and quasilinear evolution equations with bounded as well as unbounded measurable vector fields including their optimal control. The reader is referred to [4, 9, 10, 36] for deterministic systems; [7, 8, 23, 34, 35, 38, 39] for stochastic systems; and [28, 29] for neutral systems. For detailed analysis of such systems and their optimal open-loop controls, and feedback controls, we refer to the recent book [5].

So far, in this section, we have considered semilinear systems. A large class of nonlinear parabolic partial differential equations can be formulated as an abstract quasilinear differential equation involving the Gelfand triple. Let H be a separable Hilbert space and V a dense linear subspace of H with dual V^* giving the Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ with continuous and dense embeddings. We consider the following quasilinear differential equation given by

$$\dot{x} + A(x)x = f(x), x(0) = x_0, t \in I. \tag{5.8}$$

Let $\mathscr{F}_O \equiv \{\phi \in BC(H) : D\phi \in BC(H,V)\}$ denote the class of cylindrical functions on H having first Fréchet derivatives continuous and bounded. Define the operator $\mathscr{A}$ by $D(\mathscr{A}) \equiv \{\phi \in \mathscr{F}_o : \mathscr{A}\phi \in BC(H^+)\}$ with

$$\mathscr{A}\phi = - < A^*(\xi)D\phi(\xi), \xi >_{V^*,V} + < D\phi(\xi), f(\xi) >_{V,V^*}.$$

Theorem 5.7. Consider the (quasilinear) system (5.8) and suppose $\{A, f\}$ satisfy the following conditions:

(C1): $A \in C(H, \mathscr{L}(V,V^*))$ satisfying (1): $\sup\{\| A(\xi) \|_{\mathscr{L}(V,V^*)}, \xi \in H\} < \infty$, (2): there exist $\gamma \geq 0$ and $\delta > 0$ such that $< A(\xi)v, v >_{V^*,V} + \gamma \| v \|_H^2 \geq \delta \| v \|_V^2$, $\forall\, v \in V$, $\forall\, \xi \in H$,

(C2): $f : H \longrightarrow V^*$ is continuous and bounded on bounded sets of H,

(C3): the embeddings $V \hookrightarrow H \hookrightarrow V^*$ are continuous and compact.

Then, for each $x_0 \in H$ or $\lambda_0 \in \Pi_{rba}(H^+)$, the evolution equation (5.8) has atleast one measure-valued solution $\lambda \in L_\infty^w(I, \Pi_{rba}(H^+))$ in the sense of Definition 5.4, and further, as a function of $t \in I$, it is weak star continuous taking values in $\Pi_{rba}(H^+)$.

Proof. See [9, Proposition 4.1]. For optimal control of such systems, see [4].

We mention another result related to impulsive evolution equations on a Banach space E containing measurable vector fields. In Theorem 5.5 as well as Theorem 5.7, we assumed the vector fields $\{f, g\}$ to be merely continuous and bounded on bounded sets. In the following theorem this is further relaxed. Here we admit vector fields which are Borel measurable maps from one Banch space to itself or another. The system is given by

$$\begin{aligned} &dx(t) = Ax(t)dt + f(x(t))dt + g(x(t-)\nu(dt), t \in I, \\ &x(0) = x_0 \in E, \end{aligned} \tag{5.9}$$

where A is the generator of a C_0-semigroup in E and f, g are Borel measurable maps from E to E. Let $\Sigma \equiv \Sigma_E$ denote the field or the algebra of subsets of the set E generated by closed sets and let $B(E) \equiv B(E_\Sigma)$ denote the class of scalar valued functions which are uniform limits of linear combination of characteristic functions of sets from Σ. The space $B(E)$, furnished with the supnorm topology,is a Banach space. An element f of this space is said to be Σ measurable if for every Borel set Λ in the real line (the range space), the set $f^{-1}(\Lambda) \equiv \{e \in E : f(e) \in \Lambda\} \in \Sigma$.

It is known that the topological dual of the Banach space $B(E)$ is given by the space of scalar valued finitely additive bounded measures on Σ which we denote by $M_{ba}(E)$ with $\Pi_{ba}(E) \subset M_{ba}(E)$ denoting the class of finitely additive probability measures.

We introduce the following class of test functions

$$\mathscr{F} \equiv \{\phi \in B(E) : D\phi \text{ exists } \&, D\phi \in B(E, E^*)\}.$$

Corresponding to the basic elements $\{A, f, g, \nu\}$ determining the evolution equation (5.9), we introduce the operators $\{\mathscr{A}, \mathscr{C}\}$ as follows

$$D(\mathscr{A}) \equiv \{\phi \in B(E) : \mathscr{A}\phi \in B(E) \subset B(E^+)\}$$

with

$$\begin{aligned} \mathscr{A}\phi(\xi) &= < A^* D\phi(\xi), \xi >_{E^*,E} + < D\phi(\xi), f(\xi >_{E^*,E} \;\; \xi \in E, \& \\ (\mathscr{C}\phi)(t,\xi) &\equiv \int_0^1 d\theta < D\phi(\xi + \theta g(\xi)\nu(\{t\}), g(\xi) >_{E^*,E}, (t,\xi) \in I \times E. \end{aligned}$$

Note that if the measure ν is nonatomic, the operator $\mathscr{C}$ simplifies to

$$(\mathscr{C}\phi)(t,\xi) = < D\phi(\xi), g(\xi) >_{E^*,E}.$$

Theorem 5.8. Consider the Cauchy problem (5.9) and suppose $\{A, f, g, \nu\}$ satisfy the following assumptions:

(A1): A is the generator of a C_0-semigroup $\{S(t), t \geq 0\}$ in E, and $\nu \in M_{ca}(I)$ is a countably additive signed measure having bounded variation on I,

(A2): the vector fields $\{h = f, g\} \in B(E,E)$ are bounded Borel measurable maps from E to itself satisfying the approximation properties: (i): there exists a sequence $\{h_n\} \in B(E,E)$ such that for each $n \in \mathbb{N}$, h_n is locally Lipschitz, $\| h_n(x) - h_n(y) \|_E \leq \alpha_n \| x - y \|_E, x, y \in E$ with $\alpha_n > 0$ and possibly $\alpha_n \longrightarrow \infty$. (ii): $h_n(x) \longrightarrow h(x)$ uniformly on compact subsets of E.

