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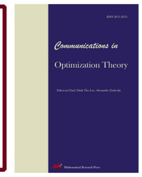

NEW NONLINEAR PRINCIPLES AND FIXED POINT THEOREMS IN LOCALLY p-CONVEX SPACES

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In Honor of the 90th Birthday of Professor R. Tyrrell Rockafellar

Abstract. The goal of this paper is to establish fixed point theorems for non-self $\mathfrak{F}$ contraction and non-self nonexpansive set-valued mappings by applying Caristi fixed point theorem as a tool in locally p-convex spaces, where $p \in (0,1]$. In particular, we first derive a few of key principles widely used in nonlinear analysis and optimization by using Caristi fixed point theorem as a tool in both complete metric spaces and locally complete p-convex spaces, where $p \in (0,1]$, respectively. Secondly, as applications of Caristi fixed point theorem being a starting point, we establish fixed point theorems for non-self $\mathfrak{F}$ contraction and non-self nonexpansive set-valued mappings in locally complete p-convex spaces. These results improve or unify corresponding results in the literature.

Keywords. Ekeland variatioal principle; Fixed point theorem; Nonconvex minimization theorem; Nonexpansive set-valued mapping; weakly inward mapping.

2020 Mathematics Subject Classification. 47H10, 49J53.

1. Introduction

It is well-known that Caristi fixed point theorem established by Caristi [8] in 1976 is undoubtedly one of the most valuable one of generalizations of the Banach contraction principle and plays a very crucial role in optimization, nonlinear analysis and related branches of applied mathematical analysis, including game theory, economics, financial markets, integral and differential equations and inclusions, dynamic systems theory, signal and image processing, and so forth. Over the past hundred years, numerous scholars made significant contributions to fixed point theory and its diverse applications. For more details on the study of fixed point theory and related applications, we refer the reader to [9, 12, 13, 40, 41, 42, 43, 46, 56, 57, 58, 62, 63] and the references therein.

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The goal of this paper is to establish new fixed point theorems for non-self $\mathfrak{F}$ contraction and non-self nonexpansive set-valued mappings by applying Caristi fixed point as a tool in locally p-convex spaces, where $p \in (0,1]$. In particular, we first derive a few of key principles widely used in nonlinear analysis and optimization by using Caristi fixed point theorem as a tool in both complete metric spaces and locally complete p-convex spaces, where $p \in (0,1]$, respectively. Secondly, as applications of Caristi fixed point theorem being a starting point, we establish fixed point theorems for non-self $\mathfrak{F}$ contraction and non-self nonexpansive set-valued mappings in locally complete p-convex spaces. These results improve or unify corresponding results in the literature. In particular, in this paper, we first discuss the famous and very important tool in nonlinear functional analysis called 'Caristi fixed point theorem' established by Caristi [8] in complete metric spaces and locally complete convex spaces. Then we show that a few of key principles established in nonlinear analysis are equivalent to Caristi fixed point theorem. By following the idea given by Ansatri [2], Ansatri and Sahu [3], we derive Ekeland variational principle [14, 15, 16], Takahashi nonconvex minimization theorem [52], and Oettli-Théra theorem in complete metric spaces and locally complete p-convex spaces, where $p \in (0,1]$ (and indeed, also equivalent to Daneš drop theorem, and flower petal theorem as discussed by Penot [37]), and see also [1, 12, 13, 40, 41, 42, 43, 46, 34, 35, 56, 57, 58, 62, 63].

This paper consists of four sections: Section one is for an introduction with literature review briefly; Section two is preliminary by including some notions and basic results; Section three is for the discussion of Caristi fixed fixed point theorems in complete metric spaces and locally complete p-convex spaces, where $p \in (0,1]$; Section four is to establish fixed point theorems for non-self $\mathfrak{F}$ contraction and nonexpansive set-valued mappings by applying Caristi fixed point theorem in locally complete p-convex spaces, and locally complete convex spaces, where $p \in (0,1]$, and finally with a summary including some related remarks.

2. Preliminaries

For the convenience of our discussion, throughout this paper, for a given set X, we denotes by 2^X the family of all subsets of X. If X is a metric space (X,d), we denote by $CB(X)$ the family of all non-empty closed bounded subsets of (X,d), where d is the metric of X. In addition, all topological vector spaces are assumed to be Hausdorff unless specified, and we denote by $\mathbb{R}$ and $\mathbb{N}$ the real line, and the all positive integers, respectively. For any set $D \subset X$, the diameter of D is defined by $diam(D) := \sup\{d(x,y), x,y \in D\}$.

We now recall some standard notions and concepts which would be used or helpful for related discussion or topic in this paper.

Definition 2.1. Let $p \in (0,1]$. A set A in a vector space X is said to be p-convex if for any $x,y \in A$, we have $sx+ty \in A$, whenever $0 \leq s,t \leq 1$ with $s^p + t^p = 1$; the set A is said to be absolutely p-convex if for any $x,y \in A$, we have $sx+ty \in A$, whenever $|s|^p + |t|^p \leq 1$. In the case $p = 1$, the concept of the (absolutely) 1-convexity is simply the usually (absolutely) convexity defined in vector spaces.

Remark 2.1. Here we may point out that the concept of ***p*-convexity** with its **historical origin** (originally termed as *degree r* by Landsberg) was introduced by Landsberg [27] in 1953 to study non-locally convex topological vector spaces. This framework was later systematized in the works of Jarchow [23] in 1981 and Kalton et al. [24] in 1984, and later used by Yuan [59] to introduce the new concept called "Yuan's Sharpness Operator" in topological vector spaces as a powerful tool to study and answer Schauder Conjecture completely in the affirmative, which standards there openly around one century since it was raised by Schauder [49] 1930's (see also Yuan [56]-[59] and references wherein for more in details).

Definition 2.2. Let $p \in (0,1]$. If A is a subset of a topological vector space X, the closure of A is denoted by $\overline{A}$, then the p-convex hull of A and its closed p-convex hull denoted by co_pA, and $\overline{co_pA}$, respectively, which is the smallest p-convex set containing A, and the smallest closed p-convex set containing A, respectively.

For the convenience of our discussion, we will use the symbols co_pA, $\overline{co_pA}$, and $\overline{aco_pA}$ to denote the p-convex hull of A, its closed p-convex hull, and its closed absolute p-convex hull, respectively, where $p \in (0,1]$.

Definition 2.3. Let $p \in (0,1]$ and A be p-convex. Let $x_1, \cdots, x_n \in A$, $t_i \geq 0$ with $\sum_1^n t_i^{\mathrm{p}} = 1$.

(i) $\sum_1^n t_i x_i$ is called a p-convex combination of $\{x_i\}$ for $i = 1, 2, \cdots, n$.
(ii) If $\sum_1^n |t_i|^{\mathrm{p}} \leq 1$, then $\sum_1^n t_i x_i$ is called an absolutely p-convex combination.

It is easy to see that $\sum_1^n t_i x_i \in A$ for a p-convex set A.

Definition 2.4. A subset A of a vector space X is called balanced (or circled) if $\lambda A \subset A$ holds for all scalars λ satisfying $|\lambda| \leq 1$. We say that A is absorbing if for each $x \in X$, there is a real number $\rho_x > 0$ such that $\lambda x \in A$ for all $\lambda > 0$ with $|\lambda| \leq \rho_x$.

Definition 2.5. Let X is a vector space and $\mathbb{R}^+$ is a non-negative part of a real line $\mathbb{R}$. Then a mapping $P : X \longrightarrow \mathbb{R}^+$ is said to be a p-seminorm if it satisfies the following requirements for $p \in (0,1]$:

(i) $P(x) \geq 0$ for all $x \in X$;
(ii) $P(\lambda x) = |\lambda|^p P(x)$ for all $x \in X$ and $\lambda \in R$;
(iii) $P(x+y) \leq P(x) + P(y)$ for all $x, y \in X$.

A p-seminorm P is called an p-norm if $x = 0$ whenever $P(x) = 0$. A topological vector space with a specific p-norm is called a p-normed space. Of course if $p = 1$, X is a usual normed space. By Lemma 3.2.5 of Balachandra [4], the following proposition gives a necessary and sufficient condition for a p-seminorm to be continuous.

Proposition 2.1. Let X be a topological vector space, P is a p-seminorm on X and $V := \{x \in X : P(x) < 1\}$. Then P is continuous if and only if the zero element $\theta \in int(V)$, where $int(V)$ is the interior of V.

Now given a p-seminorm P, the p-seminorm topology determined by P (in short, the p-topology) is the class of unions of open balls $B(x, \varepsilon) := \{y \in X : P(y-x) < \varepsilon\}$ for $x \in X$ and $\varepsilon > 0$.

We also need following notion for the so-called p-gauge (see Balachandra [4]).

Definition 2.6. Let A be an absorbing subset of a vector space X. For $x \in X$ and $p \in (0,1]$, set $P_A = \inf\{\alpha > 0 : x \in \alpha^{\frac{1}{p}} A\}$, then the non-negative real-valued function P_A is called the p-gauge (gauge if $p = 1$). The p-gauge of A is also known as the Minkowski p-functional.

By Proposition 4.1.10 of Balachandra [4], we have the following proposition.

Proposition 2.2. Let A be an absorbing subset of X and $p \in (0,1]$. Then p-gauge P_A has the following properties:
(i) $P_A(0) = 0$;
(ii) $P_A(\lambda x) = |\lambda|^p P_A(x)$ if $\lambda \geq 0$;
(iii) $P_A(\lambda x) = |\lambda|^p P_A(x)$ for all $\lambda \in R$ provided that A is balanced (circled);
(iv) $P_A(x+y) \leq P_A(x) + P_A(y)$ for all $x, y \in A$ provided that A is p-convex.
In particular, P_A is a p-seminorm if A is absolutely p-convex (and also absorbing).

Remark 2.2. It is worthwhile noting that a zero-neighborhood in a topological vector space being an absolutely θ-neighborhood is also absorbing (by Lemma 2.1.16 of Balachandran [4], or Proposition 2.2.3 of Jarchow [23]), this leads us to have the following definition for a topological vector space E being a topological p-vector space (in short, p-vector space) for $p \in (0,1]$ by using the concept of the Minkowski p-functional as given below.

Definition 2.7. A vector space X is said to be a topological p-vector space (in short, p-vector space) if the base of the origin in X is generated by a family of Minkowski p-functionals (p-gauges) (see Definition 2.6 above), where $p \in (0,1]$.

By incorporating Proposition 2.2, it seems that the following one is in a nature way to have the definition for locally p-convex spaces, where $p \in (0,1]$.

