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DUALITY OF NONSMOOTH OPTIMIZATION PROBLEMS USING τ-SUBDIFFERENTIAL

SHASHI KANT MISHRA, UMASHANKAR BAJPEI*

Department of Mathematics, Banaras Hindu University, Varanasi 221005, India

Abstract. In this paper, we consider a nonsmooth optimization problem and formulate the Wolfe type dual and the Mond-Weir type dual problems. We also establish weak, strong and strict converse duality theorems for the Wolfe type dual and the Mond-Weir type dual problems using τ-subdifferential under generalized invexity. Further, some illustrative examples are also given.

Keywords. Constrained optimization; Duality conditions; Invex function; Mond-Weir type dual; Wolfe type dual.

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1. INTRODUCTION

In optimization, the concept of duality is a powerful tool that offers a strategic approach to finding the optimal solution for a primal problem. In 1961, Wolfe [1] formulated the dual problem for nonlinear mathematical programming problems and in 1981, Mond and Weir [2] formulated Mond-Weir type duality for nonlinear mathematical programming problems. Further, in 1982, Hanson and Mond [3] established weak duality theorem (WDT) and strong duality theorems (SDT) for Wolfe type dual (WTD) problems under generalized convexity. Egudo and Mond [4] established WDT, SDT and strict converse duality theorem (SCDT) for Mond-Weir type dual (MWTD) problems under generalized convexity. In 1987, Hanson and Mond [5] considered constrained optimization problems and obtained necessary and sufficient optimality conditions for a dual problem. In 1991, Yang and Craven [6] considered nonsmooth constrained optimization problems and defined the τ-subdifferential for locally Lipschitz functions to establish necessary optimality conditions and various results for generalized invex functions [7]. Mishra and Giorgi [8] extended τ-subdifferential for generalized semi-univexity to obtain sufficient optimality conditions and duality for nonsmooth optimization problems. Weir and Jayakumar [9], Egudo and Hanson [10], Brown et al. [11], Mishra et al. [12, 13], Fajardo and

*Corresponding author.
E-mail address: shashikant.mishra@bhu.ac.in (S.K. Mishra), usbajpei@bhu.ac.in (U. Bajpei).
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Vidal [14], Luu and Mai [15], Upadhyay et al. [16, 17, 18], Hooker [19] and many other researchers also explored the concept of duality for different optimization problems and produced various results.

Motivated by the work of Van Luu and Mai [15], and Mishra et al. [12, 13], we consider nonsmooth constrained optimization problem and formulate WTD and MWTD problems. Further, we also derive WDT, SDT and SCDT for WTD and MWTD problems under generalized invexity using τ-subdifferential. Moreover, to best of our knowledge, no research has been done till now for WTD and MWTD problem using τ-subdifferential. This research has an effort in the field of nonsmooth optimization.

This paper is organized as follows. In Section 2, we provide some definitions, propositions and results which are crucial for our sequel. In Section 3.1, we formulate WTD problems and prove WDT, SDT and SCDT. In Section 3.2, we formulate MWTD problems and prove WDT, SDT, SCDT and some other results by further relaxing invexity conditions and give some examples to verify our results. Section 4 ends this paper.

2. PRELIMINARIES

Definition 2.1. [12] A function $\bar{\Phi} : S \to \mathbb{R}$ is defined as locally Lipschitz at $\bar{u^*} \in S$, where $S \subset \mathbb{R}^n$ if there exists $W > 0$ and a neighbourhood N of $\bar{u^*}$ such that, for any $\bar{v^*}, \bar{w^*} \in N$, $| \bar{\Phi}(\bar{v^*}) - \bar{\Phi}(\bar{w^*}) | \leq W \|\bar{v^*} - \bar{w^*}\|$.