Then, for every $x_0 \in E$, the evolution equation (5.9) has at least one measure solution $\lambda^o \in L_\infty^w(I, \Pi_{ba}(E)) \cap L_\infty^w(|\nu|, \Pi_{ba}(E))$ in the sense that for every $\phi \in D(\mathscr{A})$,

$$\lambda_t^o(\phi) = \lambda_0(\phi) + \int_0^t \lambda_s^o(\mathscr{A}\phi) ds + \int_0^t \lambda_{s-}^o(\mathscr{C}(s)\phi)\nu(ds), t \in I.$$

Further, $t \rightarrow \lambda_t^o$ is piece wise weak-star continuous on I except at the set of atoms $a(\nu)$ of the measure ν.

Proof. See Ahmed [41, Theorem 3.2].

Remark 5.9. (i): Theorem 5.8 has been further generalized admitting measurable vector fields $\{h = f, g\}$ which are bounded only on bounded sets; see Ahmed [41, Theorem 3.3]. (ii): The conclusion of Theorem 5.8, and its generalization, as stated in cite[Theorem 3.3]39, remain valid for any initial state $\pi_0 \in \Pi_{ba}(E)$. (iii): For uniqueness of measure solution, the interested reader is referred to cite[Theorem 4.1]39 and the related discussion.

In a recent book by Ahmed and Wang [5], one can find comprehensive study of measure valued solutions for Deterministic, Impulsive, Stochastic, and Neutral evolution equations on infinite dimensional spaces including their optimal control. Also, considered are many examples from physical sciences. It is difficult to cover these topics in a brief review paper like this.

To complete this section we present very briefly a result on stochastic differential equation on Hilbert spaces driven by a martingale measure covering Wiener process. We need the Banach space $L_{1,2}(I \times \Omega, BC(H^+))$ and its topological dual $L_{\infty,2}^w(I \times \Omega, M_{rba}(H^+))$. More precisely, we need their $\{\mathscr{F}_t, t \geq 0\}$ progressively measurable completed subspaces denoted by $M_{1,2}(I \times \Omega, BC(H^+))$ with topological dual $M_{\infty,2}^w(I \times \Omega, M_{rba}(H^+))$.

Let $\{H, E\}$ be a pair of real separable Hilbert spaces and consider the stochastic system given by

$$dx(t) = Ax(t)dt + F(x(t))dt + G(x(t-))M(dt), t \in I, x(0) = x_0, \tag{5.10}$$

where A is the generator of C_0-semigroup, $F : H \longrightarrow H$, and $G : H \longrightarrow \mathscr{L}(E,H)$ are continuous maps satisfying the following approximation properties:

(A1): There exists a sequence of maps $\{F_n\}$ from H to H, and $\{G_n\}$ from H to $\mathscr{L}(E,H)$ such that $F_n(x) \longrightarrow F(x)$ in H, and $G_n(x) \longrightarrow G(x)$ in $\mathscr{L}(E,H)$ uniformly on compact subsets of H.

(A2): There exists a sequence $\{\alpha_n, \beta_n > 0\}$ possibly $\alpha_n, \beta_n \rightarrow \infty$ such that

$$\| F_n(x) \|_H \leq \alpha_n (1 + \| x \|_H), \| F_n(x) - F_n(y) \|_H \leq \alpha_n \| x - y \|_H$$

and

$$\| G_n(x) \|_{\mathscr{L}_2(E,H)} \leq \beta_n (1 + \| x \|_H), \| G_n(x) - G_n(y) \|_{\mathscr{L}_2(E,H)} \leq \beta_n \| x - y \|_H,$$

where $\mathscr{L}_2(E,H)$ denotes the space of Hilbert-Schmidt operators from E to H.

(A3): M is an E-valued martingale measure defined on the sigma algebra $\mathscr{B}_0$ of Borel subsets of $R_0 = [0,\infty)$ so that for every $t \in R_0$ and for any $J \in \mathscr{B}_0$ with $J \subset [0,t]$, $M(J)$ is $\mathscr{F}_t$ adapted where $(\Omega, \mathscr{F}, \mathscr{F}_{t\geq 0} \subset \mathscr{F}, P)$ is a complete filtered probability space. Further, there exists a nonatomic countably additive regular positive measure π such that for any $e_1, e_2 \in E$,

$$\mathbb{E}\{< e_1, M(J) >< e_2, M(K) >\} =< e_1, e_2 >_E \pi(J \cap K)$$

for every $J, K \in \mathscr{B}_0$.

Let $\Psi \equiv \{\varphi \in BC(H) : D\varphi, D^2\varphi$ continuous and bounded on $H\}$ denote the class of test functions and define the operators $\{\mathscr{A}, \mathscr{B}, \mathscr{C}\}$ as

$$D(\mathscr{A}) \equiv \{\varphi \in \Psi : (\mathscr{A}\varphi)(\xi) \equiv (1/2)Tr(G^*(\xi)D^2\varphi(\xi)G(\xi)), \mathscr{A}\varphi \in BC(H^+)\},$$
$$D(\mathscr{B}) \equiv \{\varphi \in \Psi : D\varphi \in D(A^*) \ \& \ (\mathscr{B}\varphi)(\xi) \equiv< A^*D\varphi(\xi), \xi >_H + < D\varphi(\xi), F(\xi) >_H,$$
$$\mathscr{B}\varphi \in BC(H^+)\}, \qquad (\mathscr{C}\varphi)(\xi) = G^*(\xi)D\varphi(\xi), \xi \in H.$$