Definition 2.8. A topological p-vector space X is said to be locally p-convex if the origin in X has a fundamental set of absolutely p-convex θ-neighborhoods. This topology can be determined by p-seminorms which are defined in the obvious way (see pp.52 of Bayoumi [5], Jarchow [23] or Rolewicz [48]). When $p = 1$, a locally p-convex space X is reduced to be a usual locally convex space.

Remark 2.3. As highlighted by Kalton et al.[24], L^p spaces for $0 < p < 1$ are the prototypical examples of p-convex spaces. While they are not locally convex (their dual spaces may even be trivial, like $L^p[0,1]$), they are **locally p-convex**, providing a structured way to analyze non-locally convex linear metric spaces as shown by a few examples below.
Here are several classic examples of p-convex sets that are not convex, where $p \in (0,1)$.

Example 2.1. (The Unit Ball in $L^p(0,1)$). In the L^p space for $0 < p < 1$, the unit ball is defined by $B_p = \left\{ f \in L^p(0,1) : \int_0^1 |f(t)|^p \, dt \leq 1 \right\}$.

- **Properties**: This set is p-convex but *not* convex. Due to the reverse Jensen inequality, the convex hull of B_p is the entire space $L^p(0,1)$ in the sense of the only convex neighborhood of zero.
- **Non-local Convexity**: This demonstrates that the space has no non-trivial continuous linear functionals (i.e., its dual space is $\{0\}$).

Example 2.2 (The Unit Ball in the Sequence Space ℓ^p). For the sequence space ℓ^p with $0 < p < 1$, the unit ball is given by: $B_{\ell^p} = \left\{ (a_n) : \sum_{n=1}^{\infty} |a_n|^p \leq 1 \right\}$. In the two-dimensional case ($n = 2$), the boundary $\sqrt[p]{|x|^p + |y|^p} = 1$ curves inward (concave), forming a "star-shaped" geometry which is the hallmark of non-convexity in p-normed environments.

Example 2.3 (p-Convex Hull of Finite Points). Given a set of points $\{x_1,\dots,x_n\}$ in a p-normed space, their **p-convex combinations** are of the form:

$$\sum_{i=1}^{n}\alpha_i x_i, \quad \text{where } \sum_{i=1}^{n}\alpha_i^p = 1 \text{ and } \alpha_i \geq 0$$

The set of all such combinations forms the smallest p-convex set containing these points.

Example 2.4 (The Unit Ball in Hardy Spaces H^p). For $0<p<1$, the unit ball of the Hardy space H^p on the unit disk is:

$$B_{H^p} = \left\{ f\in H^p : \|f\|_p^p = \sup_{r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta \leq 1 \right\}$$

Similar to L^p, these balls are p-convex and represent a fundamental structure in the study of F-spaces that are not locally convex.

By Proposition 4.1.12 of Balachandra [4], we also have the following proposition.

Proposition 2.3. Let A be a subset of a vector space X, which is absolutely p-convex for $p\in(0,1]$ and absorbing. Then, we have:

(i) The p-gauge P_A is a p-seminorm such that if $B_1 := \{x\in X : P_A(x)<1\}$, and $\overline{B_1} = \{x\in X : P_A(x)\leq 1\}$. then $B_1\subset A\subset \overline{B_1}$; in particular, $kerP_A\subset A$, where $kerP_A := \{x\in X : P_A(x)=0\}$.

(ii) $A = B_1$ or $\overline{B_1}$ according as A is open or closed in the P_A-topology.

Remark 2.4. Let X be a topological vector space and let U be an open absolutely p-convex neighborhood of the origin, and let ε be any given positive number. If $y\in \varepsilon^{\frac{1}{p}}U$, then $y=\varepsilon^{\frac{1}{p}}u$ for some $u\in U$ and $P_U(y) = P_U(\varepsilon^{\frac{1}{p}}u) = \varepsilon P_U(u)\leq \varepsilon$ (as $u\in U$ implies that $P_U(u)\leq 1$). Thus, P_U is continuous at *zero*, and therefore, P_U is continuous everywhere. Moreover, we have $U = \{x\in X : P_U(x)<1\}$.

Indeed, since U is open and the scalar multiplication is continuous, we have that for any $x\in U$, there exists $0<t<1$ such that $x\in t^{\frac{1}{p}}U$ and so $P_U(x)\leq t<1$. This shows that $U\subset\{x\in X : P_U(x)<1\}$. The conclusion follows by Proposition 2.3 above.

The following result is a very important and useful result which allows us to make the approximation for convex subsets containing zero in topological vector spaces by p-convex subsets in locally p-convex spaces (see Lemma 2.1 of Ennassik and Taoudi [17], or Lemma 2.2.4 of Yuan [57]).

Lemma 2.1. Let A be a subset of a vector space X, then we have:

(i) If A is r-convex, with $r\in(0,1)$, then $\alpha x\in A$ for any $x\in A$ and any $\alpha\in(0,1]$;

(ii) If A is convex and $\theta\in A$, then A is s-convex for any $s\in(0,1]$;

(iii) If A is r-convex for some $r\in(0,1)$, then A is s-convex for any $s\in(0,r]$.

Proof. The conclusions follow by Lemma 2.1 of Ennassik and Taoudi [17]. But here, for the convenience of readers, we give a brief proof for (ii) as follows.

First if A is a convex subset of X with $\theta\in A$ and take a real number $s\in(0,1]$, we show that A is s-convex. Indeed, let $x_i\in A$, for $i=1,\cdots,n$. Now assume $\alpha_i>0$ with $\sum_{i=1}^n\alpha_i^s=1$. Since A is convex, then $\sum_{i=1}^n\frac{\alpha_i}{\sum_{j=1}^n\alpha_j}x_i\in A$. Keeping in mind that $0<\sum_{i=1}^n\alpha_i<\sum_{i=1}^n\alpha_i^s=1$, it follows that $\sum_{i=1}^n \alpha_i x_i = (\sum_{i=1}^n \alpha_i)(\sum_{i=1}^n \frac{\alpha_i}{\sum_{j=1}^n \alpha_j} x_i) + (1 - \sum_{i=1}^n \alpha_i) \cdot 0 \in A$. Thus we have $\sum_{i=1}^n \alpha_i x_i \in A$, this means A is s-convex.

Thus, this completes the proof. $\square$

Remark 2.5. We like to point out that the results (i) and (iii) of Lemma 2.1 do not hold for $p = 1$. Indeed, any singleton $\{x\} \subset X$ is convex in topological vector spaces; but if $x \neq \theta$, then it is not p-convex for any $p \in (0,1)$.

We also need the following Proposition 2.4 which is proposition 6.7.2 of Jarchow [23].

Proposition 2.4. Let K be a compact in a locally p-convex space X and $p \in (0,1]$. Then the closure $\overline{co_p K}$ of the p-convex hull, and the closure $\overline{aco_p K}$ of absolutely p-convex hull of K are compact if and only if $\overline{co_p K}$ and $\overline{aco_p K}$ are complete, respectively (thus they both are compact by the illustration given before the end of Section 1).

We recall that a set A in a vector space X is said to be convex if for any $x, y \in A$, $s, t \geq 0$, we have $(1-t)x + ty \in A$, whenever $t \in [0,1]$.

A seminorm P is called a norm if $x = 0$ whenever $P(x) = 0$. A vector space X with a specific norm (denoted by $\|\cdot\|$) is called a normed space, which is always denoted by $(X, \|\cdot\|)$ throughout this paper unless specified.

By following Pérez Carreras and Bonet [38, chapter 5] and Horváth [21] (see also Qiu and Rolewicz [47]), we have the following definition and notions.

Definition 2.9. Let X be a (real) locally p-convex Hausdorff space, where $p \in (0,1]$.

By following Pérez Carreras and Bonet [38, Chapter 3] and also Horváth [21], we call a bounded absolutely convex set B in X a disc, and denoted by "span[B]" the linear subspace spanned by B and denote p_B the Minkowski p-functional of B. Then $E_B := (\text{span[B]}, p_B)$ is a p-normed space. If E_B is a Banach space, then B is called a Banach disc (or, Banach disk used by some existing literature).

Definition 2.10. A locally p-convex space X is said to be sequently complete if every Cauchy sequence in X is convergent, where $p \in (0,1]$. A locally p-convex space X is said to be a locally complete p-convex space if every locally Cauchy sequence is locally convergent. Let $A \subset X$ be nonempty. Then A is said to be locally closed if for any locally convergent sequence in A, its local limit point belongs to A.

For more on the notions of locally complete p-convex spaces and its applications, see Qiu [42, 43, 44], Qiu and Rolewicz [47] and references wherein. Finally, we also recall the following Cantor intersection theorem which will be used in this paper.

Cantor Intersection Theorem. Let (X,d) be a complete metric space and let $\{D_n\}$ be a decreasing sequence (that is, $D_{n+1} \subset D_n$) of nonempty closed subsets of X such that the diameter of D_n for $n \in \mathbb{N}$, with $\lim_{n\to\infty} diam(D) = 0$. Then, the the intersection $\cap_{n=1}^{\infty} D_n$ contains exactly one point.

3. Nonlinear Principles related to Caristi Fixed Point Theorems in Complete Metric Spaces and locally Complete Convex Spaces

The goal of this section is first to discuss the famous and very important tool in nonlinear functional analysis called 'Caristi fixed point theorem' established by Caristi [8] in complete metric spaces and locally complete p-convex spaces, where $p \in (0,1]$. As applications, by following the idea given by Ansatri [2], Ansatri and Sahu [3], we derive a few (of its equivalent) nonlinear principles in nonlinear analysis and optimization by including Ekeland variational principle [14], Takahashi nonconvex minimization theorem [52] and Oettli-Théra theorem [33]. In addition, we also give the Leray-Schauder type alternative for Caristi type mappings in p-normed spaces, where $p \in (0,1]$. By following Goebel and Kirk [20], we recall that for a given complete metric space (M,d), we denote by $CB(X)$ the family of all non-empty closed bounded subsets of X. Then a mapping $T : M \to M$ is said to be a contraction mapping with the Lipschitz constant $k \in (0,1)$ if $d(T(x),T(y)) \leq kd(x,y)$ for each $x,y \in M$. If we define a mapping $\phi : M \to [0,+\infty)$ by $\phi(x) := (1-k)^{-1}d(x,Tx)$ for each $x \in M$, as T is k contractive, where $k \in (0,1)$, then $d(x,Tx) - kd(x,Tx) \leq d(x,Tx) - d(Tx,T^2x)$, then it follows $d(x,Tx) \leq \phi(x) - \phi(Tx)$ for each $x \in M$, and this implies that T has a unique fixed point in M as T is a contraction mapping with the Lipschitz constant $k \in (0,1)$. We know if the mapping ϕ above is only lower semicontinuous bounded below, for any mapping $T : M \to M$ which may not be a contraction mapping, but satisfies above inequality with ϕ, it then has at least one fixed point as shown by Caristi fixed point theorem established by Caristi [8] in 1976, for which we list below as Theorem 3.1 by applying a smart proof given by Du [12, 13] (see also related different proofs given by Almezel et al. [1], Ansari [2], Ansari and Sahu [3] for related discussion with more in details).