Definition 2.2. [20] Let $\bar{\Phi} : S \to \mathbb{R}$ be a convex function where $S \subseteq \mathbb{R}^n$ and w be the subgradient of $\bar{\Phi}$ at $\bar{v^*}$ if it satisfies $\bar{\Phi}(\bar{u^*}) \geq \bar{\Phi}(\bar{v^*}) + \langle w, \bar{u^*} - \bar{v^*} \rangle$ for all $\bar{u^*} \in \mathbb{R}^n$. The collection of all such subgradient of $\bar{\Phi}$ at $\bar{v^*}$ is defined as subdifferential of $\bar{\Phi}$ at $\bar{v^*}$ and denoted as $\partial^c \bar{\Phi}(\bar{v^*})$.

In 1991, Yang and Craven [6] extended the concept of subgradient to τ-subgradient of locally Lipschitz functions where $\tau : S \times S \to \mathbb{R}$.

Definition 2.3. Let an open subset $S \subseteq \mathbb{R}^n$, $\bar{\Phi} : S \to \mathbb{R}$ be a locally Lipschitz function on S. Then $w \in \mathbb{R}^n$ is a τ-subgradient of $\bar{\Phi}$ at $\bar{v^*}$ if $\bar{\Phi}(\bar{u^*}) - \bar{\Phi}(\bar{v^*}) \geq \langle w, \tau(\bar{u^*}, \bar{v^*}) \rangle$ for all $\bar{u^*} \in S$, and the collection of all τ-subgradient of $\bar{\Phi}$ at any point $\bar{v^*}$ is defined as τ-subdifferential of $\bar{\Phi}$ at $\bar{v^*}$ and denoted as $\partial^\tau \bar{\Phi}(\bar{v^*})$.

Example 2.4. Let $\bar{\Phi}$ is a locally Lipschitz function defined on $\mathbb{R}$ such that

$$\bar{\Phi}(v) = \begin{cases} 5 & v \leq 0 \\ v+5 & v > 0 \end{cases}.$$

This function is not differentiable at 0. Then, for $\tau = u + v$, where $u, v \in \mathbb{R}$

$$\partial^\tau \bar{\Phi}(0) = \{w \mid \bar{\Phi}(\bar{u}) - \bar{\Phi}(0) \geq \langle w, \tau(\bar{u}, 0) \rangle\} = [0, 1].$$

Definition 2.5. [6] Let $\bar{\Phi} : S \to \mathbb{R}$ be a locally Lipschitz function. Then, τ-generalized directional derivative of $\bar{\Phi}$ at $\bar{u^*}$ in the direction k with respect to $\tau(\bar{u^*}, \bar{v^*})$ is denoted as $\bar{\Phi}^0_\tau(\bar{u^*}; k)$ and defined as :

$$\bar{\Phi}^0_\tau(\bar{u^*}; k) = \begin{cases} -\infty \; if \; \partial^\tau \bar{\Phi}(\bar{u^*}) = \emptyset; \\ \sup \langle y, k \rangle \mid y \in \partial^\tau \bar{\Phi}(\bar{u^*}) \; if \; \partial^\tau \bar{\Phi}(\bar{u^*}) \neq \emptyset. \end{cases}$$

Definition 2.6. [6] Let an open subset $S \subseteq \mathbb{R}^n$, $\bar{\Phi} : S \to \mathbb{R}$ be a locally Lipschitz function and U be a nonempty subset of S. Then, $\bar{\Phi}$ is called an invex function with respect to $\tau(\bar{u^*}, \bar{v^*})$ at $\bar{v^*} \in U$ if, for all $\bar{u^*} \in U$, $\bar{\Phi}(\bar{u^*}) - \bar{\Phi}(\bar{v^*}) \geq \langle w, \tau(\bar{u^*}, \bar{v^*})\rangle$, $w \in \partial^\tau \bar{\Phi}(\bar{v^*})$. If $\bar{\Phi}$ is invex for each point of U, then $\bar{\Phi}$ is invex on U.

