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FIXED POINT RESULTS ON GENERALIZED CONTRACTIVE MAPPINGS AND THEIR APPLICATIONS IN DYNAMIC PROGRAMMING

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Abstract. In this paper, some innovative concepts regarding γ-FG-contractive conditions are introduced by using T-cyclic (α,β)-admissible mappings in metric-like spaces. In complete metric-like spaces, fixed point theorems are established for the newly introduced mappings. The current findings are also helpful to obtain fixed point theorems for cyclic mappings. Further, the consequences of the above results are described by several corollaries. An application of these results is show-cased through the existence of a theorem for solutions to a specific system of functional equations appearing in dynamic programming. Moreover, relevant examples are presented to validate and support the current investigations.

Keywords. γ-FG-contractive; Metric-like space; T-cyclic (α,β)-admissible; Point of coincidence; Fixed point theorems.

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1. INTRODUCTION

In applied mathematics as well as in other domains, fixed-point theorems are essential for developing methods to find the solutions of different problems. This has led to a number of researchers focusing on this particular subject matter [1, 2, 3, 5, 6, 7, 8, 9, 10, 14, 15, 16, 18, 19, 22, 23, 24, 25]. The Banach contraction mapping principle has often been considered as the foundation of metric fixed point theory. It stands as a cornerstone in the field. Samet *et al.* [24] introduced several extensions of the Banach contraction principle. Wardowski [26] developed the concept of F-contraction mappings, which expanded the scope of fixed point theory for such mappings. Wardowski and Dung [27], Parvateesam *et al.* [21] as well as Piri and Kumam [22] established various fixed point and common fixed point theorems. They also summarized the ideas of F-contractions. The concept of metric-like (dislocated) spaces was introduced by Seda and Hitzler [11], that serves as a generalization of metric spaces. Recently Kumar *et al.* [17] presented some new ideas about $\gamma-FG$-contractive mappings of T-cyclic (α,β)-type. Further,

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Sastry [25] developed the concept of TAC- contractions and established some results on it in complete $b-$metric like spaces.

Building upon these research works, this study aims to introduce the γ-FG-contractive conditions for T-cyclic (α,β)-type mappings. For this class of mappings, corresponding fixed point theorems are established in complete metric like spaces. Further, these results are applied to the functionals involved in dynamic programming. In addition, suitable examples are given to verify the new findings.

2. Preliminaries

Throughout this paper, $\mathbb{R}_0^+$ and $\mathbb{N}$ are denoted as the set of non-negative real numbers and positive integers, respectively.

Definition 2.1. [4] Let $\mathscr{M} \neq \phi$. For all $a,b,c \in \mathscr{M}$, a mapping $d_{\mathscr{M}} : \mathscr{M} \times \mathscr{M} \to \mathbb{R}_0^+$ is defined by:

(σ_1) $d_{\mathscr{M}}(a,b) = 0$ implies $a = b$,
(σ_2) $d_{\mathscr{M}}(a,b) = d_{\mathscr{M}}(b,a)$,
(σ_3) $d_{\mathscr{M}}(a,c) \leq d_{\mathscr{M}}(a,b) + d_{\mathscr{M}}(b,c)$.

Then $(\mathscr{M}, d_{\mathscr{M}})$ is known as a metric-like space.

Example 2.2. Let $\mathscr{M} = \{a,b,c\}$, and let a mapping $d_{\mathscr{M}} : \mathscr{M} \times \mathscr{M} \to \mathbb{R}_0^+$ be define by

$$d_{\mathscr{M}}(a,b) = d_{\mathscr{M}}(b,a) = \frac{7}{5} \qquad d_{\mathscr{M}}(b,c) = d_{\mathscr{M}}(c,b) = \frac{3}{2},$$

$$d_{\mathscr{M}}(a,c) = d_{\mathscr{M}}(c,a) = \frac{1}{3} \qquad d_{\mathscr{M}}(a,a) = 0, \qquad d_{\mathscr{M}}(b,b) = 2, \qquad d_{\mathscr{M}}(c,c) = \frac{1}{2}.$$

Then $d_{\mathscr{M}}$ is a metric-like, but not a metric, as $d_{\mathscr{M}}(b,b) \neq 0$ and also $d_{\mathscr{M}}(c,c) \neq 0$.

Example 2.3. Let $\mathscr{M} = \{1,2,3\}$, and define the mapping $d_{\mathscr{M}} : \mathscr{M} \times \mathscr{M} \to \mathbb{R}_0^+$ by

$$d_{\mathscr{M}}(1,1) = 1, \qquad d_{\mathscr{M}}(2,2) = 0, \qquad d_{\mathscr{M}}(3,3) = \frac{1}{3}, \qquad d_{\mathscr{M}}(1,2) = d_{\mathscr{M}}(2,1) = \frac{9}{10},$$

$$d_{\mathscr{M}}(3,1) = d_{\mathscr{M}}(1,3) = \frac{7}{10}, \qquad d_{\mathscr{M}}(3,2) = d_{\mathscr{M}}(2,3) = \frac{4}{5}.$$

Then $d_{\mathscr{M}}$ is a metric like, but not a metric as $d_{\mathscr{M}}(1,1) \neq 0$, and also it is not a partial metric as $d_{\mathscr{M}}(1,1) > d_{\mathscr{M}}(2,1)$.

A topology τ_δ is generated by every metric-like $d_{\mathscr{M}}$ on $\mathscr{M}$ that base is the set of all open $d_{\mathscr{M}}$-balls $\{B_{d_{\mathscr{M}}}(a,\rho) : a \in \mathscr{M}, \rho > 0\}$.

Definition 2.4. [4] Let $\{a_n\}$ be any sequence in $\mathscr{M}$ and $a \in \mathscr{M}$. Consider a metric-like space $(\mathscr{M}, d_{\mathscr{M}})$ with the following:

(i) $\{a_n\}$ is convergent to a w.r.t. $\tau_{d_{\mathscr{M}}}$ iff $\lim_{n\to\infty} d_{\mathscr{M}}(a_n,a) = d_{\mathscr{M}}(a,a)$.
(ii) $\{a_n\}$ is a Cauchy sequence in $(\mathscr{M}, d_{\mathscr{M}})$ if $\lim_{n,m\to\infty} d_{\mathscr{M}}(a_n,a_m)$ exists (it is finite).
(iii) $(\mathscr{M}, d_{\mathscr{M}})$ is complete if every Cauchy sequence $\{a_n\}$ in $\mathscr{M}$ converges to a point $a \in \mathscr{M}$ such that

$$\lim_{n,m\to\infty} d_{\mathscr{M}}(a_n,a_m) = \lim_{n\to\infty} d_{\mathscr{M}}(a_n,a) = d_{\mathscr{M}}(a,a). \tag{2.1}$$

(iv) A function $g:(\mathscr{M},d_{\mathscr{M}})\to(\mathscr{M},d_{\mathscr{M}})$ is continuous if, for any sequence $\{a_n\}$ in $\mathscr{M}$ such that $d_{\mathscr{M}}(a_n,a)\to d_{\mathscr{M}}(a,a)$ as $n\to\infty$, $d_{\mathscr{M}}(ga_n,ga)\to d_{\mathscr{M}}(ga,ga)$ as $n\to\infty$.