Theorem 3.1 (Caristi fixed point theorem for single-valued mappings). Let (X,d) be a complete metric space. Assume that $T : X \to X$ is a mapping and $\phi : X \to \mathbb{R}$ is a lower semicontinuous mapping which is bounded below such that, for any $x \in X$, $d(x,T(x)) \leq \phi(x) - \phi(T(x))$. Then T has a fixed point in X.

Proof. It is indeed Theorem 2.1$'$ of Caristi [8] (see different proof given by Siegel [50], Brézis and Browder [6], Goebel and Kik [20], Wong [53], and another direct proof given by Kozlowski [26]). But here, we provide a directly proof given by Du [12, 13] without using Zorn's lemma, transfinite induction, or any other well-known axiom as follows. We first define a set-valued mapping $F : X \to 2^X$ by $F(x) := \{y \in X : d(x,y) \leq \phi(x) - \phi(y)\}$ for each $x \in X$. Clearly, we have that $Tx \in F(x)$ and hence $F(x) \neq \emptyset$ for each $x \in X$.

We now claim that, for each $y \in F(x)$, $\phi(y) \leq \phi(x)$ and $F(y) \subset F(x)$. Let $y \in F(x)$ be any given point. Then $y \neq x$ and $d(x,y) \leq \phi(x) - \phi(y)$, thus $\phi(y) \leq \phi(x)$. As $F(y) \neq \emptyset$, let $z \in F(y)$. Then $d(y,z) \leq \phi(y) - \phi(x)$. It follows that $\phi(z) \leq \phi(y) \leq \phi(x)$, and $d(x,z) \leq d(x,y) + d(y,z) \leq \phi(x) - \phi(z)$. Thus $z \in F(x)$, therefore we prove that $F(y) \subset F(x)$ for each $y \in F(x)$. We now construct a sequence $\{x_n\}$ in X by induction, starting with any point $x_1 \in X$ as follows: If $x_n \in X$ is known for $n \in \mathbb{N}$, then we choose $x_{n+1} \in F(x_n)$ such that $\phi(x_{n+1}) \leq \inf_{z \in F(x_n)} \phi(z) + \frac{1}{n}$. For each $n \in \mathbb{N}$, since $x_{n+1} \in F(x_n)$, we obtain $d(x_n,x_{n+1}) \leq \phi(x_n) - \phi(x_{n+1})$. Thus $\phi(x_{n+1}) \leq \phi(x_n)$ for each $n \in \mathbb{N}$. Since ϕ is bounded below, the number $\lambda := \lim_{n \in \mathbb{N}} \phi(x_n)$ exists. Now for $m > n$, with $m,n \in \mathbb{N}$, by inequalities above, we obtain

$$d(x_n,d_m) \leq \Sigma_{i=n}^{m-1} d(x_i,x_{i+1}) \leq \phi(x_n) - \lambda.$$

Since $\lim_{n\to\infty}\phi(x_n)=\lambda$, we have $\sup_{n\to\infty}\{d(x_x,x_m):m>n\}=0$. Hence $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists some $v\in X$ such that $\lim_{n\to\infty}x_n=v$. As ϕ is lower semicontinuous, it follows that, for all $i\in\mathbb{N}$,

$$\phi(v)\le\liminf_{n\to\infty}\phi(x_n)=\inf_{n\in\mathbb{N}}\phi(x_n)\le\phi(x_i).$$

Now we claim that $\bigcap_{n=1}^{\infty}F(x_n)=\{v\}$. Indeed, for $m>n$ with $m,n\in\mathbb{N}$, by inequalities above we have $d(x_n,x_m)\le\Sigma_{i=n}^{m-1}d(x_i,x_{i+1})\le\phi(x_n)-\phi(v)$. Since $\lim_{m\to\infty}x_m=v$, the inequality above implies, for each $n\in\mathbb{N}$, $d(x_n,v)\le\phi(x_n)-\phi(v)$, which implies that $v\in\bigcap_{n=1}^{\infty}F(x_n)$ and thus $\bigcap_{n=1}^{\infty}F(x_n)\ne\emptyset$. Moreover, we can see that $F(v)\subset\bigcap_{n=1}^{\infty}F(x_n)$. Now, for any $c\in\bigcap_{n=1}^{\infty}F(x_n)$, we have

$$d(x_n,c)\le\phi(x_n)-\phi(c)\le\phi(x_n)-\inf_{z\in F(x_n)}\phi(z)\le\phi(x_n)-\phi(x_{n+1})+\frac{1}{n}.$$

for all $n,m\in\mathbb{N}$. Therefore, $\lim_{n\to\infty}d(x_n,c)=0$, or, equivalently to say, $\lim_{n\to\infty}x_n=v$. By the uniqueness of the limit of a convergent sequence, we have $c=v$. This means $\bigcap_{n=1}^{\infty}F(x_n)=\{v\}$.

On the other hand, since $F(v)\ne\emptyset$, and $F(v)\subset\bigcap_{n=1}^{\infty}F(x_n)=\{v\}$, we have $F(v)=\{v\}$. For the element $v\in X$, as $Tv\in F(v)$, it implies $v=Tv$, which is a fixed point of T. Therefore T must have a fixed point in X, and this completes the proof. □

Here, we like to remark that as mentioned above, a different proof without using transfinite induction skill has been given by Brézis and Browder [6], and also see Park [34, 35] gave a comprehensive review for the study of Caristi fixed point theorem and related metric fixed pint theory under the general framework of quasi-metric spaces. In addition, see also Feng and Liu [19], Lau and Yao [29], Zhou et al. [63] and references wherein for some recent results in some other new directions which are related to Caristi type fixed point theorem and related content, not covered by this paper.

Now as a consequence of Theorem 3.1, we have the following conclusion which is Proposition of Wong [53].

Theorem 3.2. Let (X,d) be a nonempty complete metric space. Let $T:X\to X$ be a single-valued mapping with non-empty values. Suppose there exists a lower semicontinuous function $\phi:X\to[0,+\infty)$ such that for any $x\in X$ with $x\ne Tx$, there exists $y\in X\setminus\{x\}$ satisfying

$$d(x,y)\le\phi(x)-\phi(y).$$

Then T has a fixed point.

Proof. Suppose T has not fixed point, then for each $x\in X$, $T(x)\ne x$, but there exists $y\in X$ with $y\ne x$ such that $d(x,y)\le\phi(x)-\phi(y)$. We now define a new mapping $T^*:X\to X$ with $T^*(x):=y$ for each $x\in X$, were y is a fixed in $X\setminus\{x\}$ satisfying $d(x,y)\le\phi(x)-\phi(y)$ as mentioned by the assumption. It is then clear that T^* is well defined, and we have $d(x,T^*(x))\le\phi(x)-\phi(T^*(x))$ for each $x\in X$. By Theorem 3.1, T^* has a fixed point $x^*\in X$ such that $T^*(x^*)=x^*$. But by the definition of T^*, $x^*\ne T^*(x^*)$, this is a contradiction. Thus T mush have a fixed point in X. This completes the proof. □

Now by following the idea used by Ansatri [2] and Ansari and Sahu [3], we show that how Caristi fixed point theorem plays a key role to its equivalent nonlinear principles in nonlinear analysis and optimization in complete metric spaces.

Theorem 3.3 (Caristi fixed point theorem for set-valued mappings). Let (X,d) be a nonempty complete metric space. Let $T : X \to 2^X$ be a set-valued mappings with non-empty values. Suppose there exists a lower semicontinuous function $\phi : X \to [0,+\infty)$ such that for any $x \in X$, there exists $y \in T(x)$ such that $d(x,y) \leq \phi(x) - \phi(y)$. Then T has a fixed point.

Proof. For each $x \in X$, we define a single-valued mapping $f : X \to X$ by $f(x) := y$, where choosing one $y \in X$ such that $y \in T(x)$ and $d(x,y) + \phi((y) \leq \phi(x)$. Then f is well defined and $f(x) \in T(x)$ for each $x \in X$, and we also have $d(x, f(x)) + \phi(f(x)) \leq \phi(x)$. Now by Theorem 3.1 for single-valued mapping $f : X \to X$, it follows that there exists $x \in X$ such that $x = f(x)$. By the definition of f, we have $x = f(x) \in T(x)$, thus x is a fixed point of the set-valued mapping T. This completes the proof. □

As an application of Caristi fixed point theorem, we can prove the following Nadler's fixed point for contraction set-valued mappings.

Theorem 3.4 (Nadler's fixed point theorem for contraction set-valued mappings). Let (X,d) be a complete metric space. Then, each contraction set-valued mapping $T : X \to CB(X)$ has a fixed point, where $CB(X)$ denotes the family of all non-empty closed bounded subsets of X.

Proof. Let $T : X \to CB(X)$ be a Nadler contraction set-valued mapping with contraction constant h. Choose real number $\alpha \in \mathbb{R}$ such that $h < \alpha < 1$. Let $x \in X$, then the set $\{y \in T(x) : \alpha d(x,y) \leq d(x,T(x))\} \neq \emptyset$. By the axiom of choice, there is a single-valued mapping $f : X \to X$ such that $f(x) \in T(x)$ for each $x \in X$. and $\alpha d(x, f(x)) \leq d(x,T(x))$. Thus, we have

$$d(f(x),T(f(x))) \leq d_H(T(x),T(f(X))) \leq h \cdot d(x, f(x)).$$

Note that

$$\begin{aligned} d(x, f(x)) &= \frac{1}{\alpha - h}(\alpha d(x, f(x)) - h \cdot d(x, f(x))) \\ &\leq \frac{1}{\alpha - h}(d(x,T(x)) - d(f(x),T(f(x)))). \end{aligned}$$

Now, we set $\phi(x) := \frac{1}{\alpha-h} d(x,T(x))$ for each $x \in X$. Then, ϕ is continuous from X to $[0,\infty)$ and $d(x, f(x)) \leq \phi(x) - \phi(f(x))$ for each $x \in X$. In view of Caristi fixed point theorem, there exists $x \in X$ such that $x = f(x)$, which is a fixed point of f, so $x = f(x) \in T(x)$ is a fixed point of T. This completes the proof. □

Now we prove Ekeland's variational principle by using Theorem 3.3, which is also called Caristi type (also called Caristi-Kirk Theorem) by Caristi and Kirk [9].