Definition 2.7. Let an open subset $S \subseteq \mathbb{R}^n$, $\bar{\Phi} : S \to \mathbb{R}$ be a locally Lipschitz function and U be a nonempty subset of S. Then, $\bar{\Phi}$ is called:
(1) pseudo-invex with respect to $\tau(\bar{u^*}, \bar{v^*})$ at $\bar{v^*} \in U$ if, for all $\bar{u^*} \in U$,

$$\langle w, \tau(\bar{u^*}, \bar{v^*})\rangle \geq 0 \implies \bar{\Phi}(\bar{u^*}) \geq \bar{\Phi}(\bar{v^*})\ w \in \partial^\tau \bar{\Phi}(\bar{v^*});$$

(2) strictly pseudo-invex with respect to $\tau(\bar{u^*}, \bar{v^*})$ at $\bar{v^*} \in U$ if, for all $\bar{u^*} \in U$,

$$\langle w, \tau(\bar{u^*}, \bar{v^*})\rangle \geq 0 \implies \bar{\Phi}(\bar{u^*}) > \bar{\Phi}(\bar{v^*})\ w \in \partial^\tau \bar{\Phi}(\bar{v^*});$$

(3) quasi-invex with respect to $\tau(\bar{u^*}, \bar{v^*})$ at $\bar{v^*} \in U$ if, for all $\bar{u^*} \in U$,

$$\bar{\Phi}(\bar{u^*}) \leq \bar{\Phi}(\bar{v^*}) \implies \langle w, \tau(\bar{u^*}, \bar{v^*})\rangle \leq 0\ w \in \partial^\tau \bar{\Phi}(\bar{v^*}).$$

We need following propositions and theorems to prove our results.

Proposition 2.8. [6] *Let $\bar{\Phi}$ be a locally Lipschitz function. If $\bar{v^*}$ minimizes $\bar{\Phi}(\bar{v^*})$ over S, then* $0 \in \partial^\tau \bar{\Phi}(\bar{v^*})$.

Proposition 2.9. [6] *For any $k > 0$, we have (1) $\partial^{k\tau} \bar{\Phi}(\bar{v^*}) = (1/k)\partial^\tau \bar{\Phi}(\bar{v^*})$; (2) $\partial^\tau (k\bar{\Phi})(\bar{v^*}) = k\partial^\tau \bar{\Phi}(\bar{v^*})$; (3) $\partial^\tau \bar{\Phi}_1(\bar{v^*}) + \partial^\tau \bar{\Phi}_2(\bar{v^*}) \subset \partial^\tau (\bar{\Phi}_1 + \bar{\Phi}_2)(\bar{v^*})$*

We consider a nonsmooth constrained optimization problem:

$$\inf \bar{\Phi}(v) \text{ subject to } v \in B, \tag{P}$$

where $B := \{v \in \mathbb{R}^n \mid \bar{\Psi}_\iota(v) \leq 0, \iota = 1, 2, ...n\}$ and $\bar{\phi} : S \to \mathbb{R}$, $\bar{\psi_\iota} : S \to \mathbb{R}$ (ι=1,2,....n) are locally Lipschitz functions, where $S \subset \mathbb{R}^n$.

Definition 2.10. The τ-normal cone to feasible set B at v is denoted as $N_B^\tau(v)$ and defined as

$$N_B^\tau(v) := \{w^* \in \mathbb{R}^n \mid \langle w^*, \tau(u, v)\rangle \leq 0, \forall u \in B\}.$$

Yang and Craven [6] gave the necessary condition for nonsmooth optimization problem P. They also consider an assumption (A) for their results which is define as

$$\limsup_{a \downarrow 0} [\frac{\bar{\Phi}(u + ak) - \bar{\Phi}(k)}{a}] \leq \bar{\Phi}_\tau^0(u; k) \text{ for each } k. \tag{A}$$