Lemma 2.5. [13] *Let $(\mathscr{M},d_{\mathscr{M}})$ be a metric-like space. Let $\{a_n\}$ be a sequence in $\mathscr{M}$ like $a_n\to a$, where $a\in\mathscr{M}$ and $d_{\mathscr{M}}(a,a)=0$. Then, for all $b\in\mathscr{M}$, $\lim_{n\to\infty}d_{\mathscr{M}}(a_n,b)=d_{\mathscr{M}}(a,b)$.*

The idea about T-cyclic (α,β)-admissible mapping was given by Isik *et al.* [12] and defined below.

Definition 2.6. Consider the functions $g,T:\mathscr{M}\to\mathscr{M}$ and $\alpha,\beta:\mathscr{M}\to\mathbb{R}_0^+$. A mapping g is said to be T-cyclic (α,β)-admissible if, for $a\in\mathscr{M},\alpha(Ta)\geq 1\ \Rightarrow\ \beta(ga)\geq 1$ and $\beta(Ta)\geq 1\ \Rightarrow\ \alpha(ga)\geq 1$.

3. Main Results

Here, some new concepts are introduced on γ-FG- contractive mapping of T-cyclic (α,β)-type. Again, several theorems on fixed point are to be introduced with metric-like space. Let us consider the following symbols presented in Parvaneh *et al.* [20].

$\Delta_{\mathscr{F}}$ represent the set of every mapping $\mathscr{F}:[0,\infty)\to(\infty,\infty)$ such that:

(Δ_1) for every sequence $\{s_n\}\subseteq[0,\infty)$, $\lim_{n\to\infty}s_n=0$ iff $\lim_{n\to\infty}\mathscr{F}(s_n)=-\infty$.

(Δ_2) $\mathscr{F}$ is continuous and strictly non-decreasing;

$\Delta_{G,\gamma}$ represent the set of all pairs (G,γ), where $\gamma:[0,\infty)\to[0,1)$ and $G:[0,\infty)\to(-\infty,\infty)$ such that:

(Δ_3) for every sequence $\{s_n\}\subseteq\mathbb{R}_0^+$, $\limsup\lim_{n\to\infty}\gamma(s_n)=1\Rightarrow\lim\limits_{n\to\infty}s_n=0$;

(Δ_4) for every sequence $\{s_n\}\subseteq\mathbb{R}_0^+$, $\sum_{n=1}^{\infty}G(\gamma(s_n))=-\infty$;

(Δ_5) for every sequence $\{s_n\}\subseteq[0,\infty)$, $\limsup\lim_{n\to\infty}G(s_n)\geq 0$ if and only if $\limsup\lim_{n\to\infty}s_n\geq 1$;

Definition 3.1. Let $(\mathscr{M},d_{\mathscr{M}})$ be a metric-like space. Let $\alpha,\beta:\mathscr{M}\times\mathscr{M}\to[0,\infty)$ and $f:\mathscr{M}\to\mathscr{M}$ be mapping. Then f is said to be γ-fG- contractive of T-cyclic (α,β)-type if there exists $(G,\gamma)\in\Delta_{G,\gamma}$ and $\mathscr{F}\in\Delta_{\mathscr{F}}$ such that, for all $a,b\in\mathscr{M}$,

$$\begin{aligned}&\alpha(Ta)\beta(Tb)\geq 1,d_{\mathscr{M}}(fa,fb)>0\\&\Rightarrow\alpha(Ta)\beta(Tb)\mathscr{F}(d_{\mathscr{M}}(fa,fb))\leq\mathscr{F}(\tilde{\mathscr{M}}(a,b))+G(\gamma(\tilde{\mathscr{M}}(a,b))),\end{aligned}\tag{3.1}$$

where

$$\tilde{\mathscr{M}}(a,b)=\max\left\{\begin{array}{c}d_{\mathscr{M}}(Ta,fa),d_{\mathscr{M}}(Tb,fb),d_{\mathscr{M}}(Ta,Tb),\\ \dfrac{d_{\mathscr{M}}(Tb,fa)+d_{\mathscr{M}}(Ta,fb)}{2},\\ \dfrac{d_{\mathscr{M}}(Tb,fb)[1+d_{\mathscr{M}}(Ta,fa)]}{1+d_{\mathscr{M}}(Ta,Tb)}\end{array}\right\}.\tag{3.2}$$

Theorem 3.2. Let $(\mathscr{M},d_{\mathscr{M}})$ be a complete metric-like space. Let $T,f:\mathscr{M}\to\mathscr{M}$ such that $f\mathscr{M}\subset T\mathscr{M}$, $\alpha,\beta:\mathscr{M}\to[0,\infty)$. If a closed subset $T\mathscr{M}$ of $\mathscr{M}$ which holds all the following assertions:

(i) f is continuous;
(ii) $\alpha(Tu)\geq 1$, $\beta(Tv)\geq 1$, whenever $Tu=fu$, $Tv=fv$;
(iii) f is a γ-FG-contractive mapping of T-cyclic (α,β)-type.

(iv) if a sequence $\{a_n\}$ in $\mathscr{M}$ satisfies $a_n \to a$ and for every n, $\beta(a_n) \geq 1$ then $\beta(a) \geq 1$;
(v) there exists $a_0 \in \mathscr{M}$ such that $\alpha(Ta_0) \geq 1$ and $\beta(Ta_0) \geq 1$,

then T and f have a unique point of co-incidence in $\mathscr{M}$. Moreover, f and T have a unique common fixed point, if f and T are weakly compatible.