Theorem 3.5 (Strong form of Ekeland's variational principle). Let (X,d) be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, bounded below and lower semicontinuous functional. Let $\varepsilon > 0$ and $\hat{x} \in X$ be given such that $f(\hat{x}) \leq \inf_{x \in X} f(x) + \varepsilon$. Then, for a given $\lambda > 0$, there exists $\bar{x} \in X$ such that

(1) $f(\bar{x}) \leq f(\hat{x})$;
(2) $d(\hat{x},\bar{x}) \leq \lambda$; and
(3) $f(\bar{x}) < f(x) + \frac{\varepsilon}{\lambda} d(x,\bar{x})$, for all $x \in X \setminus \{\bar{x}\}$.

Proof. Without loss of generality, we may assume that $\lambda = 1$. Let $\varepsilon > 0$ be given and choose $\hat{x} \in X$ such that $f(\hat{x}) \leq \inf_{x\in X} f(x) + \varepsilon$. Now let $X' = \{x \in X : f(x) \leq f(\hat{x}) - \varepsilon d(\hat{x},x)\}$. Then, X' is nonempty. By lower semicontinuity of f, it follows that X' is closed subset of a complete metric space, and hence, a complete metric space. For each $x \in X'$,

$$S(x) := \{y \in X : y \neq x, f(y) \leq f(x) - \varepsilon d(x,y)\},$$

for each $x \in X$. Now we define a set-valued mapping $T : X \to 2^X$ by

$$T(x) = \begin{cases} x, & \text{if } S(x) = \emptyset, \\ S(x), & \text{if } S(x) \neq \emptyset; \end{cases} \tag{3.1}$$

Then, T is a set-valued mapping from X' to itself with nonempty values. Indeed, $T(x) = x \in X'$ if $S(x) = \emptyset$. Since $T(x) = S(x)$ if not, we have, for all $y \in T(x)$,

$$\varepsilon d(\hat{x},y) \leq \varepsilon d(\hat{x},x) + \varepsilon(x,y) \leq f(\hat{x}) - f(x) + f(x) - f(y) = f(\hat{x}) - f(y),$$

and hence $y \in X'$. Moreover, for all $x \in X$ and $y \in T(x)$, we have $\frac{1}{\varepsilon} f(y) + d(x,y) \leq \frac{1}{\varepsilon} f(x)$. From Theorem 3.3, we see that T has a fixed point $\bar{x} \in X'$. Consequently, $S(\bar{x}) = \emptyset$, that is, $f(x) > f(\bar{x}) - \varepsilon d(\bar{x},x)$ for all $x \in X$ with $x \neq \bar{x}$. Since $\bar{x} \in X'$, we have $f(\bar{x}) \leq f(\hat{x}) - \varepsilon(\hat{x},\bar{x}) \leq f(\hat{x})$. Further, we have $\varepsilon d(\hat{x},\bar{x}) \leq f(\hat{x}) - f(\bar{x}) \leq f(\hat{x}) - \inf_{x\in X} f(x) \leq \varepsilon$. Hence, $d(\hat{x},\bar{x}) \leq 1$. This completes the proof. $\square$

Now as an application of Theorem 3.5, we have the following Takahashi minimization theorem established by Takahashi in [51, 52].

Theorem 3.6 (Takahashi minimization theorem). Let (X,d) a complete metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper, bounded below and lower semicontinuous functional. Suppose that, for each $\hat{x} \in X$ with $\inf_{x\in X} f(x) < f(\hat{x})$, there exists $z \in X$ such that $z \neq \hat{x}$ and $f(z) + d(\hat{x},z) \leq f(\hat{x})$. Then, there exists $\bar{x} \in X$ such that $f(\bar{x}) = \inf_{x\in X} f(x)$.

Proof. We prove the conclusion by using the Strong form of Ekeland's variational principle as a tool. By Theorem 3.5 above, for any given $\varepsilon > 0$, there exists $\bar{x} \in X$ such that $f(\bar{x}) < f(x) + \varepsilon d(x,\bar{x})$ for all $x \in X$ with $x \neq \bar{x}$. We now claim that $f(\bar{x}) = \inf_{x\in X} f(x)$ as follows.

Assume to the contrary that there exists $w \in X$ such that $f(w) > \inf_{x\in X} f(x)$. Now by the hypothesis, there exists $z \in X$ such that $z \neq w$ and $f(z) + d(w,z) \leq f(w)$ which contradicts inequality above. Hence, we have some $\bar{x} \in X$ such that $f(\bar{x}) = \inf_{x\in X} f(x)$. This completes the proof. $\square$

In what follows, we also give the following theorem which characterizes the completeness of the underlying metric space which is as an application of Takahashi minimization theorem due to Takahashi [51].

Theorem 3.7. A metric space (X,d) is complete if, for every uniformly continuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ and every $\hat{x} \in X$ with $\inf_{x\in X} f < f(\hat{x})$, there exists $z \in X$ such that $z \neq \hat{x}$ and $f(z) + d(\hat{x},z) \leq f(\hat{x})$, there exists $\bar{x} \in X$ such that $f(\bar{x}) = \inf_{x\in X} f(x)$.

Proof. Let $\{x_n\}$ be a Cauchy sequence in X. Consider the function $f : X \to \mathbb{R} \cup \{+\infty\}$ defined by $f(x) := \lim_{n\to\infty} d(x_n,x)$ for each $x \in X$. The f is uniformly continuous and $\inf_{x\in X} f(x) = 0$.

Let $f(\hat{x}) = 0$. Then, there exists an $x_m \in X$ such that

$$x_m \neq \hat{x}, \quad f(x_m) < \frac{1}{3} f(\hat{x}) \quad \text{and} \quad d(x_m, \hat{x}) - f(\hat{x}) < f(\hat{x}).$$

Thus $3f(x_m) + d(x_m, \hat{x}) < f(\hat{x}) + 2f(\hat{x}) = 3f(\hat{x})$. Therefore, there exists an $\bar{x} \in X$ such that $f(\bar{x}) = \inf_{x \in X} f(x) = 0$, and so, $0 = f(\bar{x}) = \lim_{n\to\infty} d(x_n, \bar{x})$. Thus, $\{x_n\}$ converges to $\bar{x}$, and hence, X is complete. This completes the proof. □

Here we like to refer interested readers to related study for the characterization of complete metrics given by Cobzaş [11], Jachymski et al. [22], Park [34], Park and Rhoades [36] and related references therein and related comprehensive review on the discussion of various circumstances in which fixed point results imply completeness for metric spaces by involving Ekeland variational principle and of its equivalent, Caristi fixed point theorem; and other fixed point results having this property in metric spaces, including quasi-metric spaces and in partial metric spaces; and in addition for the discussion on topology and order and on fixed points in ordered structures and their completeness properties.

By using Cantor intersection theorem in complete metric spaces as a tool, we have following so-called extended version for Ekeland's variational principle which is equivalent to the Caristi type fixed point theorem [9] as shown below.

Theorem 3.8 (Extended Ekeland's variational principle). Let K be a nonempty closed subset of a complete metric space (X,d) and $F : K \times K \to \mathbb{R}$ be a bifunction mapping. Assume that $\varepsilon > 0$ and the following assumptions are satisfied:

(i) For all $x \in K$, the set $L := \{y \in K : F(x,y) + \varepsilon d(x,y) \leq 0\}$ is closed;

(ii) $F(x,x) = 0$ for all $x \in K$;

(iii) $F(x,y) \leq F(x,z) + F(z,y)$ for all $x,y,z \in K$.

Now if $\inf_{y\in K} F(x_0,y) > -\infty$ for some $x_0 \in K$, then there exists $\bar{x} \in K$ such that

(a) $F(x_0,\bar{x}) + \varepsilon d(x_0,\bar{x}) \leq 0$; and

(b) $F(\bar{x}.x) + \varepsilon d(\bar{x},x) > 0$ for all $x \in K$ with $x \neq \bar{x}$.

Proof. For the sake of convenience, we set $d_\varepsilon(u,v) := \varepsilon d(u,v)$. Then d_ε is equivalent to d and (X,d_ε) is complete. For each $x \in X$, we define a set $S(x) := \{y \in K : F(x,y) + d_\varepsilon(x,y) \leq 0\}$. By condition (i), $S(x)$ is closed for each $x \in K$. From condition (ii), $x \in S(x)$ for all $x \in K$, thus $S(x)$ is nonempty for each $x \in X$. So, there exists $y \in S(x)$, that is, $F(x,y) + d_\varepsilon(x,y) \leq 0$, and, also there exists $z \in S(y)$, such that $F(y,z) + d_\varepsilon(z,y) \leq 0$. By adding inequalities above together, and using condition (iii), we obtain $0 \geq F(x,y) + d_\varepsilon(x,y) + F(y,z) + d_\varepsilon(z,y) \geq F(x,z) + d_\varepsilon(x,z)$. Therefore, $z \in S(x)$, which implies that $S(y) \subset S(x)$. Now we show that there is a sequence $\{x_n\}$ of K such that for each $n \in \mathbb{N}$,

$$x_{n+1} \in S(x_n), F(x_n, x_{n+1}) < \inf_{z \in S(x_n)} F(x_n, z) + \frac{1}{n+1}.$$

Let $\nu(x_0) := \inf_{z\in S(x_0)} F(x_0,z) > -\infty$, and construct a sequence in the following manner:

For x_1, let $x_1 \in S(x_0)$ such that $F(x_0,x_1) < \nu(x_0) + 1$. Since $x_1 \in S(x_0)$, we have $S(x_1) \subset S(x_0)$. By condition (iii), we have $\nu(x_1) = \inf_{z\in S(x_1)} F(x_1,z) \geq \inf_{z\in S(x_1)} F(x_0,z) - F(x_0,x_1) \geq \nu(x_0) - F(x_0,x_1) > -\infty$. Then, there exists $x_2 \in S(x_1)$ such that $F(x_1,x_2) \leq \nu(x_1) + \frac{1}{2}$. Continuing in this way, we obtain a sequence $\{x_n\}$ such that for each $n \in \mathbb{N}$,

$$x_{n+1} \in S(x_n), F(x_n, x_{n+1}) < \nu(x_n) + \frac{1}{n+1},$$

and $\nu(x_{n+1}) \geq \nu(x_n) - F(x_n, x_{n+1})$, which implies that

$$-\nu(x_n) \leq -F(x_n, x_{n+1}) + \frac{1}{n+1} \leq \nu(x_{n+1}) - \nu(x_n) + \frac{1}{n+1}.$$

Thus, for $n \in \mathbb{N}$,

$$\nu(x_{n+1}) + \frac{1}{n+1} \geq 0. \tag{3.2}$$

If $z_1, z_2 \in S(x_n)$, then $d(z_1, z_2) \leq d(x_n, z_1) + d(x_n, z_2) \leq -F(x_n, z_1) - F(x_n, z_2) \leq -2\nu(x_n)$. Thus it implies that the diameter of $S(x_n)$, denoted by $diam(S(x_n))$, $diam(S(x_n)) \leq -2\nu(x_n)$. Thus, by the inequality (2) above, we have $\lim_{n\to\infty} diam(S(x_n)) = 0$. By the fact that $\{S(x_n)\}$ is a family of closed sets such that $S(x_{n+1}) \subset S(x_n)$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} diam(S(x_n)) = 0$, now by Cantor intersection theorem in complete metric spaces, there exists exactly one point $\bar{x} \in X$ such that $\bigcap_{n=1}^{\infty} S(x_n) = \{\bar{x}\}$. This implies that $\bar{x} \in S(x_0)$, that is, $F(x_0, \bar{x}) + d_\varepsilon(x_0, \bar{x}) \leq 0$, which shows that the conclusion (a) holds.