Theorem 2.11. [6] *Let $\bar{\Phi}$ and $\bar{\Psi}_\iota$ $(\iota = 1, 2, ...n)$ be locally Lipschitz functions on B and v be an optimal solution of problem P. Assume that the assumption A is satisfied for $\bar{\Phi}$ and $\bar{\Psi}_\iota (\iota = 1, 2, ...n)$ at v. Then, there exists nonnegative constants $C_\iota (\iota = 0, 1, 2, ...)$ not all zero, such that $0 \in C_0 \partial^\tau \bar{\Phi}(v) + \sum_{\iota=1}^l C_\iota \partial^\tau \bar{\Psi}_\iota(v) + N_B^\tau(v)$, $C_\iota \bar{\Psi}_\iota(v) = 0,\ \iota = 1, 2, ...n$.*

Theorem 2.12. [6] *Let $\bar{\Phi}$ and $\bar{\Psi}_\iota$ $(\iota = 1, 2, ...n)$ be locally Lipschitz functions on B and v be an optimal solution of problem P. Assume that the assumption A is satisfied for $\bar{\Phi}$ and $\bar{\Psi}_\iota$ $(\iota = 1, 2, ...n)$ at v and that $\bar{\Psi}_\iota$ $(\iota = 1, 2, ...n)$ satisfy $\bar{\Psi}_{i\tau}^0(v; h) \leq \bar{\Psi}_\iota(u) - \bar{\Psi}_\iota(v)$, where $u \in B, h = \tau(u, v) \in H(v)$, and Slater condition: for some $u^* \in B, \bar{\Psi}_\iota(u^*) < 0$. Then, there exists nonnegative constants C_ι $(\iota = 0, 1, 2, ...n)$ not all zero, such that $0 \in \partial^\tau \bar{\Phi}(v) + \sum_{\iota=1}^l C_\iota \partial^\tau \bar{\Psi}_\iota(v) + N_B^\tau(v)$, $C_\iota \bar{\Psi}_\iota(v) = 0,\ \iota = 1, 2, ...n$.*

3. Main Results

3.1. **Wolfe type dual.** For problem P, WTD problem (W):

$$\begin{gathered}\text{Maximize } \bar{\Phi}(v)+\sum_{\iota=1}^{l}\bar{\Psi}_\iota(v)\\ \text{Subject to } 0\in\partial^\tau\bar{\Phi}(v)+\sum_{\iota=1}^{l}C_\iota\partial^\tau\bar{\Psi}_\iota(v)+N_B^\tau(v),\\ \text{where } C_\iota\geq 0 \text{ and not all } C_\iota=0, \iota=1,2,....n.\end{gathered}\tag{W}$$

Theorem 3.1. *(WDT) Let u^* be feasible for problem P and (v^*,C) where $C=(C_\iota),\ \iota=1,2,...n$, be feasible for problem W. Let $\bar{\Phi}$ and $\bar{\Psi}_\iota,\ \iota=1,2,.....n$, be all invex functions with respect to a common τ. Then, $\bar{\Phi}(u^*)\geq\bar{\Phi}(v^*)+\sum_{\iota=1}^{l}C_\iota\bar{\Psi}_\iota(v^*)$.*

Proof. Let u^* be feasible for problem P and (v^*,C), where $C=(C_\iota),\ \iota=1,2,...n$, is feasible for problem W. Then, we have

$$\bar{\Psi}_\iota(u^*)\leq 0,\ \forall\iota=1,2,....n,\tag{3.1}$$

and

$$0\in\partial^\tau\bar{\Phi}(v^*)+\sum_{\iota=1}^{l}C_\iota\partial^\tau\bar{\Psi}_\iota(v^*)+N_B^\tau(v^*).\tag{3.2}$$