Proof. Case-I. Take $a_0 \in \mathscr{M}$ such that $1 \leq \alpha(Ta_0)$, $1 \leq \beta(Ta_0)$. The sequences $\{a_n\}$ and $\{b_n\}$ in $\mathscr{M}$ are defined by

$$a_n = fa_n = Ta_{n+1},\ n \in \mathbb{N}. \tag{3.3}$$

If $a_n = a_{n+1}$, then a_{n+1} is the point of co-incidence of T and f. Assume that $a_n \neq a_{n+1}$, for all $n \in \mathbb{N}$. Since f is a γ-FG-contractive mapping of T-cyclic (α,β)-type, $\alpha(Ta_0) \geq 1$, then $\beta(fa_0) = \beta(Ta_0) \geq 1 \Rightarrow \alpha(Ta_2) = \alpha(fa_1) \geq 1$. In similar manner, we have $1 \leq \alpha(Ta_{2n})$ and $1 \leq \beta(Ta_{2n+1})$, for all $n \in \mathbb{N}$. Equivalently, as f is a T-cyclic (α,β)-type admissible mapping and $1 \leq \beta(Ta_0)$, we get $1 \leq \beta(Ta_{2n})$ and $1 \leq \alpha(Ta_{2n+1})$, $\forall n \in \mathbb{N}$, that is, $1 \leq \alpha(Ta_n)$ and $1 \leq \beta(Ta_n)$ $\forall n \in \mathbb{N}$. Similarly, $\alpha(Ta_n)\beta(Ta_{n+1}) \geq 1$, for all $n \in \mathbb{N}$. Therefore, by (3.1) and (3.3), we have

$$\begin{aligned}\mathscr{F}(d_{\mathscr{M}}(fa_n,fa_{n+1})) &\leq \alpha(Ta_n)\beta(Ta_{n+1})\mathscr{F}(d_{\mathscr{M}}(fa_n,fa_{n+1}))\\ &\leq \mathscr{F}(\tilde{\mathscr{M}}(a_n,a_{n+1})) + G(\gamma(\tilde{\mathscr{M}}(a_n,a_{n+1}))).\end{aligned} \tag{3.4}$$

Now, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\tilde{\mathscr{M}}(a_n,a_{n+1}) &= \max\left\{\begin{array}{c} d_{\mathscr{M}}(Ta_n,Ta_{n+1}), d_{\mathscr{M}}(Ta_n,fa_n), d_{\mathscr{M}}(Ta_{n+1},fa_{n+1}),\\ \dfrac{d_{\mathscr{M}}(Ta_n,fa_{n+1}) + d_{\mathscr{M}}(Ta_{n+1},fa_n)}{2},\\ \dfrac{d_{\mathscr{M}}(Ta_{n+1},fa_{n+1})[1+d_{\mathscr{M}}(Ta_n,fa_n)]}{d_{\mathscr{M}}(Ta_n,Ta_{n+1})+1}\end{array}\right\}\\ &= \max\left\{\begin{array}{c} d_{\mathscr{M}}(a_{n-1},a_n), d_{\mathscr{M}}(a_{n-1},a_n), d_{\mathscr{M}}(a_n,a_{n+1}),\\ \dfrac{d_{\mathscr{M}}(a_{n-1},a_{n+1}) + d_{\mathscr{M}}(a_n,a_n)}{2},\\ \dfrac{d_{\mathscr{M}}(a_n,a_{n+1})[1+d_{\mathscr{M}}(a_{n-1},a_n)]}{d_{\mathscr{M}}(a_{n-1},a_n)+1}\end{array}\right\}\\ &\leq \max\{d_{\mathscr{M}}(a_{n-1},a_n), d_{\mathscr{M}}(a_n,a_{n+1})\}.\end{aligned} \tag{3.5}$$

If $\tilde{\mathscr{M}}(a_n,a_{n+1}) = d_{\mathscr{M}}(a_n,a_{n+1})$ then (3.3) gives that

$$\begin{aligned}F(d_{\mathscr{M}}(a_n,a_{n+1})) = F(d_{\mathscr{M}}(fa_n,fa_{n+1})) &\leq \alpha(Ta_n)\beta(Ta_{n+1})F(d_{\mathscr{M}}(fa_n,fa_{n+1}))\\ &< F(\tilde{\mathscr{M}}(a_n,a_{n+1})) + G(\gamma(\tilde{\mathscr{M}}(a_n,a_{n+1}))),\end{aligned}$$

for some $n \in \mathbb{N}$. So, $G(\gamma(\tilde{\mathscr{M}}(a_n,a_{n+1}))) \geq 0 \Rightarrow \gamma(\tilde{\mathscr{M}}(a_n,a_{n+1})) \geq 1$, which gives a contradiction. Hence, for all $n \in \mathbb{N}$, $\tilde{\mathscr{M}}(a_n,a_{n+1}) = d_{\mathscr{M}}(a_{n-1},a_n)$. From (3.1), we have

$$\begin{aligned}F(d_{\mathscr{M}}(a_n,a_{n+1})) =&\alpha(Ta_n)\beta(Ta_{n+1})F(d_{\mathscr{M}}(fa_n,fa_{n+1}))\\ \leq& F(d_{\mathscr{M}}(a_{n-1},a_n)) + G(d_{\mathscr{M}}(\tilde{\mathscr{M}}(a_n,a_{n+1}))),\ \forall n \in \mathbb{N}.\end{aligned} \tag{3.6}$$

Consequently, we obtain that

$$\begin{aligned}F(d_{\mathscr{M}}(a_n,a_{n+1})) &\leq \alpha(Ta_n)\beta(Ta_{n+1})F(d_{\mathscr{M}}(fa_n,fa_{n+1}))\\ &\leq F(d_{\mathscr{M}}(a_{n-1},a_n)) + G(d_{\mathscr{M}}(\tilde{\mathscr{M}}(a_{n-1},a_n))) + G(d_{\mathscr{M}}(\tilde{\mathscr{M}}(a_n,a_{n+1}))).\end{aligned} \tag{3.7}$$

Iteratively, we find that

$$F(d_{\mathscr{M}}(a_n,a_{n+1})) \leq F(d_{\mathscr{M}}(a_0,a_1)) + \sum_{i=1}^{n} G(\gamma(\tilde{\mathscr{M}}(a_{i-1},a_i))). \tag{3.8}$$