Secondly, $\bar{x} \in \cap_{n=1}^{\infty} S(x_n)$ and, since $S(\bar{x}) \subset S(x_n)$ for all $n \in \mathbb{N}$, we thus have $S(\bar{x}) = \{\bar{x}\}$. It then follows that any $x \notin S(\bar{x})$ whenever $x \neq \bar{x}$ implies $F(\bar{x}, x) + d_\varepsilon(\bar{x}, x) > 0$, that is, the conclusion (b) holds. This completes the proof. □

As mentioned above, the following result shows the equivalences among extended Ekelend's variational principle, extended Takahashi's minimization theorem, Caristi-Kirk fixed point theorem for set-valued mappings, and Oettli-Théra theorem in complete metric spaces, which is Theorem 3.13 of Ansari[2] and Theorem 5.16 of Ansariand Sahu [3], thus we omit its proof here by saving the space.

Theorem 3.9. Let K be a nonempty closed subset of a complete metric space (X, d) and $F : K \times K \to \mathbb{R}$ be a (bifunction) mapping such that it is lower semicontinuous in the second argument and the following conditions hold:

(i) $F(x, x) = 0$ for all $x \in X$; and

(ii) $F(x, y) \leq F(x, z) + F(z, y)$ for all $x, y, z \in X$.

Assume that there exists $\hat{x} \in X$ such that $\inf_{x\in X} F(\hat{x}, x) > -\infty$. Let

$$\hat{S} := \{x \in X : F(\hat{x}, x) + d(\hat{x}, x) \leq 0. \tag{3.3}$$

Then (by (i) it follows that $\hat{x} \in \hat{S} \neq \emptyset$), the following statements are equivalent:

(a) (Extended Ekeland's variational principle): There exists $\bar{x} \in \hat{S}$ such that

$$F(\bar{x}, x) + d(\bar{x}, x) > 0, \text{ for all } x \neq \bar{x}. \tag{3.4}$$

(b) (Extended Takahashi's minimization theorem): Assume that

$$\begin{cases} \text{for every } \hat{x} \in \hat{S} \text{ with } \inf_{x\in X} F(\hat{x}, x) < 0 \text{ there exist} \\ x \in X \text{ such that } F(\hat{x}, x) + d(\hat{x}, x) \leq 0 \text{ for all } x \neq \hat{x} \end{cases} \tag{3.5}$$

Then, there exists $\bar{x} \in \hat{S}$ such that $F(\bar{x}, x) \geq 0$ for all $x \in X$.

(c) (Caristi-Kirk fixed point theorem): Let $T : X \to 2^X$ be a set-valued mapping such that

$$\begin{cases} \text{for every } \hat{x} \in S \text{ there exists} \\ x \in T(\hat{x}) \text{ satisfying } F(\hat{x},x) + d(\hat{x},x) \leq 0. \end{cases} \tag{3.6}$$

Then, there exists $\bar{x} \in S$ such that $\bar{x} \in T(\bar{x})$.

(d) (Oettli-Théra theorem): Let $D \subset X$ have the property that

$$\begin{cases} \text{for every } \hat{x} \in \hat{S} \setminus D \text{ there exists} \\ x \in X \text{ such that } F(\hat{x},x) + d(\hat{x},x) \leq 0, \text{ for all } x \neq \hat{x}, \end{cases} \tag{3.7}$$

Then, there exists $\bar{x} \in \hat{S} \cap D$.

Theorem 3.9 shows that Caristi type fixed point theorem would be a fundamental tool for nonlinear analysis and optimization. Here, before ending this section, we will show that Ekeland variational principle holds in locally complete convex spaces as discuss below. But first we need recall some facts and definition used for locally complete convex spaces.

We know that every (sequentially) complete locally convex space is locally complete and every (sequentially) closed set is locally closed, but none of the converses are true, for further details, we refer to Horváth [21], Pérez Carreras and Bonet [38], and Qiu [40, 41, 42, 43, 44, 45] for discussion in details. By using the notion of locally closed sets, we also introduce the following class of functions, which is wider than the class of (sequentially) lower semicontinuous functions.

Definition 3.1. Let X be a locally p-convex space, where $p \in (0,1]$. Let $f : X \to (-\infty,+\infty]$ be a proper function (i.e., the set $dom f := \{x \in X : f(x) \neq +\infty\} \neq \emptyset$). Then f is said to be locally lower semicontinuous if for each $r \in \mathbb{R}$, the set $\{x \in X : f(x) \leq r\}$ is locally closed in X.

We now recall the following result which is Corollary 3.1 of Qiu [42], and stated it as Lemma 3.1 below under the framework of locally p-convex spaces, where $p \in (0,1]$.

Lemma 3.1. Let X be a locally complete p-convex space, where $p \in (0,1]$. Assume $\{p_\lambda\}_{\lambda \in I}$ is a family of seminorms defining the topology on the locally p-convex space X, and $\{\alpha_\lambda\}_{\lambda \in I}$ a family of positive real numbers. Let $f : X \to (-\infty,+\infty]$ be a locally lower semicontinuous, bounded from below, proper function and let $\{\alpha_\lambda\}_{\lambda \in I}$ be a family of positive real numbers. Then we have that for any given $x_0 \in dom f$, there exists $z \in X$ such that for all $\lambda \in I$,

(i) $f(z) + \alpha_\lambda p_\lambda(z - x_0) \leq f(x_0)$, and

(ii) for any $x \neq z$, there exists $\mu \in I$ such that $f(z) < f(x) + \alpha_\mu p_\mu(x - z)$.

Proof. It is Theorem 1.1 of Qiu [41] (which is indeed Corollary 3.1 of Qiu [42]). The proof is complete. □.

As an application of Lemma 3.1, we have the following Caristi fixed point theorem in locally complete p-convex spaces for a single-valued mapping in locally p-convex space, where $p \in (0,1]$.

Theorem 3.10. Let X be a locally complete p-convex space, where $p \in (0,1]$, $\{p_\lambda\}_{\lambda \in I}$ be a family of seminorms defining the topology on X, and $\{\alpha_\lambda\}_{\lambda \in I}$ a family of positive real numbers. Let $f : X \to (-\infty,+\infty]$ be a locally lower semicontinuous, bounded from below, proper function and let $\{\alpha_\lambda\}_{\lambda\in I}$ be a family of positive real numbers. Assume $T : X \to X$ is a single-valued mapping with non-empty values, and for each $x \in X$ such that for all $\lambda \in I$, we have

$$\alpha_\lambda p_\lambda(x-Tx)+f(Tx) \leq f(x),$$

then, there exists $z \in domf \subset X$ such that $T(z) = z$, i.e., T has a fixed point $z \in X$.

Proof. Taking any $x_0 \in domf$, by Lemma 3.1, we see that there exists $z \in X$ such that, for any given $\lambda \in I$, $f(z)+\alpha_\lambda p_\lambda(z-x_0) \leq f(x_0)$, and for any $x \in X$ with $x \neq z$, $f(z) < f(x)+\sup_{\lambda\in I}\alpha_\lambda p_\lambda(x-z)$. Thus $z \in domf$. We now prove that z is a fixed point of T. If not, $T(z) \neq z$, then $f(z) < f(T(z))+\sup_{\lambda\in I}\alpha_{\lambda\in I} p_\lambda(T(z)-z)$. Thus there exists $\mu \in I$ such that $f(z) < f(T(z))+\alpha_\mu p_\mu(T(z)-z)$. Now by the assumption on mapping T, we have

$$f(z) < f(T(z))+\alpha_\mu p_\mu(T(z)-z) \leq f(z),$$

which is a contradiction, thus we must have $T(z) = z$. This completes the proof. □.

We now have the following general Caristi type fixed point theorem which shows that the conclusion of Theorem 3.10 is true for mapping T is a set-valued mapping in locally complete convex spaces but with weaker conditions on the mapping T by comparing with Theorem 5.A of Qiu [41] (which is Corollary 3.3 of Qiu [42], and see also Theorem 4.1, Theorem 4.2 and Theorem 4.3 of Qiu [40]).

Theorem 3.11. Let X be a locally complete p-convex space, where $p \in (0,1]$, $\{p_\lambda\}_{\lambda\in I}$ be a family of seminorms defining the topology on X, and $\{\alpha_\lambda\}_{\lambda\in I}$ a family of positive real numbers. Let $f : X \to (-\infty,+\infty]$ be a locally lower semicontinuous, bounded from below, proper function; and let $\{\alpha_\lambda\}_{\lambda\in I}$ be a family of positive real numbers. Assume that $T : X \to 2^X$ is a set-valued mapping with non-empty values such that, for each given $\lambda \in I$, any $x \in X$, there exists $y \in T(x)$, $\alpha_\lambda p_\lambda(x-y)+f(y) \leq f(x)$. Then there exists $z \in domf \subset X$ such that $z \in T(z)$, i.e., T has a fixed point $z \in X$.