Then, there exists $t\in\partial^\tau\bar{\Phi}(v^*),\ t_\iota\in\partial^\tau\bar{\Psi}_\iota(v^*)$ and $\bar{t}\in N_B^\tau(v^*),\iota=1,2,....n$, such that $0=t+\sum_{\iota=1}^{l}C_\iota t_\iota+\bar{t}$, $t+\sum_{\iota=1}^{l}C_\iota t_\iota=-\bar{t}$. Since $\bar{\Phi}$ is invex at v^* w.r.t. τ, then

$$\bar{\Phi}(u^*)-\bar{\Phi}(v^*)\geq\langle t,\tau(u^*,v^*)\rangle.\tag{3.3}$$

Note that $\bar{\Psi}_\iota$ is invex at v^* w.r.t. τ and $C_\iota\geq 0$. Then

$$\sum_{\iota=1}^{l}C_\iota\bar{\Psi}_\iota(u^*)-\sum_{\iota=1}^{l}C_\iota\bar{\Psi}_\iota(v^*)\geq\left\langle\sum_{\iota=1}^{l}C_\iota t_\iota,\tau(u^*,v^*)\right\rangle.$$

From (3.1), we can write

$$0\geq\sum_{\iota=1}^{l}C_\iota\bar{\Psi}_\iota(v^*)+\left\langle\sum_{\iota=1}^{l}C_\iota t_\iota,\tau(u^*,v^*)\right\rangle.\tag{3.4}$$

By adding Eqs. (3.3) & (3.4), we get

$$\bar{\Phi}(u^*)-\bar{\Phi}(v^*)\geq\sum_{\iota=1}^{l}C_\iota\bar{\Psi}_\iota(v^*)+\left\langle t+\sum_{\iota=1}^{l}C_\iota t_\iota,\tau(u^*,v^*)\right\rangle.$$

Thus $\bar{\Phi}(u^*)-\bar{\Phi}(v^*)\geq\sum_{\iota=1}^{l}C_\iota\bar{\Psi}_\iota(v^*)-\left\langle\bar{t},\tau(u^*,v^*)\right\rangle$. Since $\bar{t}\in N_B^\tau(v^*)$, $\langle\bar{t},\tau(u,v)\rangle\leq 0$. It follows that $\bar{\Phi}(u^*)\geq\bar{\Phi}(v^*)+\sum_{\iota=1}^{l}C_\iota\bar{\Psi}_\iota(v^*)$. □

Theorem 3.2. *(SDT) Let $\bar{\Phi}$ and $\bar{\Psi}_\iota,\ \iota=1,2,....n$, satisfy the invexity assumption of Theorem 3.1. If u^* is an optimal solution for problem P and a suitable constraint qualification satisfies for problem P. Then, there exists $C=(C_\iota),\ \iota=1,2,.....n$, such that ($u^*$,C) is an optimal solution of problem W and the respective objective values are equal.*

Proof. As u^* is an optimal solution for problem P and a suitable constraint qualification is satisfied. Then, there exists $C = C_\iota,\ \iota = 1,2,...n$, such that KKT condition for problem P is satisfied. That is, $0 \in \partial^\tau \bar{\Phi}(u^*) + \sum_{\iota=1}^{l} C_\iota \partial^\tau \bar{\Psi}_\iota(u^*) + N_B^\tau(u^*)$ and $\sum_{\iota=1}^{l} C_\iota \bar{\Psi}_\iota(u^*) = 0$. Therefore (u^*, C) is feasible for W. It follows from Theorem 3.1 that

$$\bar{\Phi}(u^*) \geq \bar{\Phi}(v^*) + \sum_{\iota=1}^{l} C_\iota \bar{\Psi}_\iota(v^*). \tag{3.5}$$

For any feasible solution (v^*, C) for problem W,

$$\bar{\Phi}(u^*) = \bar{\Phi}(u^*) + \sum_{\iota=1}^{l} C_\iota \bar{\Psi}_\iota(u^*). \tag{3.6}$$

By using (3.6) in (3.5), we have

$$\bar{\Phi}(u^*) + \sum_{\iota=1}^{l} C_\iota \bar{\Psi}_\iota(u^*) \geq \bar{\Phi}(v^*) + \sum_{\iota=1}^{l} C_\iota \bar{\Psi}_\iota(v^*).$$

Thus (u^*, C) is optimal for problem W and the respective objective values are equal. □