By taking $n\to\infty$, we deduce that $\lim_{n\to\infty} F(d_{\mathscr{M}}(a_n,a_{n+1})) = -\infty$. Since $F\in\Delta_F$ and $(G,\gamma)\in\Delta_{G,\gamma}$, we have

$$\lim_{n\to\infty} d_{\mathscr{M}}(a_n,a_{n+1}) = 0. \tag{3.9}$$

We see that $\{a_n\}$ is a Cauchy sequence. Indeed, we can let $\{a_{m(p)}\}$ and $\{a_{n(p)}\}$ be the subsequences of $\{a_n\}$ such that, for $m(p)>n(p)\geq p$, there exists $\varepsilon_0>0$ such that

$$d_{\mathscr{M}}(a_{m(p)},a_{n(p)}) \geq \varepsilon_0 \tag{3.10}$$

and $n(p)$ is the smallest number which holds (3.10). $d_{\mathscr{M}}(a_{m(p)},a_{n(p)-1}) < \varepsilon_0$. By (σ_3) and (3.9) we have

$$\begin{aligned}\varepsilon_0 \leq d_{\mathscr{M}}(a_{m(p)},a_{n(p)}) &\leq d_{\mathscr{M}}(a_{m(p)},a_{n(p)-1}) + d_{\mathscr{M}}(a_{n(p)-1},a_{n(p)})\\ &< \varepsilon_0 + d_{\mathscr{M}}(a_{n(p)-1},a_{n(p)}).\end{aligned} \tag{3.11}$$

Taking the limit sup as $p\to\infty$ in above inequality, we see by (3.9) that

$$\limsup d_{\mathscr{M}}(fa_{m(p)},fa_{n(p)}) = \limsup d_{\mathscr{M}}(a_{m(p)},a_{n(p)}) = \varepsilon_0,\ \forall\,\mathbb{N}. \tag{3.12}$$

By using (3.9), we have

$$\limsup d_{\mathscr{M}}(a_{m(p)-1},a_{n(p)}) = \limsup d_{\mathscr{M}}(a_{m(p)},a_{n(p)-1}) = \varepsilon_0 \tag{3.13}$$

and

$$\limsup d_{\mathscr{M}}(a_{m(p)-1},a_{n(p)-1}) = \varepsilon_0. \tag{3.14}$$

By using the condition of T-cyclic α, β, we have $\alpha(Ta_{m(p)})\beta(Ta_{n(p)}) \geq 1$,

$$\begin{aligned}F(d_{\mathscr{M}}(fa_{m(p)},fa_{n(p)})) &\leq \alpha(Ta_{m(p)})\beta(Ta_{n(p)})F(d_{\mathscr{M}}(fa_{m(p)},fa_{n(p)}))\\ &\leq F(\tilde{\mathscr{M}}(a_{m(p)},a_{n(p)})) + G(\gamma(\tilde{\mathscr{M}}(a_{m(p)},a_{n(p)}))),\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}\tilde{\mathscr{M}}(a_{m(p)},a_{n(p)}) &= \max\left\{\begin{array}{c} d_{\mathscr{M}}(Ta_{m(p)},Ta_{n(p)}), d_{\mathscr{M}}(Ta_{m(p)},fa_{m(p)}), d_{\mathscr{M}}(Ta_{n(p)},fa_{n(p)}),\\ \dfrac{d_{\mathscr{M}}(Ta_{m(p)},fa_{n(p)}) + d_{\mathscr{M}}(Ta_{n(p)},fa_{m(p)})}{2},\\ \dfrac{d_{\mathscr{M}}(Ta_{n(p)},fa_{n(p)})[1+d_{\mathscr{M}}(Ta_{m(p)},fa_{m(p)})]}{d_{\mathscr{M}}(Ta_{m(p)},Ta_{n(p)})+1}\end{array}\right\},\\ &= \max\left\{\begin{array}{c} d_{\mathscr{M}}(a_{m(p)-1},a_{n(p)-1}), d_{\mathscr{M}}(a_{m(p)-1},a_{m(p)}), d_{\mathscr{M}}(a_{n(p)-1},a_{n(p)}),\\ \dfrac{d_{\mathscr{M}}(a_{m(p)-1},a_{n(p)}) + d_{\mathscr{M}}(a_{n(p)-1},a_{m(p)})}{2}\\ \dfrac{d_{\mathscr{M}}(a_{n(p)-1},a_{n(p)})[1+d_{\mathscr{M}}(a_{m(p)-1},a_{m(p)})]}{d_{\mathscr{M}}(a_{m(p)-1},a_{n(p)-1})+1}\end{array}\right\},\end{aligned} \tag{3.16}$$

Taking lim sup as $p\to+\infty$ in (3.14) and using (3.8), (3.11), (3.12), and (3.13), we have

$$\tilde{\mathscr{M}}(a_{m(p)},a_{n(p)}) = \max\left\{\varepsilon_0,0,0,\frac{\varepsilon_0+\varepsilon_0}{2},0\right\} = \varepsilon_0 \tag{3.17}$$

and

$$\begin{aligned} F(\varepsilon_0) &\leq F(\limsup d_{\mathscr{M}}(fa_{m(p)},fa_{n(p)}) \\ &\leq \limsup F(\tilde{\mathscr{M}}(a_{m(p)},a_{n(p)})) + \limsup G(\gamma(\tilde{M}(a_{m(p)},a_{n(p)}))) \\ &\leq F(\varepsilon_0) + \limsup G(\gamma(\tilde{\mathscr{M}}(a_{m(p)},a_{n(p)}))), \end{aligned} \tag{3.18}$$

which implies that $\limsup G(\gamma(\tilde{\mathscr{M}}(a_{m(p)},a_{n(p)}))) \geq 0$. This gives $\limsup \gamma(\tilde{\mathscr{M}}(a_{m(p)},a_{n(p)})) \geq 1$. Since $\gamma(s) < 1$, for all $s \geq 0$, we obtain $\limsup \gamma(\tilde{\mathscr{M}}(a_{m(p)},a_{n(p)})) = 1$. Therefore,

$$\limsup \tilde{\mathscr{M}}(a_{m(p)},a_{n(p)}) = 0,$$

which is a contradiction of (3.10) and (3.16). Therefore, $\{a_n\}$ is a Cauchy sequence. Since $(\mathscr{M},d_{\mathscr{M}})$ is complete, there exists $u \in \mathscr{M}$ such that

$$d_{\mathscr{M}}(u,u) = \lim_{n\to\infty} d_{\mathscr{M}}(a_n,u) = \lim_{n,m\to\infty} d_{\mathscr{M}}(a_n,a_m) = 0. \tag{3.19}$$