Theorem 5.10. Suppose that A is the infinitesimal generator of a C_0 -semigroup in H and the maps $F : H \longrightarrow H$, $G : H \longrightarrow \mathscr{L}(E,H)$ are continuous, and bounded on bounded subsets of H, satisfying the approximation properties (A1) and (A2); and M is a nonatomic E valued vector measure satisfying (A3). Then, for every initial state x_0 for which $P\{\omega \in \Omega : |x_0|_H < \infty\} = 1$, the evolution equation (5.10) has at least one measure valued solution $\lambda^o \in M^w_{\infty,2}(I \times \Omega, M_{rba}(H^+)) \subset L^w_{\infty,2}(I \times \Omega, M_{rba}(H^+))$ in the sense that, for every $\varphi \in D(\mathscr{A}) \cap D(\mathscr{B})$,

$$\lambda^o_t(\varphi) = \varphi(x_0) + \int_0^t \lambda^o_s(\mathscr{A}\varphi)\pi(ds) + \int_0^t \lambda^o_s(\mathscr{B}\varphi)ds + \int_0^t < \lambda_{s-}(\mathscr{C}\varphi), M(ds) >_E, t \in I \quad (5.11)$$

Proof. See [7, 8, 39].

In Theorem 5.10, we assumed that the vector fields $\{F, G\}$ are continuous and bounded on bounded sets. These conditions are further relaxed proving existence of measure valued solutions with Borel measurable vector fields $\{F, G\}$ as seen in [9, Theorem 4.1]. Existence of optimal feedback controls are also considered in [8, Theorems 5.2, 5.3, 5.4, 5.5]. Recently, Theorem 5.10 was further extended covering a larger class of stochastic systems driven by Brownian motion and Poisson process [23]. Also, several interesting standard and nonstandard optimal control problems have been treated there.

Here, we consider three examples of practical importance in three different areas (A1) Artificial Heart (A2) Nonlinear Filtering (A3) Immunotherapy.

(A1) Artificial Heart: First we consider an example dealing with artificial heart. The cardiac cycle of the human heart covers the time from the beginning of one heartbeat to the beginning of the next. It consists of two periods: one during which the heart muscle relaxes and refills with blood, called diastole, following a period of contraction and pumping of blood, called systole. After emptying, the heart relaxes and expands to receive another influx of blood returning from the lungs and other systems of the body, before contracting again.The lungs and circulatory system work together to ensure a continuous supply of oxygen to the body and remove carbon dioxide. In essence the circulatory system, comprised of the heart and blood vessels, delivers oxygen-poor blood to the lungs, where it picks up oxygen and releases carbon dioxide. This refreshed, oxygen-rich blood is then pumped by the heart to the rest of the body.

An artificial heart is a miniature hydraulic pump consisting of a chamber $D \subset R^3$ (an open bounded domain) having an inlet and an outlet port, and a control port $\Gamma \subset \partial D$ connected to the pump whereby the space in the interior of the heart can be contracted and expanded. The control domain is chosen as

$$U \equiv \{z \in L_\infty(\partial D, R^3) : z(\zeta) = 0 \text{ for } \zeta \in \partial D \setminus \Gamma, \| z \|_\infty \leq \gamma < \infty\}.$$

We are concerned with the dynamics of blood flow inside the chamber and its control. The standard function spaces used in the study of Navier-Stokes equation (NSE) are given by the Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ having the structure of Reflexive Banach spaces. For details, see Ahmed [38] and Ahmed and Teo [11]. The objective of control is to perform the function of the heart while minimizing hemolysis (rupture of red blood cells leading to haemoglobin) and blood clot. Most likely, hemolysis is caused by shear stress, and blood clot by curl of the flow velocity. Under divergence free and incompressibility assumptions, the hemodynamics in the heart chamber is governed by the Navier-Stokes equation with boundary control as follows

$$\begin{aligned} dv/dt - \kappa A v &= f(t,v,u), v(0) = v_0, t \geq 0, \\ f(t.v,u) &= -B(v) + A\mathscr{R}G(u) + g, \end{aligned} \tag{5.12}$$

where $\kappa > 0$ is the kinematic viscosity, A is the diffusion operator representing the Stokes operator, B is a nonlinear operator mapping V to V^* arising from the convective term, g the volume force, $\mathscr{R}$ is the Dirichlet map (inverse of trace map), and G is a continuous map from U to $H^{1/2}(\partial D, R^3)$ producing the control force. In view of our objective, an appropriate cost integrand can be chosen as

$$\ell(t,v) \equiv \{\alpha_1(t) \| \nabla v \|_H^2 + \alpha_2(t) \| curl(v) \|_H^2\}, \alpha_1, \alpha_2 \in L_\infty^+(I). \tag{5.13}$$

Let $M_{rba}(U)$ denote the space of regular bounded Borel measures on U and

$$\mathscr{U}_{ad} \equiv L_\infty^w(I, \Pi_{rba}(U)) \subset L_\infty^w(I, M_{rba}(U))$$

denote the set of admissible relaxed controls (weak star measurable functions defined on I and taking values in the space of regular probability measure $\Pi_{rba}(U)$). Under the standard assumptions as seen above, for each initial state $x_0 \in H$ (or $\mu_0 \in \Pi_{rba}(H)$) and control $u \in \mathscr{U}_{ad}$, the evolution equation (5.12) has a measure valued solution $\mu \in L_\infty^w(I, \Pi_{rba}(H^+)) \subset L_\infty^w(I, M_{rba}(H^+))$ satisfying the following equation

$$\mu_t(\varphi) = \mu_0(\varphi) + \int_0^t \mu_s(\mathscr{A}\varphi)ds + \int_0^t \mu_s(\mathscr{B}^u(s)\varphi)ds, t \in I, \tag{5.14}$$

for every $\varphi \in D(\mathscr{A}) \cap D(\mathscr{B}^u)$ having bounded supports where

$$\begin{aligned} (\mathscr{A}\varphi)(\xi) = \kappa < A^* D\varphi(\xi), \xi >_H \text{ and } (\mathscr{B}^u\varphi)(\xi) &\equiv < D\varphi(\xi), f(t,\xi,u_t) >_H \\ &= \int_U < D\varphi(\xi), f(t,\xi,\zeta) >_H u_t(d\zeta). \end{aligned}$$

Let $\mu = \mu^u$ in equation (5.14), corresponding to the control $u \in \mathscr{U}_{ad}$, denote the measure solution. The corresponding cost functional is given by

$$J(u) = \int_I \int_V \ell(t,\xi) \mu_t^u(d\xi) dt. \tag{5.15}$$

The problem is to find a control policy $u \in \mathscr{U}_{ad}$ that minimizes the cost functional (5.15) thereby minimizing hemolysis and blood clot. We present the following result proving existence of optimal control. For necessary conditions of optimality and computational algorithm, see [14, Theorem 4A and Theorem 4B].