Theorem 3.3. *(SCDT) Let $\bar{\Phi}$ and $\bar{\Psi}_\iota,\ \iota = 1,2,.....n$, are invex with respect to a common τ. Let u^* be optimal for problem P and (v^*, C) be optimal for problem W. If suitable constraint qualification is satisfied for problem P and $\bar{\Phi}$ is strictly invex at v^*, then $u^* = v^*$.*

Proof. Let $u^* \neq v^*$. Then, by SDT for W, there exists $C^* = (C_\iota^*),\ \iota = 1,2,....n$, and $C_\iota^* \geq 0$ such that (u^*, C^*) is optimal for problem W. Then,

$$\bar{\Phi}(u^*) = \bar{\Phi}(u^*) + \sum_{\iota=1}^{l} C_\iota^* \bar{\Psi}_\iota(u^*) = \bar{\Phi}(v^*) + \sum_{\iota=1}^{l} C_\iota \bar{\Psi}_\iota(v^*). \tag{3.7}$$

In view of (3.2), there exists $t \in \partial^\tau \bar{\Phi}(v^*),\ t_\iota \in \partial^\tau \bar{\Psi}_\iota(v^*)$, and $\bar{t} \in N_B^\tau(v^*),\ \iota = 1,2,....n$, such that

$$t + \sum_{\iota=1}^{l} C_\iota t_\iota = -\bar{t}. \tag{3.8}$$

Since $\bar{\Phi}$ is a strictly invex at v^*, then

$$\bar{\Phi}(u^*) - \bar{\Phi}(v^*) > \langle t, \tau(u^*, v^*) \rangle, \tag{3.9}$$

and $\bar{\Psi}_\iota,\ \iota = 1,2,....n$, are invex and $C \geq 0$. It follows that $\sum_{\iota=1}^{l} C_\iota \bar{\Psi}_\iota(u^*) - \sum_{\iota=1}^{l} C_\iota \bar{\Psi}_\iota(v^*) \geq \langle \sum_{\iota=1}^{l} C_\iota t_\iota, \tau(u^*, v^*) \rangle$. Since $\bar{\Psi}_\iota(u^*) \geq 0$, then

$$0 \geq \sum_{\iota=1}^{l} C_\iota \bar{\Psi}_\iota(v^*) + \left\langle \sum_{\iota=1}^{l} C_\iota t_\iota, \tau(u^*, v^*) \right\rangle. \tag{3.10}$$

From (3.9)-(3.10), one has $\bar{\Phi}(u^*) - \bar{\Phi}(v^*) > \sum_{\iota=1}^{l} C_\iota \bar{\Psi}_\iota(v^*) + \left\langle t + \sum_{\iota=1}^{l} C_\iota t_\iota, \tau(u^*, v^*) \right\rangle$. By using (3.8), one has

$$\bar{\Phi}(u^*) - \bar{\Phi}(v^*) > \sum_{\iota=1}^{l} C_\iota \bar{\Psi}_\iota(v^*) - \langle \bar{t}, \tau(u^*, v^*) \rangle.$$

Since $\bar{t} \in N_B^\tau(v^*)$, $\langle \bar{t}, \tau(u, v) \rangle \leq 0$. It follows that $\bar{\Phi}(u^*) > \bar{\Phi}(v^*) + \sum_{\iota=1}^{l} C_\iota \bar{\Psi}_\iota(v^*)$. This contradicts (3.7). Hence $u^* = v^*$. □

3.2. **Mond-Weir type dual.** For problem P, MWTD problem (M):

$$\begin{aligned} &\text{Maximize } \bar{\Phi}(v) \\ &\text{Subject to } 0 \in \partial^{\tau}\bar{\Phi}(v) + \sum_{\iota=1}^{l} C_{\iota}\partial^{\tau}\bar{\Psi}_{\iota}(v) + N_B^{\tau}(v), \\ &\sum_{\iota=1}^{l} C_{\iota}\bar{\Psi}_{\iota}(v) \geq 0, \\ &\text{where } C_{\iota} \geq 0 \text{ and not all } C_{\iota} = 0, \iota = 1,2,.....n. \end{aligned} \tag{M}$$