From (3.3) and (3.19), we have

$$fa_n \to u \text{ and } Ta_{n+1} \to u. \tag{3.20}$$

Since $T\mathscr{M}$ is closed, by (3.20), $u \in T\mathscr{M}$. Therefore, there exists $c \in \mathscr{M}$ such that $Tc = u$. As $a_n \to u$, $\beta(a_n) = \beta(Ta_{n+1}) \geq 1, \forall n \in \mathbb{N}$ and by (iii), $\beta(u) = \beta(Tc) \geq 1$. Hence, $\alpha(Ta_n)\beta(Tc) \geq 1$, $\forall n \in \mathbb{N}$. If the continuity of f is applied, then it is deduced

$$\lim_{n\to\infty} d_{\mathscr{M}}(a_{n+1},fu) = \lim_{n\to\infty} d_{\mathscr{M}}(Ta_n,fu) = d_{\mathscr{M}}(Tu,fu). \tag{3.21}$$

By (3.19) and Lemma 2.5,

$$\lim_{n\to\infty} d_{\mathscr{M}}(a_{n+1},fu) = \lim_{n\to\infty} d_{\mathscr{M}}(u,fu). \tag{3.22}$$

Comparing (3.21) and (3.22), we have $d_{\mathscr{M}}(Tu,fu) = d_{\mathscr{M}}(u,fu)$. Now using (3.4), we get $1 \leq \alpha(Ta_n)\beta(Tc)$, and then

$$\begin{aligned} F(d_{\mathscr{M}}(fa_n,fz)) &\leq \alpha(Ta_n)\beta(Tc)F(d_{\mathscr{M}}(fa_n,fc)) \\ &\leq F(\tilde{\mathscr{M}}(a_n,c)) + G(\gamma(\tilde{\mathscr{M}}(a_n,c))), \end{aligned} \tag{3.23}$$

where

$$\tilde{\mathscr{M}}(a_n,c) = \max\left\{ \begin{array}{c} d_{\mathscr{M}}(Ta_n,Tc), d_{\mathscr{M}}(Ta_n,fa_n), d_{\mathscr{M}}(Tc,fc), \\ \frac{d_{\mathscr{M}}(Ta_n,fc)+d_{\mathscr{M}}(fa_n,Tc)}{2}, \frac{[1+d_{\mathscr{M}}(Ta_n,fa_n)]d_{\mathscr{M}}(fc,Tc)}{1+d_{\mathscr{M}}(Ta_n,Tc)} \end{array} \right\}.$$

Taking $n \to \infty$ and using (3.19), we have

$$\lim_{n\to\infty} \tilde{\mathscr{M}}(a_n,c) = \max\left\{ \begin{array}{c} d_{\mathscr{M}}(u,u), d_{\mathscr{M}}(u,u), d_{\mathscr{M}}(u,fc), \\ \frac{d_{\mathscr{M}}(u,fc)+d_{\mathscr{M}}(u,u)}{2}, \frac{[1+d_{\mathscr{M}}(u,u)]d_{\mathscr{M}}(fc,u)}{1+d_{\mathscr{M}}(u,u)} \end{array} \right\} = d_{\mathscr{M}}(u,fc).$$

It follows that

$$F(d_{\mathscr{M}}(u,fc)) \leq \alpha(Ta_n)\beta(Tc)F(d_{\mathscr{M}}(fa_n,fc)) \leq F(d_{\mathscr{M}}(u,fc)) + G(\gamma(d_{\mathscr{M}}(u,fc))).$$

So, $G(\gamma(d_{\mathscr{M}}(u,fc))) \geq 0$, which gives that $\gamma(d_{\mathscr{M}}(u,fc)) \geq 1$, a contradiction. Hence, $fc = u$ and u is the coincidence point of T and f.

The uniqueness properties of coincidence point follows from conditions (3.1) and (3.2). Let a,b be fixed points of f, such that $a \neq b$. Applying (3.4), we get $\alpha(a)\beta(b) \geq 1$. In view of

$$\begin{aligned} F(d_{\mathscr{M}}(fa,fb)) &\leq \alpha(Ta)\beta(Tb)F(d_{\mathscr{M}}(fa,fb)) \\ &\leq F(\tilde{\mathscr{M}}(a,b)) + G(\gamma(\tilde{\mathscr{M}}(a,b))), \end{aligned} \tag{3.24}$$

where

$$\tilde{\mathscr{M}}(a,b) = \max \left\{ \begin{array}{c} d_{\mathscr{M}}(Ta,Tb), d_{\mathscr{M}}(Ta,fa), d_{\mathscr{M}}(Tb,fb), \\ \frac{d_{\mathscr{M}}(Ta,fb)+d_{\mathscr{M}}(fa,Tb)}{2}, \frac{d_{\mathscr{M}}(fb,Tb)[1+d_{\mathscr{M}}(fa,Ta)]}{d_{\mathscr{M}}(Ta,Tb)+1} \end{array} \right\} = d_{\mathscr{M}}(a,b),$$

we have

$$F(d_{\mathscr{M}}(a,b)) \leq F(d_{\mathscr{M}}(a,b)) + G(\gamma(d_{\mathscr{M}}(a,b))).$$

So $G(\gamma(d_{\mathscr{M}}(a,b))) \geq 0$, which gives $\gamma(d_{\mathscr{M}}(a,b)) \geq 1$. This leads to a contradiction. Therefore, $a = b$. T and f have a unique common fixed point. Using weak compatibility of T and f, we get $fc = fTu = Tfu = Tc$, which implies $fc = Tc$. By using the uniqueness properties, $c = fc = Tc$. T and f have a unique common fixed point. □

Taking $T = I_{\mathscr{M}}$ in Theorem 3.2, we obtain the following result.

Corollary 3.3. Let $(\mathscr{M}, d_{\mathscr{M}})$ be a complete metric-like space. Suppose that the mappings $f : \mathscr{M} \to \mathscr{M}$ and $\alpha, \beta : \mathscr{M} \to \mathbb{R}_0^+$ satisfy the following conditions

(1) f is continuous;
(2) if $a_0 \in \mathscr{M}$, then $\alpha(a_0) \geq 1$ and $\beta(a_0) \geq 1$;
(3) $fu = v$ and $fv = u$ such that $1 \leq \alpha(u)$, $1 \leq \beta(v)$;
(4) f is a γ-FG-contractive mapping of (α, β)-type;
(5) if a sequence $\{a_n\}$ in $\mathscr{M}$ such that $a_n \to a$ and $1 \leq \beta(a_n)$, then $1 \leq \beta(a)$.

Then f has a unique fixed point.

Remark 3.4. The results of [26, 27] are the particular cases of Theorem 3.2, for $G(s) = \ln s, \gamma(s) = q$, where $q \in (0,1)$ and $\tau = -\ln q$.