Theorem 5.11. For each initial state $\mu_0 \in \Pi_{rba}(H)$ and control policy $u \in \mathscr{U}_{ad}$, the system given by equation (5.12) has a measure valued solution $\mu \in L^w_\infty(I, \Pi_{rba}(H^+))$ in the sense that it satisfies the identity (5.14). Further the control to solution map $\mathscr{U}_{ad} \ni u \longrightarrow \mu^u \in L^w_\infty(I, \Pi_{rba}(H^+))$ is continuous with respect to the relative weak star topologies on $\mathscr{U}_{ad}$ and $L^w_\infty(I, \Pi_{rba}(H^+))$ respectively.

Proof. See [14, Theorem 2A] and [4, Theorem 4.1].

It is interesting to observe that because of the special structure of the NSE, equation (5.14) has measure valued solutions satisfying some stronger regularity properties. This is presented in the following corollary. For $Z = \{V, H\}$, let $M_2(Z)$ denote the space of Borel measures on Z having finite second moments in the sense that

$$\parallel \mu \parallel_{M_2(Z)} \equiv \int_Z |z|^2_Z |\mu|(dz)$$

where $|\mu|$ denotes the positive measure induced by the variation of μ, and $\Pi_2(Z) \subset M_2(Z)$ is the space of probability measures.

Corollary 5.12. Under the assumptions of Theorem 5.11, for every $\mu_0 \in M_2(H)$ and $u \in \mathscr{U}_{ad}$, (i): Equation (5.14) has a unique measure-valued solution $\mu^u \in L^w_\infty(I, \Pi_2(H)) \subset L^w_\infty(I, \Pi_2(H^+))$. (ii): The solution $\mu^u \in L^w_\infty(I, M_2(H)) \cap L_1(I, M_2(V))$ for each $u \in \mathscr{U}_{ad}$. (iii): There exists an optimal control minimizing the cost functional (5.15).

Proof. Part (i) follows from Theorem 5.11. We present a brief outline of the proof of (ii) and (iii). Recall the Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ and take $\varphi(\xi) = (1/2)(\xi, \xi)_H$ for $\xi \in V$. Substituting this φ in equation (5.14) and carrying out some simple computations, one can verify that

$$\parallel \mu_t \parallel_{M_2(H)} + 2\kappa \int_0^t \parallel \mu_s \parallel_{M_2(V)} ds = \parallel \mu_0 \parallel_{M_2(H)} + 2\int_0^t \mu_s(\mathscr{B}^u \varphi) ds, t \in I. \tag{5.16}$$

Recall equation (5.12) and note that, for any given $u \in \mathscr{U}_{ad}$,

$$\mathscr{B}^u(t) = -B(v) + A\mathscr{R}G(u_t) + g(t) = -B(v) + \int_U A\mathscr{R}G(\zeta) u_t(d\zeta) + g(t) \equiv -B(v) + \hat{g}(t, u_t),$$

where $\hat{g}(t, u_t) = g(t) + \int_U A\mathscr{R}G(\zeta) u_t(d\zeta), t \in I$. The operator B representing the convective term arises from the well known trilinear form given by $b(v, w, z) \equiv < (v \cdot \nabla) w, z >$ where $\{v, w, z\}$ are the (fluid) velocity vectors. From this one can identify the operator $B(v) = (v \cdot \nabla) v$. It is well known that in divergence free vector fields, $b(v, w, w) = 0$ for all velocity vectors v, w. Hence $< v, B(v) >_{V,V^*} = 0$ and thus, for φ as given above, we have

$$(\mathscr{B}^u(s)\varphi) = < \xi, \hat{g}(s, u_s) >_{V,V^*}, s \in I.$$

It follows from elementary Cauchy inequality that, for $\varepsilon > 0$,

$$| < \xi, \hat{g}(s, u_s) >_{V,V^*} | \leq (\varepsilon/2) \parallel \xi \parallel^2_V + (1/2\varepsilon) \parallel \hat{g}(s, u_s) \parallel^2_{V^*}.$$

Hence, for every $\varepsilon > 0$,

$$2\Big| \int_0^t \mu_s(\mathscr{B}^u \varphi) ds \Big| \leq \varepsilon \int_0^t \parallel \mu_s \parallel_{M_2(V)} ds + (1/\varepsilon) \int_0^t \parallel \hat{g}(s, u_s) \parallel^2_{V^*} ds, t \in I. \tag{5.17}$$

Choosing $\varepsilon = \kappa$ in the expression (5.17) and using it in the expression (5.16) we arrive at the following inequality

$$\| \mu_t \|_{M_2(H)} + \kappa \int_0^t \| \mu_s \|_{M_2(V)} \, ds \leq \| \mu_0 \|_{M_2(H)} + (1/\kappa) \int_0^t \| \hat{g}(s,u_s) \|_{V^*}^2 \, ds, t \in I. \quad (5.18)$$