Theorem 3.4. *(WDT) Let u^* be feasible for problem P and (v^*,C), where $C=(C_{\iota}), \iota = 1,2,.....n$, be feasible for problem M. Let $\bar{\Phi}$ and $\bar{\Psi}_{\iota},\ \iota = 1,2,.....n$, be all invex with respect to a common τ. Then, $\bar{\Phi}(u^*) \geq \bar{\Phi}(v^*)$.*

Proof. Let u^* be feasible for P and (v^*,C) be feasible for problem M. Then, we have

$$\bar{\Psi}_{\iota}(u^*) \leq 0, \iota = 1,2,....n. \tag{3.11}$$

$$0 \in \partial^{\tau}\bar{\Phi}(v^*) + \sum_{\iota=1}^{l} C_{\iota}\partial^{\tau}\bar{\Psi}_{\iota}(v^*) + N_B^{\tau}(v), \tag{3.12}$$

$$\sum_{\iota=1}^{l} C_{\iota}\bar{\Psi}_{\iota}(v^*) \geq 0. \tag{3.13}$$

Then, there exists $t \in \partial^{\tau}\bar{\Phi}(v^*),\ t_{\iota} \in \partial^{\tau}\bar{\Psi}_{\iota}(v^*)$ and $\bar{t} \in N_B^{\tau}(v^*)$ such that

$$t + \sum_{\iota=1}^{l} C_{\iota}t_{\iota} = -\bar{t}. \tag{3.14}$$

Since $\bar{\Phi}$ is invex at v^* w.r.t. τ, then

$$\bar{\Phi}(u^*) - \bar{\Phi}(v^*) \geq \langle t, \tau(u^*,v^*)\rangle, \tag{3.15}$$

and $\bar{\Psi}_{\iota}$ is invex at v^* w.r.t. τ and $C_{\iota} \geq 0$. It follows that

$$\sum_{\iota=1}^{l} C_{\iota}\bar{\Psi}_{\iota}(u^*) - \sum_{\iota=1}^{l} C_{\iota}\bar{\Psi}_{\iota}(v^*) \geq \left\langle \sum_{\iota=1}^{l} C_{\iota}t_{\iota}, \tau(u^*,v^*) \right\rangle. \tag{3.16}$$

From (3.11), we can write $0 \geq \sum_{\iota=1}^{l} C_{\iota}\bar{\Psi}_{\iota}(v^*) + \langle \sum_{\iota=1}^{l} C_{\iota}t_{\iota}, \tau(u^*,v^*)\rangle$. By (3.15)-(3.16), one has $\bar{\Phi}(u^*) - \bar{\Phi}(v^*) \geq \sum_{\iota=1}^{l} C_{\iota}\bar{\Psi}_{\iota}(v^*) + \left\langle t + \sum_{\iota=1}^{l} C_{\iota}t_{\iota}, \tau(u^*,v^*)\right\rangle$. By using (3.14), one has

$$\bar{\Phi}(u^*) - \bar{\Phi}(v^*) \geq \sum_{\iota=1}^{l} C_{\iota}\bar{\Psi}_{\iota}(v^*) - \left\langle \bar{t}, \tau(u^*,v^*)\right\rangle.$$

Since $\bar{t} \in N_B^{\tau}(v^*)$, $\langle \bar{t}, \tau(u,v)\rangle \leq 0$. It follows that $\bar{\Phi}(u^*) \geq \bar{\Phi}(v^*) + \sum_{\iota=1}^{l} C_{\iota}\bar{\Psi}_{\iota}(v^*)$. From (3.13), one has $\bar{\Phi}(u^*) \geq \bar{\Phi}(v^*)$. □