Corollary 3.5. *Let $(\mathscr{M}, d_{\mathscr{M}})$ be a complete metric-like space. Suppose that the mappings $\alpha, \beta : \mathscr{M} \to [0,\infty)$ and $\mathrm{f}, T : \mathscr{M} \to \mathscr{M}$ are such that $f\mathscr{M} \subset T\mathscr{M}$. Let $T\mathscr{M}$ be a closed subset of $\mathscr{M}$ which satisfies the following conditions:*

(1) if $a_0 \in \mathscr{M}$, then $1 \leq \alpha(Ta_0)$ and $1 \leq \beta(Ta_0)$;
(2) f is continuous;
(3) $\mathrm{f}u = Tu$ and $\mathrm{f}v = Tv$ such that $1 \leq \alpha(Tu)$, $1 \leq \beta(Tv)$;
(4) f is a γ-FG-contractive mapping of T-cyclic (α, β)-type;
(5) if a sequence $\{a_n\}$ in $\mathscr{M}$ such that $a_n \to a$ and $1 \leq \beta(a_n)$, then $1 \leq \beta(a)$.

Then T and f have a unique point of coincidence in $\mathscr{M}$. Furthermore, T and f have a unique common fixed point provided that T and f are weakly compatible.

The following result can be obtained by taking $\alpha(Ta)\beta(Tb) = 1$, $F(s) = G(s) = \ln(s)$, and $T = I_{\mathscr{M}}$ in Theorem 3.2.

Corollary 3.6. *Let $f : \mathscr{M} \to \mathscr{M}$ and $\alpha, \beta : \mathscr{M} \to [0,\infty)$ be functions. Suppose that the complete metric-like space $(\mathscr{M}, d_{\mathscr{M}})$ satisfies the following conditions*

(1) if $a_0 \in \mathscr{M}$, then $1 \leq \alpha(a_0)$ and $1 \leq \beta(a_0)$;

(2) f *is continuous;*
(3) fu $= v$ *and* fv $= u$ *such that* $1 \leq \alpha(u)$, $1 \leq \beta(v)$*;*
(4) *If a sequence* $\{a_n\}$ *in* $\mathscr{M}$ *such that* $a_n \to a$ *and* $1 \leq \beta(a_n)$*, then* $1 \leq \beta(a)$*;*
(5) *For* $\tilde{\mathscr{M}}$ *defined in* (3.2)*,* $d_{\mathscr{M}}(\mathrm{fa},\mathrm{fb}) \leq \gamma(\tilde{\mathscr{M}}(a,b))\tilde{\mathscr{M}}(a,b)$ *and* $d_{\mathscr{M}}(\mathrm{fa},\mathrm{fb}) > 0, \forall a,b \in \mathscr{M}$.

Then f *has a unique fixed point.*

Taking $G(t) = F(s) = \ln(s)$, $\alpha(Ta)\beta(Tb) = 1$, and $T = I_{\tilde{M}}$ in Theorem 3.2, we have the result.

Corollary 3.7. Let $(\mathscr{M}, d_{\mathscr{M}})$ be a metric-like space. Let $\alpha, \beta : \mathscr{C} \to [0,\infty)$ and $f : \mathscr{M} \to \mathscr{M}$ be mappings such that

(1) f is continuous;
(2) if $a_0 \in \mathscr{M}$, then $1 \leq \alpha(a_0)$ and $1 \leq \beta(a_0)$;
(3) $fu = v$ and $fv = u$ such that $1 \leq \alpha(u)$, $1 \leq \beta(v)$;
(4) if a sequence $\{a_n\}$ in $\mathscr{C}$ such that $a_n \to a$ and $1 \leq \beta(a_n)$, then $1 \leq \beta(a)$;
(5) For $\tilde{\mathscr{M}}$ defined in (3.2), $d_{\mathscr{M}}(fa,fa) \leq \gamma(\tilde{\mathscr{M}}(a,b))\tilde{\mathscr{M}}(a,b)$ and $d_{\mathscr{M}}(fa,fb) > 0$, $\forall a,b \in \mathscr{M}$.

Then f has a unique fixed point.

Substituting $\alpha(Ta)\beta(Tb) = 1$, $G(s) = \ln(s)$, and $F(s) = -\frac{1}{\sqrt{s}}$ into Theorem 3.2, we have the result.

Corollary 3.8. Let $(\mathscr{M}, d_{\mathscr{M}})$ be a metric-like space. Let $f : \mathscr{M} \to \mathscr{M}$ and $\alpha, \beta : \mathscr{M} \to [0,\infty)$ be such that

(1) if $a_0 \in \mathscr{M}$, then $\alpha(a_0) \geq 1$ and $\beta(a_0) \geq 1$;
(2) f is continuous;
(3) if a sequence $\{a_n\}$ in $\mathscr{M}$ such that $a_n \to a$ and $1 \leq \beta(a_n)$, then $1 \leq \beta(a)$;
(4) if $fu = v$ and $fv = u$, then $1 \leq \alpha(u)$, $1 \leq \beta(v)$;
(5) for the same $\tilde{\mathscr{M}}$ defined in (3.2) and $(\ln t, \gamma) \in \Delta_{G,\gamma}$; for some $d_{\mathscr{M}}(fa,fb) > 0$, $d_{\mathscr{M}}(fa,fb) \leq \frac{\tilde{\mathscr{M}}(a,b)}{[1-\sqrt{\tilde{\mathscr{M}}(a,b)}\ln\gamma(\tilde{\mathscr{M}}(a,b))]^2}$, $\forall a,b \in \mathscr{M}$.

Then f has a unique fixed point.

Take $\gamma(s) = r$ and $\alpha(Ta)\beta(Tb) = 1$, where $r \in (0,1)$, into Theorem 3.2, we have the following result.

Corollary 3.9. Let $(\mathscr{M}, d_{\mathscr{M}})$ be a metric-like space. Suppose that the mappings $f : \mathscr{M} \to \mathscr{M}$ and $\alpha, \beta : \mathscr{M} \to [0,\infty)$. satisfy the following:

(1) if $a_0 \in \mathscr{M}$, then $1 \leq \alpha(a_0)$ and $1 \leq \beta(a_0)$;
(2) f is continuous;
(3) for a sequence $\{a_n\}$ in $\mathscr{M}$ such that $a_n \to a$ and $1 \leq \beta(a_n)$, $1 \leq \beta(a)$;
(4) If $fu = v$ and $fv = u$,then $1 \leq \alpha(u)$, $1 \leq \beta(v)$;
(5) for $\tilde{\mathscr{M}}$ defined in (3.2), $d_{\mathscr{M}}(fa,fb) \leq \frac{\tilde{\mathscr{M}}(a,b)}{[1+\sqrt{\tilde{\mathscr{M}}(a,b)}]^2}$, $\forall a,b \in \mathscr{M}$ and some $d_{\mathscr{M}}(fa,fb) > 0$.