Proof. For each $x \in X$, we define a single-valued mappingf $T_1 : X \to X$ by $T_1(x) := \{y\}$, where y is an element in $T(x)$ such that, for all $\lambda \in I$, $\alpha_\lambda p_\lambda(x-y)+f(y) \leq f(x)$. By the assumption on mapping T, we have $T_1(x) \neq \emptyset$, well-defined, and $T_1(x) \subset T(x)$ for all $x \in X$. In addition, for each $x \in X$, for all $\lambda \in I$, $\alpha_\lambda p_\lambda(x-T_1x)+f(T_1x) \leq f(x)$. Now applying Theorem 3.10 to mapping T_1, we see that T_1 has one fixed point $z \in X$ such that $z = T_1(z) \subset T(z)$, which means z is a fixed point of T. This completes the proof. □

Before the ending of this section, we consider some nonlinear alternative for Caristi-type mapping in complete p-normed spaces, where $p \in (0,1]$, by following the idea presented in Precup [39].

Let X be a p-convex space, where $p \in (0,1]$. In general, we recall that, for a subset Y of X, a retract of X through the retraction $P : X \to Y$, $P|_Y = id_Y$. We now have the nonlinear alternative for Caristi-type mappings in p-normed space, where $p \in (0,1]$.

Theorem 3.12. Let $(X, \|\cdot\|_p)$ be a complete p-normed space, where $p \in (0,1]$. Assume that $Y \subset X$ and $P : X \to Y$ is a retraction; and $\phi : X \to \mathbb{R}^+$ is lower semicontinuous and $T : Y \to X$ is a mapping such that the following conditions are satisfied (here the metric d defined by $d(x,y) := \|x-y\|_p$ for each $x,y \in X$):

(1) $d(x,T(x)) \leq \phi(x)-\phi(T(x))$ for all $x \in Y$;

(2) $d(x,TP(x)) \leq (x) - \phi(TP(x))$ for all $x \in X \setminus Y$; and
(3) $TP(x) \neq x$ for all $x \in X \setminus Y$ (i.e., the Leray - Schauder-type condition).
Then T has a fixed point $x \in X$ such that $x = T(x)$.

Proof. By assumption (1) and (2), it is clear that Caristi's theorem (i.e., Theorem 3.11 above) applies to the mapping: $TP : X \to X$ defined by $TP(x) = T(P(x))$ for each $x \in X$. Thus, there exists $x \in X$ with $x = TP(x)$ by Theorem 3.11 above. Now by the condition (3), we must have $x \in Y$, which implies that $x \in Y$. From fact that $P(x) = x$, where P is a retraction of X to Y, we see that $x = T(x)$. This completes the proof. □

As an application of Theorem 3.12, we have the following nonlinear alternative for Caristi-type mappings in p-normed spaces, where $p \in (0,1]$.

Theorem 3.13 (Nonlinear Leray-Schauder type alternative). Let $(X, \|\cdot\|_p)$ be a complete p-normed space, where $p \in (0,1]$. Assume that $Y \subset X$, and $P : X \to Y$ is a retraction, and $\phi : X \to \mathbb{R}^+$ is lower semicontinuous and $T : Y \to X$ is a mapping satisfying the following condition (here the metric d defined by $d(x,y) := \|x-y\|_p$ for each $x,y \in X$):
(1) $d(x,T(x)) \leq \phi(x) - \phi(T(x))$ for all $x \in Y$;
Then at least one of the following statements holds:
(a) there exists $x \in Y$ that $x = T(x)$;
(b) there exists $x \in X \setminus Y$ such that $x = TP(x))$;
(c) there exists $x \in X \setminus Y$ such that $d(x,TP(x)) > \phi(x) - \phi(TP(x))$.

Proof. The conclusions follow by Theorem 3.12. This completes the proof. □

Remark 3.12. We first notes that Theorem 3.11 shows that: 1) Theorem 3.5 and Theorem 3.6 of Ansari[2] hold in locally complete p-convex spaces; 2) the general Caristi type fixed point theorem holds in locally complete p-convex spaces; 3) Theorem 3.11 shows the corresponding results for Caristi type fixed point theorem hold by weakening the condition on the set-valued mappings established by Qiu [40]-[43] and references wherein; 4) in particular, Theorem 3.11 improves Theorem 5.A of Qiu [41], which is Corollary 3.3 of Qiu [42].

4. Fixed Point Theorems of Non-self $\mathfrak{F}$ Contraction and Nonexpansive Set-valued Mappings in Locally p-Convex Spaces

The aim of this section is to establish fixed point theorems for non-self contraction and non-expansive set-valued mappings under the general locally complete p-convex spaces by using Caristi fixed point theorem [8] and its generalization given in last section as a tool, where $p \in (0,1]$. In this section we focus on the study of general fixed point theorems for non-self contraction and nonexpansive set-valued mappings under the framework of locally complete convex spaces, which indeed unify or improve corresponding results in the existing literature.

For a given space E, we denote by $K(E)$ the family of all nonempty compact subsets E. For a given Hausdroff locally p-convex space, where $p \in (0,1]$, we denote by $\mathfrak{F}$ the family of all continuous seminorms generating the topology of the locally p-convex space E. For each $p \in \mathfrak{F}$ and $A,B \in K(E)$, we recall the metric $\delta(A,B)$ between subsets A and B by $\delta(A,B) := \sup\{p(a-b) : a \in A, b \in B\}$, and the Hausdorff metric $d_{Hp}(A,B)$ by

$$d_{Hp}(A,B) := \max\{\sup_{a \in A} \inf_{b \in B} p(a-b), \sup_{b \in B} \inf_{a \in A} p(a-b)\}.$$

It is clear that when $p \in \mathfrak{F}$ is the only seminorn in E (thus E is a normed space), d_{Hp} is a traditional Hausdroff metric on $K(E)$ (see, e.g., Ko and Tsai [25]). Secondly, by the definition of Hausdorff metric d_{Hp} for each $p \in \mathfrak{F}$, we know that if $A, B \in K(E)$, then each $a \in A$, there exists $b \in B$ such that $p(a-b) \leq d_{Hp}(A,B)$ (e.g., by the definition of Hausdorff metric and the compactness of subsets, as the same idea used by Nadler [32]).

Let E denote a Hausdroff locally p-convex topological vector space, where $p \in (0,1]$. We denote by $\mathfrak{F}$ the family of continuous scminorms generating the topology of E, and let K be a nonempty subset of E. We recall that a mapping $T : K \to K(E)$ is said to be a $\mathfrak{F}$ **contraction** set-valued mapping if for each $p \in \mathfrak{F}$, there exists a constant $k_p \in (0,1)$ such that $d_{Hp}(T(x), T(y)) \leq k_p p(x-y)$. We note that when $k_p = 1$ for all $p \in \mathfrak{F}$, T is a non-expansive set-valued mapping if, for any $x, y \in K$, $d_{Hp}(T(x), T(y))) \leq p(x-y)$. In addition, by following the idea presented in [10], we have the following definition for a given locally p-convex space E (where $p \in (0,1]$), which is said to satisfy the "***P*-Opial condition**" if, for each $x \in E$ and every net (x_α) converging weakly to x and for each $P \in \mathfrak{F}$, the following inequality holds for any $y \neq x$, $\liminf P(x_\alpha - y) > \liminf P(x_\alpha - x)$. Here we note that the notion of "***P*-Opial condition**" (denoted by the upper case letter P) was originally introduced by Chen and Singh [10] in 1992 given for the locally convex spaces E which is a case of locally p-convex spaces when $p = 1$. Let K be a nonempty bounded closed convex subset of a normed space $(E, \|\cdot\|)$. We say that K has the fixed point property (FPP) for nonexpansive mapping T if for every nonexpansive mapping $T : K \to K$ (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$), T has a fixed point $x \in K$ such that $Tx = x$. The space E is said to have fixed point property (FPP) if any nonempty bounded closed convex subset of E has the fixed point property for nonexpansive mapping; the space E has the weak fixed point property (WFPP) if any weakly compact convex subset of E has the fixed point property for nonexpansive mapping. For a reflexive Banach space, both properties are obviously the same. The famous question whether a Banach space has the fixed point property (WFPP) had remained open for a long time. It is now well known that not all nonexpansive mappings have the FPP or WFPP, and in order to obtain positive results in the problem for the existence of fixed points for nonexpansive mappings, it is necessary to impose some restrictions either on mapping T or on the underlying space E. Now it is time for us to establish fixed point theorems for non-self contraction, and non-self expansive set-valued mappings by using Caristi fixed point theorem [8] and its generalization above as tools under the framework of locally complete p-convex spaces, where $p \in (0,1]$. Throughout this section, all locally p-convex spaces are assumed to be locally complete unless specified,where $p \in (0,1]$.

Definition 4.1. Let M be a subset of a given locally convex convex space E. Let $T : M \to K(E)$ be a set-valued mapping with non-empty values, where $K(E)$ is the family of all nonempty compact subsets of the space E. Then we say that

(1) T satisfies the boundary condition (α) if, for each $x \in M$, any $y \in T(x)$, $(x,y] \cap M \neq \emptyset$, where, $(x,y] := \{(1-\lambda)x + \lambda y : \lambda \in (0,1]\}$ (and also, we define $[x,y] := \{(1-\lambda)x + \lambda y : \lambda \in [0,1]\}$); and

(2) T is called weakly inward if, for each $x \in M$, $T(x) \subset \overline{I_M(x)}$, where

$$I_M(x) := \{z \in E : x = x + \lambda(y - x), y \in M, \text{ and } \lambda \in [1, +\infty)\}.$$

Now we recall the following result which is the famous Caristi fixed point theorem in the setting of Hausdorff locally complete convex spaces, which is actually a special case of Theorem

3.10 above in locally complete spaces (see also [8, 7, 12, 13, 18, 28, 40, 41, 42, 43, 44, 45, 46, 61, 62, 19, 29]).

Lemma 4.1. Let E be a locally complete p-convex space, where $p \in (0,1]$. Assume that $T : E \to E$ is any arbitrary (single-valued) mapping with non-empty values, and suppose that there exists a lower semicontinuous function $f : E \to [0,+\infty)$, such that, for each for each given $p \in \mathfrak{F}$, and for any $x \in E$, $p(x - T(x)) \leq f(x) - f(T(x))$. Then T has a fixed point.

Proof. This is a special case of Theorem 3.10e. The proof is complete. □

Here, we also point out that Lemma 4.1 actually is Theorem 2.1 of Latif et al. [28], which is special form of Theorem 3.2 given by Cammaroto et al. [7] with the original result of Theorem 3.1 established by Fang [18]. In addition, the general form of Lemma 4.1 is also given by Theorem 4.1 or Theorem 4.2 of Qiu [40], for which we state its another special form as Lemma 4.2 below which includes the above Lemma 4.1 as its special case.