Since $\hat{g} \in L_2(I,V^*)$ independently of the choice of control $u \in \mathscr{U}_{ad}$, it follows from the above inequality that the solution $\mu^u \in L_\infty^w(I,M_2(H)) \cap L_1(I,M_2(V))$ for all $u \in \mathscr{U}_{ad}$. In other words, the set $\{\mu^u, u \in \mathscr{U}_{ad}\}$ is a bounded subset of $L_\infty^w(I,M_2(H)) \cap L_1(I,M_2(V))$. This proves (ii). For proof of part (iii), note that the map $u \longrightarrow \mu^u$ is continuous with respect to the weak star topology on $\mathscr{U}_{ad}$, and the weak star topology on $L_\infty^w(I,\Pi_{rba}(H))$ respectively. It is clear from the expression for the cost integrand ℓ given by equation (5.13) that there exists an $\alpha \in L_\infty^+(I)$ such that $0 \leq \ell(t,\xi) \leq \alpha(t)|\xi|_V^2, \xi \in V$, and therefore

$$J(u) = \int_I \int_V \ell(t,\xi)\mu_t^u(d\xi)dt \leq \int_0^T \alpha(t) \| \mu_t^u \|_{M_2(V)} \, dt < \infty, \forall u \in \mathscr{U}_{ad},$$

and it follows from part (ii) that $\sup\{J(u), u \in \mathscr{U}_{ad}\} < \infty$. Let $\{\mu^n, \mu^o\}$ denote the solutions of equation (5.14) corresponding to controls $\{u_n, u_o\} \in \mathscr{U}_{ad}$ respectively, and suppose $u_n \xrightarrow{w*} u_o$. Then, as seen above, $\mu^n \xrightarrow{w*} \mu^o$ in $L_\infty(I,\Pi_{rba}(H))$. Using this fact and subtracting equation (5.14) corresponding to the pair $\{u_o, \mu^o\}$ from the same equation corresponding to the pair $\{u_n, \mu^n\}$ term by term and using $\varphi(\xi) = (1/2)(\xi,\xi)_H, \xi \in V$, and carrying out some simple rearrangements and using dominated convergence theorem, one can verify

$$\lim_{n\to\infty} \int_I \| \mu_t^n \|_{M_2(V)} \, dt = \int_I \| \mu_t^o \|_{M_2(V)} \, dt.$$

Hence, as $u^n \xrightarrow{w*} u^o$, we have $J(u^n) \longrightarrow J(u^o)$ proving its weak star continuity on $\mathscr{U}_{ad}$. Since $\mathscr{U}_{ad}$ is compact in the weak star topology, J attains its minimum on $\mathscr{U}_{ad}$. •

Alternate Approach: To reduce turbulence in the blood flow inside the heart chamber, one may consider reducing kinetic energy in the flow. Forcing the fluid velocity to stay within a small desirable range, reduces turbulence. So it is desirable to choose a closed ball $B_r \subset H$ of radius r (and centred at $0 \in H$) as the acceptable range of flow velocities. In this case the cost functional may be chosen as

$$J(u) = \int_I \mu_t^u(H \setminus B_r)dt$$

where $\mu^u \in L_\infty^w(I,\Pi_{rba}(H))$ is the solution of equation (5.14) corresponding to control $u \in \mathscr{U}_{ad}$. Here $u \longrightarrow J(u)$ is weak star lower semi-continuous, and $\mathscr{U}_{ad}$ is weak star compact. Thus there exists $u^o \in \mathscr{U}_{ad}$ at which J attains its minimum.

Remark 5.13. (i): For detailed study and analysis of measure valued solutions of stochastic Navier-Stokes equations and their optimal control, see [34, 35, 38], where many interesting optimal feedback control problems are considered. For necessary conditions of optimality see [14].

(ii): Given the spatial constraint, it is important to optimize the size and shape of the artificial device including its power supply. Bio-compatibility of the material used for construction of the device is another concern of critical importance.

(A2) Nonlinear Filtering: The subject of nonlinear filtering on finite and infinite dimensional spaces, is another interesting area where the dynamics of the system is given by a differential equation on the space of measures. This arises naturally from the following problem as discussed below. We are given the following pair of stochastic differential equations,

$$dx = b(x)dt + \sigma(x)dw_1, x(0) = x_0, \text{ in } R^n, \tag{5.19}$$

$$dy = h(x)dt + \sigma_0(y)dw_0, y(0) = 0, \text{ in } R^m, \text{ generaly } m \leq n, \tag{5.20}$$

where $b : R^n \longrightarrow R^n, \sigma : R^n \longrightarrow M(n \times r), h : R^n \longrightarrow R^m, \sigma_0 : R^m \longrightarrow M(m \times s)$ and w_1, w_0 are independent Brownian motions with values in R^r and R^s having covariance matrices $0 \leq Q_1 \in M(r \times r)$ and $0 < Q_0 \in M(s \times s)$ respectively. Let $\mathscr{F}_t^y, t \geq 0$, denote the history of the process $\{y\}$ till time $t \geq 0$. For any $\varphi \in BC(R^n)$, the objective is to find the best (unbiased minimum variance) estimate of $\varphi(x(t))$ given the history $\mathscr{F}_t^y$. It is known from basic theory of filtering [27] that the solution is given by the conditional expectation

$$\mathbb{E}\{\varphi(x(t)) \big| \mathscr{F}_t^y\} \equiv \int_{R^n} \varphi(\xi) Q_t^y(d\xi),$$

where Q_t^y denotes the conditional probability measure of $x(t)$ given the history of the process $\{y\}$ till time t with $Q^y \in L_\infty(I, \mathscr{M}_1(R^n))$, where $\mathscr{M}_1(R^n) \subset \mathscr{M}_b(R^n)$ denotes the class of regular probability measures on the sigma algebra $\mathscr{B}(R^n)$ of Borel sets in R^n while $\mathscr{M}_b(R^n)$ denotes the linear space of bounded signed Borel measures on R^n. It is known that the measure valued function $\mu_t^y, t > 0$, defined on $\mathscr{B}(R^n)$ by

$$\mu_t^y(\cdot) \equiv \mu_t^y(R^n)\; Q_t^y(\cdot) \tag{5.21}$$

is the un-normalized (nonnegative) measure that satisfies the following linear differential equation on $\mathscr{M}_b(R^n)$ written in the weak form

$$d\mu_t(\varphi) = \mu_t(\mathscr{A}\varphi)dt + < \mu_t(\varphi h), \mathscr{R}_0^{-1}(y)dy >_{R^m}, \mu_0(\varphi) = \pi(\varphi), \pi = \mathscr{L}(x_0), t \geq 0, \tag{5.22}$$