Then f has a unique fixed point.

Let us construct the following example for validation and give a proper strength to Theorem 3.2.

Example 3.10. Let $\mathscr{M} = \mathbb{R}$ and the metric-like space $(\mathscr{M}, d_{\mathscr{M}})$, where $d_{\mathscr{M}}$ is defined by $d_{\mathscr{M}}(a,b) = a^2 + b^2$. Let T and f be self mappings on $\mathbb{R}$ and $\gamma : \mathbb{R}^+ \to [0,1)$ be given by

$$f(a) = \begin{cases} \frac{-a}{4}, & a \in [0, \frac{1}{2}] \\ \frac{a}{5}, & \text{otherwise} \end{cases}, \quad T(a) = \begin{cases} \frac{a}{2}, & a \in [-\frac{1}{2}, 0] \\ \frac{a+1}{4}, & \text{otherwise} \end{cases} \text{ and } \gamma(t) = \frac{1}{25}.$$

Note that $(\mathscr{M}, d_{\mathscr{M}})$ is a complete metric-like space. The mappings $\alpha, \beta : \mathscr{M} \to [0, \infty)$ are defined by

$$\alpha(a) = \begin{cases} \frac{a^2}{2}, & a \in (-\infty, -\frac{1}{6}) \\ e^{-a}, & a \in [-\frac{1}{6}, 0] \\ 0, & a \in \mathbb{R}^+ \end{cases} \quad \text{and } \beta(a) = \begin{cases} e^{-a}, & a \in [\frac{-1}{10}, 0] \\ 0, & \text{otherwise.} \end{cases}$$

Now, we can show that f is a T-cyclic (α, β)-admissible mapping. Taking $a \in \mathscr{M}$ such that $\alpha(Ta) \geq 1$, one sees that $Ta \in [-\frac{1}{4}, 0]$ and hence $a \in [-\frac{1}{2}, 0]$. By the definitions of β and f, we get $fa \in [-\frac{1}{10}, 0]$, so $\beta(fa) \geq 1$. Similarly, we have to prove that if $\beta(Ta) \geq 1$, then $\alpha(fa) \geq 1$. Hence, f is a T-cyclic (α, β)-admissible mapping.

Now, consider a sequence $\{a_n\}$ in $\mathscr{M}$ such that $\beta(a_n) \geq 1$ and $a_n \to a$ as $n \in \mathbb{N}$. Again, the conditions $\alpha(Ta_0) \geq 1$ and $\beta(Ta_0) \geq 1$ holds with $a_0 = \frac{-1}{6}$. Hence, by the definition of β, we obtain $a_n \in [-\frac{1}{10}, 0]$, and $a \in [-\frac{1}{10}, 0]$, which implies $\beta(a) \geq 1$. Again, we can show that f is a $\gamma - FG$-contractive mapping of T-cyclic (α, β)-type which is defined in (3.1), with $G, F : [0, \infty) \to (-\infty, \infty)$ as $F(s) = G(s) = \ln s,, \forall s \in [0, \infty)$. Let $a, b \in \mathscr{M}$ be such that $\alpha(Ta)\beta(Tb) \geq 1$. Then $Ta \in [-\frac{1}{6}, 0]$, $Tb \in [-\frac{1}{10}, 0]$ and hence $a \in [-\frac{1}{2}, 0]$, $b \in [-\frac{1}{4}, 0]$. Now,

$$d_{\mathscr{M}}(fa, fb) \leq \frac{1}{25}(a^2 + b^2) \leq \frac{1}{25} d_{\mathscr{M}}(a,b) \leq \gamma(\tilde{\mathscr{M}}(a,b)) d_{\mathscr{M}}(a,b) \leq \gamma(\tilde{\mathscr{M}}(a,b)) \tilde{\mathscr{M}}(a,b),$$

where

$$\tilde{\mathscr{M}}(a,b) = \max \left\{ \begin{array}{c} d_{\mathscr{M}}(Ta, fa), d_{\mathscr{M}}(Tb, fb), d_{\mathscr{M}}(Ta, Tb), \\ \frac{d_{\mathscr{M}}(Tb, fa) + d_{\mathscr{M}}(Ta, fb)}{2}, \\ \frac{d_{\mathscr{M}}(Tb, fb)[1 + d_{\mathscr{M}}(Ta, fa)]}{1 + d_{\mathscr{M}}(Ta, Tb)} \end{array} \right\}.$$

Hence, $F(d(fa, fb)) \leq F(\tilde{\mathscr{M}}(a,b)) + G(\gamma(\tilde{\mathscr{M}}(a,b)))$ and satisfy all the assertions of Theorem 3.2. Hence, $a^* = 0$ is the unique fixed point of f and T.

4. Application in Dynamic Programming

In [6], Bellman and Lee investigated solutions of functional equations and their systems in dynamic programming by using a range of fixed point theorems. In this section, our aim is to solve the functional equations that emerge in dynamic programming. Throughout this section, R and S are considered as Banach spaces, with $Z \subseteq R$ representing the state space and $E \subseteq S$ the decision space. By applying the results, the existence and uniqueness of the common solutions to these functional equations are demonstrated

$$X = \sup_{b \in E} \{p(a,b) + G(a,b,X(\tau(a,b)))\}, \ a \in Z \tag{4.1}$$

and

$$Y = \sup_{b\in E}\{q(a,b) + K(a,b,Y(\tau(a,b)))\},\ a \in Z, \tag{4.2}$$

where $G, K : Z \times E \times \mathbb{R} \to \mathbb{R}$, $\tau : Z \times E \to Z$ and $p, q : Z \times E \to \mathbb{R}$. It is well-established that the equations of the forms (4.1) and (4.2) serve as valuable tools in computer programming, mathematical optimization, and dynamic programming [6, 7]. Let us consider the space of all bounded real-valued functions, denoted by $D(Z)$ defined on R. Furthermore, $D(Z)$ is a complete metric-like space. $d_{\mathscr{M}}(h,k) = \sup_{a\in W}|ha - ka|$, for each $h,k \in D(Z)$ is a complete metric-like space. Again, consider the self mappings T and T define on $D(z)$ and given by:

$$fh(a) = \sup_{b\in E}\{p(a,b) + G(a,b,h(\tau(a,b)))\}, a \in Z, h \in D(Z),$$

$$Th(a) = \sup_{b\in E}\{q(a,b) + K(a,b,h(\tau(a,b)))\}, a \in Z, h \in D(Z).$$

Suppose that the following condition holds:

(C_1) For every $h \in D(Z)$, there exists $k \in D(Z)$ such that $fh(a) = Tk(a)\ \forall a \in Z$;
(C_2) There exists $h \in D(Z)$, such that $fh(a) = Th(a) \Rightarrow Tfh(a) = fTh(a),\ \forall a \in Z$.
(C_3) The functions $p, q : Z \times E \to (\infty, -\infty)$ and $G, K : Z \times E \times (\infty, -\infty) \to (\infty, -\infty)$ are bounded;
(C_4) There exists $h_0 \in D(Z)$ such that $\xi(Th_0)$.
(C_5) If the sequence $\{h_n\}$ converge to $h*$ in $D(Z)$ such that $\xi(h_n) \geq 0$, then $\xi(h^*) \geq 0$.
(C_6) There exist $h \in D(Z)$ such that $\xi(Th) \geq 0$, which implies $\xi(fh) \geq 0$.
(C_7) If $r \in (0,1)$ if such that for $h, k \in D(Z)$ with $\xi(Th) \geq 0$ and $\xi(Tk) \geq 0$, then $|G(a,b,h(a)) - G(a,b,k(a))| \leq r\tilde{\mathscr{M}}(h,k)$, for all $(a,b) \in Z \times E$, and $t \in Z$, where

$$\tilde{\mathscr{M}}(h,k) = \max\left\{\begin{array}{c} d_{\mathscr{M}}(Th(t),Tk(t)), d_{\mathscr{M}}(Tk(t),fk(t)), d_{\mathscr{M}}(Th(t),fh(t)), \\ \dfrac{d_{\mathscr{M}}(Th(t),fk(t)) + d_{\mathscr{M}}(Tk(t),fh(t))}{2}, \\ \dfrac{[1+d_{\mathscr{M}}(Th(t),fh(t))]d_{\mathscr{M}}(Tk(t),fk(t))}{1+d_{\mathscr{M}}(Th(t),Tk(t))} \end{array}\right\}.$$

Theorem 4.1. *Let conditions $(C_1)-(C_7)$ hold and $T(D(Z))$ be a bounded and closed subspace of $D(Z)$. Then equations (4.1) and (4.2) have a unique common bounded solution in Z.*

Proof. Let $\lambda > 0$ be an arbitrary number, and let $a \in h_1, h_2 \in D(Z)$ such that $\xi(Th_1) \geq 0$ and $\xi(Th_2) \geq 0$. Then there exist $b_1, b_2 \in E$ such that

$$fh_1(a) < p(a.b_1) + G(a,b_1,h_1(\tau(a,b_1))) + \lambda, \tag{4.3}$$

$$fh_2(a) < p(a.b_2) + G(a,b_2,h_2(\tau(a,b_2))) + \lambda, \tag{4.4}$$

$$fh_1(a) \geq p(a.b_2) + G(a,b_2,h_1(\tau(a,b_2))), \tag{4.5}$$

and

$$fh_2(a) < p(a.b_1) + G(a,b_1,h_2(\tau(a,b_1))). \tag{4.6}$$

Next, by using (4.3) and (4.6), we get

$$\begin{aligned} fh_1(a) - fh_2(a) &< G(a,b_1,h_1(\tau(a,b_1))) - G(a,b_1,h_2(\tau(a,b_1))) + \lambda \\ &\leq |G(a,b_1,h_1(\tau(a,b_1))) - G(a,b_1,h_2(\tau(a,b_1)))| + \lambda \\ &\leq r\tilde{\mathscr{M}}(h_1,h_2) + \lambda. \end{aligned} \tag{4.7}$$

Similarly, by using equations (4.4) and (4.5), we have

$$fh_2(a) - fh_1(a) < r\tilde{\mathscr{M}}(h_1,h_2) + \lambda. \tag{4.8}$$

From equation (4.7) and (4.8), we obtain

$$|fh_1(a) - fh_2(a)| < r\tilde{\mathscr{M}}(h_1,h_2) + \lambda. \tag{4.9}$$

Since the above inequality is true for any $a \in Z$, then $d_{\mathscr{M}}(fh_1, fh_2) \le r\tilde{\mathscr{M}}](h_1,h_2) + \lambda$. As $\lambda > 0$ is arbitrary, we deduce $d_{\mathscr{M}}(fh_1, fh_2) \le r\tilde{\mathscr{M}}(h_1,h_2)$. Now the mappings $\alpha, \beta : D(Z) \to \mathbb{R}_0^+$ are defined by

$$\alpha(h) = \beta(h) = \begin{cases} 1, & \text{if } \xi(h) \ge 0 \text{ where } h \in D(Z), \\ 0, & \text{otherwise} \end{cases}$$

and also, define $\gamma : [0,\infty] \to [0,\infty]$ by $\gamma(t) = r,\ r \in (0,1)$. Therefore, using inequality (4.9), we have

$$d_{\mathscr{M}}(fh_1, fh_2)) \le \gamma(\tilde{\mathscr{M}}(h_1,h_2))\tilde{\mathscr{M}}(h_1,h_2).$$

It is easy to verify that it satisfies all the conditions of Corollary 3.6. Hence, T and f have a unique common fixed point, which implies that (4.1) and (4.2) have a unique bounded common solution. □

Remark 4.2. Theorem 4.1 also holds if we consider (C_5') in place of (C_5), (C_5') is $|G(a,b,h(a)) - G(a,b,k(a))| \le \ln(1+N(h,k))$, where $h,k \in D(Z)$, $\xi(Th) \ge 0$, $\xi(TK) \ge 0$, $(a,b) \in Z \times E$, $t \in Z$ and

$$N(h,k) = \max\left\{ \begin{array}{c} d_{\mathscr{M}}(Th(t),Tk(t)), \frac{1}{2}d_{\mathscr{M}}(Th(t),fk(t)), d_{\mathscr{M}}(Tk(t),fh(t)), \\ \frac{[1+d_{\mathscr{M}}(Th(t),fh(t))]d_{\mathscr{M}}(Tk(t),fk(t))}{1+d_{\mathscr{M}}(Th(t),Tk(t))} \end{array} \right\}.$$

5. Conclusion

In metric-like spaces, T-cyclic (α,β)-admissible mappings were used to study the γ-FG-contractive conditions. For the new mappings in complete metric-like spaces, fixed point results were proved. A fixed point theorem for cyclic mappings was established, and some corollaries were also obtained. In addition, some functional equations in dynamic programming were solved.

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