We now have the following result which is a generalization of Lemma 4.1. It is also a variant of Lemma 1.2 given by Zhong et al. [62].

Lemma 4.2. Let E be a locally complete convex space. Assume that $f : E \to [0,+\infty)$ is a bounded below lower semicontinuous function and $h : [0,+\infty)$ a nonnegative continuous non-decreasing function such that $\int_0^{+\infty} \frac{dr}{1+h(r)} = +\infty$. Let $T : E \to E$ be a (single-valued) mapping with non-empty values such that, for any given $x_0 \in E$ and $p \in \mathfrak{F}$, for any $x \in E$, $\frac{p(x-f(x))}{1+h(p(x_0-x))} \leq f(x) - f(T(x))$. Then T has a fixed point.

Proof. Actually, this is a special case of Theorem 4.1 of Qiu [40]. Indeed, assume $\psi(s) := \frac{1}{1+h(s)}$, and $\phi(s) := s$ for each $s \in [0,+\infty)$ in Theorem 4.1 of Qiu [40]. Then, for each given $p_\lambda \in \mathfrak{F}$, with properties of h and f defined by Lemma 4.2 above, it follows that $\phi_\lambda(p_\lambda(x - T(x))) = p_\lambda(x - T(x))$. Let $\psi_\lambda(f(x)) := \psi_\lambda(p_\lambda(x_0 - x))$ for each $x \in E$. Then, by the definitions of ψ and ϕ, we have

$$\psi_\lambda(f(x)) = \psi_\lambda(p_\lambda(x_0 - x)) = \frac{1}{1 + h(p_\lambda(x_0 - x))} \quad \text{for each } x \in X.$$

It is clear that all hypotheses of Theorem 4.1 of Qiu [40] are satisfied,. Thus T has a fixed point by Theorem 4.1 of Qiu [40]. This completee the proof. □

Here we also point out that Lemma 4.2 is actually Theorem 2.2 of Latif et al.[28] (which is actually a special case of Theorem 4.1 or Theorem 4.2 by Qiu [40]). Moreover, it is a version of Lemma 1.2 given by Zhong et al. [62] under the framework of locally complete convex spaces which can be proved by using the same idea used for Theorem 2.1 by Zhong [61]. In addition, the corresponding results on Ekeland principle and Caristi fixed point theorm in locally p-convex spaces for $p \in (0,1]$ have been also established by Qiu and Rolewicz [46] (but we omit their discussion here for saving spaces).

Now by Lemma 4.1, we first have the following fixed point theorem for non-self $\mathfrak{F}$ contract set-valued mapping in locally complete convex spaces which is a special case of complete locally p-convex spaces when $p = 1$.

Theorem 4.3. Let M be a nonempty closed subset of locally complete convex space E and $T : M \to K(E)$ be a $\mathfrak{F}$ contraction set-valued mapping satisfying the boundary condition (α). Then T has a fixed point.

Proof. Let $p \in \mathfrak{F}$ be any given one seminorm in E. For each $x \in M$, as $T(x)$ is compact, we choose $y \in T(x)$ such that $p(x-y) = d_p(x, T(x))(:= \inf\{p(x-z) : z \in T(x)\})$. We denote by z_{xp} as a farthest point from x in $[x,y] \cap M$, which means $p(x-z_{xp}) = \max\{p(x-w) : w \in [x,y] \cap M \neq \emptyset\}$. Then it is easy to verify by using the definition of z_{xp} and the seminorm p's triangle inequality property that $p(x-y) = p(x-z_{xp}) + p(z_{xp}-y)$. Since T is a $\mathfrak{F}$ contraction set-valued mapping with non-empty compact values, it follows that, for each $p \in \mathfrak{F}$, there exists $k_p \in (0,1)$ such that

$$\begin{aligned} d_p(z_{xp}, T(z_{xp})) &\leq p(z_{xp}-y) + d_p(y, T(z_{xp})) \leq p(z_{xp}-y) + d_p(T(x), T(z_{xp})) \\ &\leq p(z_{xp}-y) + k_p p(x-z_{xp}) = d_p(x, T(x)) - (1-k_p)p(x-z_{xp}). \end{aligned}$$

Thus $p(x-z_{xp}) \leq (1-k_p)^{-1}\{d_p(x,T(x)) - d_p(z_{xp}, T(z_{xp}))\}$. Now we define the mapping $f : M \to M$ by $f_p(x) := z_{xp}$ for each $x \in M$; and also define a non-negative function $\phi_p : M \to [0, +\infty)$ by $\phi(x) := (1-k_p)^{-1} d_p(x, T(x))$ for each $x \in M$. From the definitions of f and ϕ, one sees that $p(x - f_p(x)) \leq \phi_p(x) - \phi_p(f_p(x))$, for each $x \in M$. By a fact that M is a closed subset of a complete space E, it is complete and hence by Lemma 4.1 that f has a fixed point $u \in M$ such that $u = f_p(u)$. Note that $f_p(u) = u = z_{up}$. By the definition of z_{up}, it is the farthest point from u in $[u,y] \cap M$, which means $p(u - z_{up}) = \max\{p(u-w) : w \in [u,y] \cap M \neq \emptyset\}$, which help us to prove $d(u, T(u)) = 0$. This implies that u is a fixed point of T, From the boundary condition (α) for the $\mathfrak{F}$ mapping T, we have $(u,y] \cap M \neq \emptyset$. Without loss of the generality, there exists a non-zero positive number $\lambda_0 \in (0,1]$ such that $w_0 = (1-\lambda_0)u + \lambda_0 y$, and $w_0 \in (u,y] \cap M \neq \emptyset$. By the definition of z_{up}, we have $p(u - z_{up}) = \max\{p(u-w) : w \in [u,y] \cap M \neq \emptyset\}$. It follows that $p(u - z_{up}) \geq p(u - w_0) = \lambda_0 p(u-y) = \lambda_0 d(u, T(u))$ (as given in the beginning of the proof above). Since $u = z_{up}$, it follows that $\lambda_0 d(u, T(u)) = p(u - z_{up}) = p(z_{up} - z_{up}) = 0$, which implies that $d(u, T(u)) = 0$ as $\lambda_0 > 0$. This means u is a fixed point of T. The proof is complete. $\square$

We note that Theorem 4.3 extends Theorem 2 of Massa [30] (see also Massa et al. [31]) for non-self $\mathfrak{F}$ contraction set-valued mappings in Hausdorff locally convex spaces. As an application of Lemma 4.1, we have the following fixed point result for non-self $\mathfrak{F}$ contraction set-valued mappings with different boundary conditions.

Theorem 4.4. Let M be a nonempty closed subset of locally complete convex space E, and $T : M \to K(E)$ be a $\mathfrak{F}$ contraction set-valued mapping such that, for each given $p \in \mathfrak{F}$, for any $x \in M$, $\{z \in T(x) : p(x-z) = d_p(x, T(x))\} \cap \overline{I_M(x)} \neq \emptyset$ (also called "intersection boundary condition"). Then T has a fixed in M.

Proof. Suppose that T has no fixed point. Then, for any given $p \in \mathfrak{F}$, for any $x \in M$, we have $d_p(x, T(x)) > 0$. As T is a $\mathfrak{F}$ contraction set-valued mapping, there exists $k_p \in (0,1)$ such that $d_{Hp}(T(x), T(y)) \leq k_p p(x-y)$. Now choose a constant $q \in (0,1)$ and denoted by $k := k_p < \frac{1-q}{1+q}$, where $k_p \in (0,1)$. In addition, by the assumption on the boundary condition of $\mathfrak{F}$ mapping F,

$$\{z \in T(x) : p(x-z) = d_p(x, T(x))\} \cap \overline{I_M(x)} \neq \emptyset.$$

Since $T(x)$ is also non-empty compact for $x \in M$, there exists $z \in T(x) \cap \overline{I_M(x)}$ such that $d_p(x, T(x)) = p(x-z) > 0$. In addition, by the definition of $I_M(x)$ and $z \in T(x) \cap \overline{I_M(x)}$, it follows that we can choose some $t \in (0,1)$ (e.g., see [8]) such that $t^{-1} d_p(1-t)x + tz, M) < qp(x-z)$. Let $w := (1-t) + tz$, then there exists some $y \in M$ such that

$$p(w-y) < qtp(x-z) = qp((1-t)x + tz - x) = qp(w-x).$$

Since $p(y-x) \leq p(y-w) + p(w-x)$, it follows that $p(y-x) - p(w-x) \leq p(w-y) < qp(w-x)$, which implies $p(y-x) < (1+q)p(w-x)$. Therefore, $(q-1)p(w-x) < \frac{q-1}{q+1} p(x-y)$.

On the other hand, as $w = (1-t) + tz$, where $t \in (0,1)$, it is easy to see $p(w-x) + p(w-z) = p(x-z)$. Since $T(x)$ is a $\mathfrak{F}$ contraction set-valued mapping with non-empty compact values, by the definition of the Hausdorff metric d_{Hp}, we can choose $u \in T(x)$ and $v \in T(y)$ such that $p(w-u) = d_p(w, T(x))$ and $p(u-v) \leq d_{Hp}(T(x), T(y)) \leq k_p p(x-y)$. Thus

$$\begin{aligned} d_p(y, T(y)) &\leq p(y-v) \leq p(y-w) + p(w-u) + p(u-v) \\ &\leq qp(w-x) + d_p(w, T(x)) + kp(x-y) \\ &< qp(w-x) + p(x-z) - p(w-x) + kp(x-y) \\ &< d_p(x, T(x)) - \left(\frac{1-q}{1+q} - k \right) p(x-y). \end{aligned}$$

Putting $c = \frac{1-q}{1+q} - k$, we have $d_p(y, T(y)) < d_p(x, T(x)) - cp(x-y)$. Therefore

$$p(x-y) < \frac{d_p(x, T(x))}{c} - \frac{d_p(y, T(y))}{c}.$$

Now we define two mappings $f : M \to M$, and $\phi : M \to [0, +\infty)$ by $f(x) := y$, and $\phi(x) := \frac{d_p(x, T(x))}{c}$ for each $x \in M$, respectively. It is clearly that $p(x - f(x)) < \phi(x) - \phi(f(x))$. It follows from Lemma 4.1 that f has a fixed point $x_0 \in M$. Thus $f(x_0) = x_0$. On the other hand, we have

$$0 = p(x_0 - f(x_0)) < \phi(x_0) - \phi(f(x_0)) = \phi(x_0) - \phi(x_0) = 0,$$

which is impossible. Hence, T must have a fixed point. This completes the proof. □

As a special case of Theorem 4.4, we have the following result.