where the operators $\mathscr{A}$ and $\mathscr{R}_0$ are given by the following expressions

$$\mathscr{A}\varphi \equiv (1/2Tr((D^2\varphi)(\sigma Q \sigma^*)) + < D\varphi(\xi), b(\xi) >, \;\&\; \mathscr{R}_0(y) \equiv \sigma_0(y) Q_0 \sigma_0^*(y)$$

with $\mathscr{R}_0(y)$ assumed to have inverse. This is known as Zakai equation and as seen in the expression (5.21), its solution is a variable positive multiple of the conditional probability law of the signal process x based on the signal-dependent noisy observed process y. By use of Girsanov transformation, equation (5.22) can be reformulated as

$$d\mu_t(\varphi) = \mu_t(\mathscr{A}\varphi)dt + < \mu_t(\varphi h), dV >_{R^m}, \mu_0(\varphi) = \pi(\varphi), \pi = \mathscr{L}(x_0), t \geq 0, \tag{5.23}$$

on a different probability space $(\Omega, \mathscr{F}^y, \tilde{P})$, on which V is a Wiener process called the innovation process. For detailed study and derivation of such equations the reader is referred to [27, 34, 38, 42]. Clearly, filtering equation (5.22), (equivalently (5.23)), is derived for a given set of system operators $\{b, \sigma, h, \sigma_0\}$. Here, we are interested to choose the best sensor h (see (5.20)) (from a given class of nonlinear maps) that optimizes the performance of the filter in terms of the mean square of the difference between the filter estimate and the true value, all evaluated at the terminal time T. For a given sensor h, this is given by

$$\rho(h) = \mathbb{E}\Big\{ \Big(\mathbb{E}\{\varphi(x(T)) \big| \mathscr{F}_T^y\} - \varphi(x(T)) \Big)^2 \Big| \mathscr{F}_T^y \Big\}$$
$$= \int_{R^n} \varphi^2(\xi) Q_T^h(d\xi) - \Big(\int_{R^n} \varphi(\xi) Q_T^h(d\xi) \Big)^2, \qquad (5.24)$$

where $Q_T^h(\cdot)$ denotes the conditional probability measure of $x(T)$ given the history $\mathscr{F}_T^y$ corresponding to the sensor h. Let $H_{ad} \subset C(R^n, R^m)$ denote the class of admissible sensors for the observer equation (5.20) or equivalently, the filter equation (5.23). Using the unnormalized measure related to the conditional probability measure given by (5.21), the functional $\rho(h)$ can be written as

$$\rho(h) = \Big\{ (1/(\mu_T^h(1))^2) \Big(\mu_T^h(1)\mu_T^h(\varphi^2) - (\mu_T^h(\varphi))^2 \Big) \Big\} \text{ for } h \in H_{ad}. \qquad (5.25)$$

Note that this is a random variable dependent on the path process generated by the trajectories of the innovation process V. Let $\mathscr{C} \equiv C(I, R^m)$ denote the space of continuous functions and $\mathscr{B}(\mathscr{C})$ denote the algebra of cylinder sets in $\mathscr{C}$, and $\mathscr{W}$ is the Wiener measure on it induced by the innovation process V. The objective functional is given by

$$J(h) = \int_{\mathscr{C}} \rho(h)(w) \mathscr{W}(dw).$$

In the following theorem, we prove the existence of optimal sensor.

Theorem 5.14. Consider the filter dynamics given by equation (5.23) and suppose $\{b, \sigma, \sigma_0\}$ satisfy usual Lipschitz conditions with $\mathscr{R}_0 = \sigma_0 Q_0 \sigma_0^*$ being invertible. Let H_{ad} be a compact subset of $C(R^n, R^m)$ with respect to the compact open topology in the sense of Ascoli [55, Theorem 43.15]. Then there exists an $h_o \in H_{ad}$ at which $J(h_o) \leq J(h)$, for all $h \in H_{ad}$.

Proof. For detailed proof, see Ahmed [38, Theorem 4.6.3]. We present a brief outline. First note that disregarding the dependence of μ_T on h temporarily, ρ is a function of $\mu_T \in \mathscr{M}_b(R^n)$. Thus for any sequence μ^n converging weakly (weak star) to μ^o and $\varphi \in BC(R^n)$, it follows from the expression on the right hand side of equation (5.25) that it converges to the same expression evaluated at μ^o. Now using the Markov semigroup corresponding to the operator $\mathscr{A}$, differential equation (5.23) can be written as the following equivalent integral equation

$$\mu_t(\varphi) = \pi(S(t)\varphi) + \int_0^t \mu_r(S(t-r)\varphi h), dV(r) >, t \in I. \qquad (5.26)$$

Let $\{h_n, h_o\} \in H_{ad}$ with $h_n \xrightarrow{\tau_{co}} h_o$ and $\{\mu^n, \mu^o\}$ the corresponding solutions of equation (5.26). We write equation (5.26) corresponding to the pair $\{h_o, \mu^o\}$ and subtract it term by term from the same expression corresponding to the pair $\{h_n, \mu^n\}$. Then, after minor algebraic manipulation and recalling the contraction property of Markov semigroups, and letting $h_n \longrightarrow h_o$, one can easily verify that $\mu^n \xrightarrow{w^*} \mu^o$. Thus we conclude that as h_n converges to h_o in the compact open topology, $\rho(h_n) \to \rho(h_o)$ $\mathscr{W} - a.s.$ Further, from the expression (5.25) one can easily verify that $\rho(h)$ is bounded from above. Hence it follows from Lebesgue bounded convergence theorem that $J(h_n) \to J(h_o)$. Thus J is continuous on H_{ad} in the compact open topology. Since H_{ad} is compact in the compact open topology, J attains its minimum on H_{ad}. •

Remark 5.15. Theorem 5.15 proves existence of optimal sensor (measurement strategy). Necessary conditions of optimality, whereby optimal sensor can be determined, is found in [34]. Similar results for infinite dimensional filtering can be found in Ahmed [34, 38].