Corollary 4.5. Let M be a closed subset of locally complete convex space E and let $T : M \to K(E)$ be a weakly inward $\mathfrak{F}$ contraction mapping, i.e., $T(x) \subset \overline{I_M(x)}$ for each $x \in M$. Then T has a fixed point.

Remark 4.6. We point out that Theorem 4.4 extends Theorem 3.3 gievn by Zhang [60]. Secondly, it is worth to mention that the intersection condition of Theorem 4.4 can not be replaced the condition $T(x) \cap \overline{I_M(x)} \neq \emptyset$ for each $x \in M$ even in the setting of Banach spaces as shown by Zhang [60] and Xu [54].

Next, by applying Lemma 4.2 above and combining a contractive condition basically due to Zhong [61], we have the following fixed point theorem for non-self $\mathfrak{F}$ contract weekly inward set-valued mappings in the setting of locally convex topological vector spaces.

Here we assume that $h : [0, +\infty) \to [0, +\infty)$ is a continuous nondecreasing function satisfying: $\int_0^{+\infty} \frac{dr}{1+h(r)} = +\infty$. Now we have the following fixed point theorem for non-self contract set-valued mappings for Spaces satisfying p-Opial condition.

Theorem 4.7. Let M be a closed subset of a locally complete convex space E, and let $T : M \to K(E)$ be a weakly inward mapping, for a given $x_0 \in M$ and a given constant number $\sigma \in (0,1]$. For each given $p \in \mathfrak{F}$,

$$d_{Hp}(T(x),T(y)) \leq \left(1 - \frac{\sigma}{1+h(p(x_0-x))}\right) p(x-y),$$

for any $x, y \in M$. Then T has a fixed point.

Proof. Suppose that T has no fixed point. Then, $d_p(x,T(x)) > 0$ for all $x \in M$. By the assumption that h is a continuous nondecreasing function satisfying: $\int_0^{+\infty} \frac{dr}{1+h(r)} = +\infty$, we can choose $c \in (0,\sigma)$, where $\sigma \in (0,1]$, and let

$$q(x) := \frac{\sigma - c}{2(1+h(p(x_0-x)))}$$

such that $q(x) < 1$. Since T is with non-empty values, there exists $z \in T(x)$ such that $d_p(x,T(x)) = p(x-z) > 0$. Note that $T(x) \subset \overline{I_M(x)}$ for $x \in M$. Thus there exist $y \in M$ and $\lambda \geq 1$ such that $P(z-z') = p(z-(x+\lambda(y-x))) < q(x)p(x-z)$, where $z' := x+\lambda(y-x)$. Let $t := \frac{1}{\lambda}$ and $w := (1-t)x+tz$. Note that

$$w-y = (1-t)x+tz-y = t(z-(x+\frac{1}{t}(y-x)) = t(z-z').$$

It follows that $p(w-y) = t(p(z-(x+\lambda(y-x))) = tp(z-z')$. Thus $p(w-y) = tp(z-z') < tq(x)p(x-z)$, $p(w-x) = tp(x-z)$, and $p(w-z) = (1-t)p(x-z)$.

Secondly, as $w = (1-t)x+tz$, it is easy to verify that $p(w-x)+p(w-z) = p(x-z)$. As $p(y-x) \leq p(y-w)+p(w-x)$, this implies that $p(y-x)-p(w-x) \leq p(w-y)$. Thus $p(x-y) < p(w-x)+tq(x)p(x-z) = (1+q(x))p(w-x)$. Therefore, $(q(x)-1)p(w-x) < \frac{q(x)-1}{q(x)+1}p(x-y)$, where $q(x) \in (0,1)$. Since T is with compact values, by the definition of Hausdorf metric d_{Hp}, we can choose $u \in T(x)$ and $v \in T(y)$ such that $p(w-u) = d_p(w,T(x))$ and $p(u-v) \leq d_{Hp}(T(x),T(y))$. Thus

$$\begin{aligned}
d_p(y,T(y)) &\leq p(y-v \leq p(y-w)+p(w-u)+p(u-v) \\
&< (q(x)-1)p(w-x)+p(x-z)+d_{Hp}(T(x),T(y)) \\
&< \left(\frac{q(x)-1}{q(x)+1}+1\right)p(x-y)+p(x-z)-\frac{\sigma}{1+h(p(x_0-x))p(x_0-x))}p(x-y) \\
&< 2\left(\frac{\sigma-c}{2(1+h(p(x_0-x)))}\right)p(x-y)+p(x-z)-\frac{\sigma}{1+h(p(x_0-x))}p(x-y) \\
&< d_p(x,T(x))-\frac{c}{1+h(p(x_0-x))}p(x-y).
\end{aligned}$$

Thus it follows

$$\frac{p(x-y)}{1+h(p(x_0-x))} < \frac{d_p(x,T(x))}{c} - \frac{d_p(y,T(y))}{c}.$$

Now we define two mappings for $f : M \to M$ and $\phi : M \to [0,+\infty)$ by $f(x) = y$ and $\phi(x) = \frac{d_p(x,T(x))}{c}$ for each $x \in M$. Note that ϕ is continuous and non-negative, and

$$\frac{p(x-f(x))}{1+h(p(x_0-x))} < \phi(x)-\phi(f(x)).$$

Lemma 4.2 yields that f has a fixed point. But, above inequality shows that $0 < 0$, which is impossible. This contradiction shows that T must have a fixed point. The proof is complete. □

Remark 4.8. Note that Theorem 4.7 extends Theorem 2.5 by Zhong et al.[62] to Hausdorff locally convex spaces.

As an application of Theorem 4.6, we have the following fixed point theorem for non-self expansive set-valued mappings with boundary condition (α) in locally complete convex spaces which satisfy the P-Opial condition introduced by Chen and Singh [10] in 1992.

Theorem 4.9. Let M be a non-empty weakly compact convex subset of a locally complete convex space E satisfying P-Opial condition. Let $T : M \to K(E)$ be a non-self nonexpasive set-valed mappping with the boundary condition (α) (and thus $T(M)$ is bounded). Then mapping $(I-T)$ is demiclosed, i.e., the set $(I-T)(M)$ is closed in $E_w \times E$, where E_w is E with its weak topology and I is the identify mapping; and T has at least one fixed point.

Proof. Since E satisfies the P-Opial condition, we see that if $x_n \to x$ weekly, and $y \neq x$, then $\limsup \|x_n - x\| < \limsup \|x_n - y\|$. By Theorem 1 of Chen and Singh [10], we have that $(I-T)(M)$ is closed in $E_w \times E$, where E_w is E with its weak topology and I is the identify mapping. This fact helps us to prove that T has a fixed point. In order to do so, we now show that there exist a sequence $\{x_n\} \subset M$ and a sequence $\{y_n\} \subset E$ with $y_n \in T(x_n)$ such that $\lim_{n\to\infty}(x_n - y_n) = 0$. Then $0 \in (I-T)(M)$. Now let q be a given point in M. For each $n \in \mathbb{N}$, we define a set-valued mapping $T_n : M \to K(E)$ by $T_n(x) = (1-\frac{1}{n})Tx + \frac{1}{n}q$, for each $x \in M$. Then it is clear that $T_n(x) \subset E$ and T_n satisfies the boundary condition (α). As each T_n is a $\mathfrak{F}$ contraction set-valued mapping for $n \in \mathbb{N}$, by Theorem 4.3, there exists $x_n \in M$ such that $x_n \in T_n(x_n)$. This means that there exists $y_n \in T(x_n)$ such that $x_n - y_n = \frac{1}{n}q - \frac{1}{n}y_n$ for $n \in \mathbb{N}$. As $T(M)$ is bounded, one sees that $\lim_{n\to\infty}(x_n - y_n) = 0$. Now by Theorem 1 of Chen and Singh [10], one concludes that $(I-T)(M)$ is closed in $E_w \times E$. Then it follows that $0 \in (I-T)K$. This means that there exists $x \in K$ such that $x \in T(x)$, that is, T has at least one fixed point in M. This completes the proof. □

Now we have the following fixed point result for non-self nonexpansive mappings which are weekly inward in locally convex spaces.

Theorem 4.10. Let M be a nonempty weakly compact convex subset of a localy complete convex space E, and let $T : M \to K(E)$ being a nonexpansive and weakly inward mapping. If $(I-T)$ is demiclosed, then T has a fixed point.

Proof. Leq q be a point in M. For each $n \in \mathbb{N}$, we define a set-valued mapping $T_n : M \to K(X)$ by $T_n(x) = (1-\frac{1}{n})Tx + \frac{1}{n}q$, for each $x \in M$. Then $T_n(x) \subset E$ and T_n is also weekly inward. As each T_n is a $\mathfrak{F}$ contraction set-valued mapping for $n \in \mathbb{N}$, by Corollary 4.5, there exists $x_n \in M$ such that $x_n \in T_n(x_n)$. This means that there exists $y_n \in T(x_n)$ such that $x_n = (1-\frac{1}{n})y_n + \frac{1}{n}q$ for $n \in \mathbb{N}$, which means $(x_n - y_n) = \frac{1}{n-1}(q - x_n)$. As M is weakly compact, $\{q - x_n\}$ is also bounderd. Without loss of the generality, assume that x_n converges weekly to z in M. Now, for any $p \in \mathfrak{F}$, $\lim_{n\to+\infty} p(x_n - y_n) = \lim_{n\to+\infty} \frac{1}{n-1} p(q - x_n) = 0$. By the assumption that $(I-T)$ is demiclosed, one hast $0 \in (I-T)z$, which means $z \in T(z)$. This completes the proof. □

As an application of Theorem 4.10, we have the following general fixed point theorem for nonexpansive set-valued mappings which are weekly inward in locally complete convex spaces which satisfy P-Opial comdition, see more results by Yanagi [55] and related references wherein.

Theorem 4.11. Let M be a nonempty weakly compact convex subset of a localy complete convex space E which satisfies the P-Opial condition. Let $T : M \to K(X)$ be a nonexpansive and weakly inward mapping. Then T has a fixed point.

Proof. As E satisfies the P-Opial condition and T is nonexpansive, one sees that $(I-T)$ is demiclosed. Then the conclusion follows by Theorem 4.10, and this completes the proof. $\square$.

5. CONCLUSIONS

In this paper, key principles in nonlinear analysis and optimization are derived as applications of Caristi fixed point theorem in complete metric spaces and locally complete p-convex convex spaces, where $p \in (0,1]$. Fixed point theorems of non-self contraction and nonexpansive set-valued mappings under the general locally complete convex spaces based on Caristi fixed point theorem and and its generalization were given in this paper.

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