(A3) Immunotherapy: As stated in section 2, optimal control theory for nonlinear reaction diffusion equations have been considered for application to immunotherapy [18]. This therapy empowers the body's natural immune system to fight and kill the cancer cells. There are several methods known as (1) Checkpoint modulators (2) Immune cell therapy (3) Therapeutic Antibodies (4) Cancer treatment Vaccines (5) Immune system modulators. For detailed explanation, see [18]. One of the techniques is based on synthesis of antigen-specific antibodies in laboratories which are then administered to cancer patients to boost up their immune system. Considering case (3), Therapeutic Antibodies are produced in the laboratory and linked to certain toxic substances (such as bacterial toxin). Such antibodies are called ADC (Antibody-Drug-Conjugates) and once they are administered to the patient, they bind on to cancer cells which absorb the toxic substance leading to their destruction.

Considering the presence of n distinct antibody-antigen pairs, their interaction can be described by the following reaction diffusion equation

$$\dot{y} = Ay + F(y(t), u_t), y(0) = y_0, t \in I, \tag{5.27}$$

on the Banach space $E_0 = C(\overline{D}, R^{2n})$ where $D \subset R^3$ is an open bounded domain with smooth boundary and $\overline{D}$ its closure. We consider the product space $E_0 \equiv E_B \times E_G$ with $E_B = E_G = C(\overline{D}, R^n)$ denoting their respective state spaces. In general, we may assume that m distinct antibody producing control drugs are used. For characterization of control space, we take a closed bounded set $D_0 \subset D$ and define $U \equiv C(D_0, R^m)$ with $U_b \subset U$ denoting a closed bounded set contained in the positive cone of U. Let $\mathscr{M}_1(U_b) \subset \mathscr{M}(U_b)$ denote the class of regular probability measures contained in the linear space of Borel measures on U_b. For admissible controls, we choose relaxed controls, $\mathscr{U}_{ad} \equiv L^w_\infty(I, \mathscr{M}_1(U_b))$, the set of weak star measurable functions on I taking values in the space of probability measures $\mathscr{M}_1(U_b)$ a subset of the linear space $L^w_\infty(I, \mathscr{M}(U_b))$.

Lett Υ denote the $2n \times 2n$ diagonal matrix with positive elements, the operator $A = \Upsilon\Delta$ subject to homogeneous Neumann boundary condition, is the generator of a C_0-semigroup on E_0. The nonlinear map $F : E_0 \times U_b \longrightarrow E_0$ is continuous and bounded on bounded sets of $E_0 \times U_b$. For $y \in E_0$ and $u \in \mathscr{U}_{ad}$, $F(y, u_t) = \int_{U_b} F(y, v) u_t(dv)$. It is known that under the given assumptions, for each $y_0 \in E_0$ (or $\mu_0 \in \mathscr{M}_1(E_0)$) and $u \in \mathscr{U}_{ad}$, the evolution equation (5.27) has a unique measure valued solution $\mu \in L^w_\infty(I, \mathscr{M}_1(E_0)) \subset L^w_\infty(I, \mathscr{M}(E_0))$. For proof of existence of measure valued solutions see [5, 18].

As seen in this section, for evolution equations on the space of measures, we need the following operators $\{\mathscr{A}, \mathscr{B}\}$ as defined below. Let

$$\Psi \equiv \{\varphi \in BC(E_0) : D\varphi(\xi) \in BC(E_0, E_0^*)\}$$

denote the class of test functions and

$$(\mathscr{A}\varphi)(\xi) = < A^* D\varphi(\xi), \xi >_{E_0^*, E_0}, \ \forall \ \xi \in E_0$$

$$(\mathscr{B}(u_t)\varphi)(\xi) = \int_{U_b} < F(\xi, v), D\varphi(\xi) >_{E_0, E_0^*} u_t(dv), \ \forall \ \xi \in E_0 \text{ and } u \in \mathscr{U}_{ad}$$

with the domain of $\mathscr{B}$ being independent of $v \in U_b$.

For $u \in \mathscr{U}_{ad}$, let $\mu^u \in L_\infty^w(I, \mathscr{M}_1(E_0)) \subset L_\infty^w(I, \mathscr{M}(E_0))$ denote the measure valued solution of equation (5.27) in the sense that for every $\varphi \in \mathscr{D}(\mathscr{A}) \cap \mathscr{D}(\mathscr{B})$ having bounded support, μ^u satisfies the following identity

$$\mu_t^u(\varphi) = \mu_0(\varphi) + \int_0^t \mu_s^u(\mathscr{A}\varphi)ds + \int_0^t \mu_s^u(\mathscr{B}(u)\varphi)ds, t \in I. \tag{5.28}$$

The clinical objective is to raise the population of antigen-specific antibodies to a prescribed level considered adequate to kill the pathogens (antigens). A reasonable cost functional for achieving this goal is given by

$$J(u) = \int_{I\times E_0} \ell(t,\xi)\mu_t^u(d\xi)dt + \int_{E_0} \Phi(\xi)\mu_T^u(d\xi), \tag{5.29}$$

where $\ell(t,\xi) = \ell(t,\zeta,\eta) \equiv w_r(\zeta,\eta)[(\alpha(t)|\zeta - \zeta_d|_{E_B}^2 + \beta(t)|\eta|_{E_G}^2)]$ with $\alpha, \beta \in L_1^+(I)$, $\xi = \{\zeta,\eta\} \in E_0 = E_B \times E_G$. For any finite real number $r > 0$, the weight function $w_r \in C(E_0)$ and positive with support $B_r \subset E_0$ a ball of radius r and $0 \le \Phi \in BC(E_0)$. Existence of a minimizing control follows from weak star compactness of the set $\mathscr{U}_{ad}$ and continuity of J on $\mathscr{U}_{ad}$ in the same topology. The control $u \in \mathscr{U}_{ad}$ that minimizes the cost functional (5.29) gives the optimal drug administration policy.

An alternate Approach: Another approach for treatment of cancer is to choose the best drug administration strategy over the time period I so as to eliminate (or reduce) cancer causing pathogens from the patients body. With this goal in mind, we choose the set $C \equiv E_B \times B_\varepsilon \subset E_B \times E_G \equiv E_0$ where $B_\varepsilon \subset E_G$ is a closed ball of radius $\varepsilon > 0$ centred at $0 \in E_G$. The objective is to find a control policy $u \in \mathscr{U}_{ad}$ that maximizes the measure $\mu_T^u(C)$ of the target set C. This is equivalent to reducing the population of antigens by the end of the treatment period to the level determined by the target set C. Alternatively, one may choose to maximize the functional

$$J(u) \equiv (1/T)\int_0^T \mu_t^u(C)dt. \tag{5.30}$$

In fact one can verify that as $u^n \xrightarrow{w*} u^o$ in $\mathscr{U}_{ad}$, $\mu^n \xrightarrow{w*} \mu^o$ in $L_\infty^w(I, \mathscr{M}_1(E_0))$. Since C is a closed set, we have $\overline{\lim}^n \mu_t^n(C) \le \mu_t^o(C)$ for all $t \in I$. Thus

$$\overline{\lim}^n \int_I \mu_t^n(C)dt \le \int_I \overline{\lim}^n \mu_t^n(C)dt \le \int_I \mu_t^o(C)dt. \tag{5.31}$$

Hence $\overline{\lim}^n J(u^n) \le J(u^o)$ proving upper semicontinuity of J on $\mathscr{U}_{ad}$. Given this fact, existence of optimal control follows from weak star compactness of the set $\mathscr{U}_{ad}$.

Necessary Conditions of Optimality Here, we present necessary conditions of optimality associated with the system (5.28) and the cost functional (5.29). Note that equation (5.28) is the weak form of the following evolution equation on the space of measures $\mathscr{M}(E_0)$ corresponding to control $u \in \mathscr{U}_{ad}$,

$$d\mu_t^u = \mathscr{A}^*\mu_t^u dt + (\mathscr{B}(u_t))^*\mu_t^u dt, \mu_0^u = \mu_0, t \in I. \tag{5.32}$$

Theorem 5.16. Consider the integral equation (5.28) or equivalently the evolution equation (5.32) on the space of Borel measures $\mathscr{M}(E_0)$ with the operators and admissible controls $\{\mathscr{A}, \mathscr{B}(u_t), u \in \mathscr{U}_{ad}\}$ as described above, and initial condition $\mu_0 \in \mathscr{M}_1(E_0)$. Let J denote the cost functional given by (5.29) with $0 \le \ell \in L_1(I, BC(E_0))$ and $0 \le \Phi \in BC(E_0)$. Then, for the control state pair $\{u^o,\mu^o\} \in \mathscr{U}_{ad} \times L^w_\infty(I,\mathscr{M}_1(E_0))$ to be optimal, it is necessary that there exists a $\Psi \in L_1(I,BC(E_0)) \cap C(I,BC(E_0))$ so that the triple $\{u^o,\mu^o,\Psi\}$ satisfy the following inequality and evolution equations,

$$dJ(u^o,u-u^o) = \int_I \mu^o_s(\mathscr{B}(u_s-u^o_s)\Psi)ds \geq 0, \ \forall\, u \in \mathscr{U}_{ad}, \tag{5.33}$$

$$-(d/dt)\Psi = \mathscr{A}\Psi + \mathscr{B}(u^o_t)\Psi + \ell(t,\cdot), \Psi(T,\cdot) = \Phi(\cdot), t \in I, \tag{5.34}$$

$$d\mu^o_t = \mathscr{A}^*\mu^o_t dt + (\mathscr{B}(u^o_t))^*\mu^o_t, \mu^o_0 = \mu_0, t \in I. \tag{5.35}$$

Proof. Proof is similar to that of [14, Theorem 4A].

Remark 5.17. (i): Theorem 5.16 giving the necessary conditions of optimality is very general. It dose not require any of the basic constituents of the system $\{F,\ell,\Phi\}$ to be differentiable with respect to the state variable. (ii): Another important aspect of this result is that standard necessary conditions of optimality readily follow from this general result if $\{F,\ell,\Phi\}$ satisfy standard regularity assumptions. This is presented in the following corollary.

Corollary 5.18. (i): Suppose A is the generator of C_0-semigroup and F is locally Lipschitz in the state variable and continuous in the control variable. Then for each $y_0 \in E_0$, equation (5.27) has a unique mild solution. (ii): If $\{F,\ell,\Phi\}$ are Borel measurable in all the variables and continuously Gâteaux differentiable in the state variable, the measure solution corresponding to the optimal control u^o reduces to the Dirac measure $\{\mu^o_t(d\xi) = \delta_{y^o(t)}(d\xi), t \in I\}$ evaluated along the optimal path y^o; and the necessary conditions given by Theorem 5.16 reduce to the following necessary conditions

$$dJ(u^o,u-u^o) = \int_I < F(y^o(t),u_t-u^o_t), \psi(t) >_{E_0,E_0^*} dt, \ \forall\, u \in \mathscr{U}_{ad},$$
$$-(d/dt)\psi(t) = A^*\psi(t) + (D_yF(y^o(t),u^o_t))^*\psi(t) + \ell_y(t,y^o(t)), \psi(T) = \Phi_y(y^o(T)),$$
$$(d/dt)y^o(t) = Ay^o(t) + F(y^o(t),u^o_t), y^o(0) = y_0, t \in I.$$

Conclusion. We have presented a brief review of optimal control theory for infinite dimensional systems touching on some selected topics susch as (1): deterministic systems governed by differential equations as well as differential inclusions and their optimal control (2): structural controls determined by operator valued measures and optimization (3): stochastic systems on Hilbert Spaces and their optimal control (4): measure-valued solutions for systems containing non-smooth vector fields (or generating operators) which do not admit path wise solutions such as strong, weak and mild.

We may have possibly and inadvertently missed some important contributions of other workers in the related fields for which the author offers sincere apology.

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