Theorem 3.5. *(SDT) Let $\bar{\Phi}$ and $\bar{\Psi}_{\iota},\ \iota = 1,2,.....n$, satisfy the invex assumption of Theorem 3.4. If u^* is an optimal solution for problem P and P satisfies a suitable constraint qualification, then there exists $C=(C_{\iota}), \iota = 1,2,.....n$, such that (u^*,C) is an optimal solution of problem M and respective objective values are equal.*

Proof. Since u^* is a solution of problem P and a suitable constraint qualification is satisfied, then there exists $C=(C_\iota),\ \iota=1,2,....n$, such that the necessary conditions of optimality for problem P are satisfied as $0\in\partial^\tau\bar{\Phi}(u^*)+\sum_{\iota=1}^{l}C_\iota\partial^\tau\bar{\Psi}_\iota(u^*)+N_B^\tau(u^*)$ and $\sum_{\iota=1}^{l}C_\iota\bar{\Psi}_\iota(u^*)=0$. We can see that (u^*,C) is feasible for problem M. By WDT, we can write $\bar{\Phi}(u^*)\geq\bar{\Phi}(v)$, for all feasible solution (v,C) for problem M. Thus we can conclude that (u^*,C) is optimal solution of problem M and respective objective values are equal. □

Theorem 3.6. *(SCDT) Let $\bar{\Phi}$ and $\bar{\Psi}_\iota,\ \iota=1,2,....n$, be invex functions with respect to a common τ. Let u^* be optimal for problem P and (v^*,C) be optimal for problem M. If a constraints qualification is satisfied for problem P and $\bar{\Phi}$ is strictly invex at v^*, then $u^*=v^*$.*

Proof. On the contrary, let $u^*\neq v^*$. By Theorem 3.5, we can write $\bar{\Phi}(u^*)=\bar{\Phi}(v^*)$. As (u^*,C) is optimal for problem M, one sees from (3.14) that there exists $t\in\partial^\tau\bar{\Phi}(v^*),\ t_\iota\in\partial^\tau\bar{\Psi}_\iota(v^*)$, and $\bar{t}\in N_B^\tau(v^*),\ \iota=1,2,....n$, such that $t+\sum_{\iota=1}^{l}C_\iota t_\iota=-\bar{t}$. Since $\bar{\Phi}$ is a strictly invex at v^*, then

$$\bar{\Phi}(u^*)-\bar{\Phi}(v^*)>\langle t,\tau(u^*,v^*)\rangle. \tag{3.17}$$

Since $\bar{\Psi}_\iota,\ \iota=1,2,...n$, with $C\geq 0$ are invex, then

$$\sum_{\iota=1}^{l}C_\iota\bar{\Psi}_\iota(u^*)-\sum_{\iota=1}^{l}C_\iota\bar{\Psi}_\iota(v^*)\geq\langle\sum_{\iota=1}^{l}C_\iota t_\iota,\tau(u^*,v^*)\rangle.$$

From (3.11), we can write

$$0\geq\sum_{\iota=1}^{l}C_\iota\bar{\Psi}_\iota(v^*)+\left\langle\sum_{\iota=1}^{l}C_\iota t_\iota,\tau(u^*,v^*)\right\rangle. \tag{3.18}$$

By adding (3.17) & (3.18), one further has $\bar{\Phi}(u^*)-\bar{\Phi}(v^*)>\sum_{\iota=1}^{l}C_\iota\bar{\Psi}_\iota(v^*)-\langle\bar{t},\tau(u^*,v^*)\rangle$. Since $\bar{t}\in N_B^\tau(v^*)$, $\langle\bar{t},\tau(u,v)\rangle\leq 0$. It follows that $\bar{\Phi}(u^*)>\bar{\Phi}(v^*)+\sum_{\iota=1}^{l}C_\iota\bar{\Psi}_\iota(v^*)$. From (3.13), one has $\bar{\Phi}(u^*)>\bar{\Phi}(v^*)$, which contradicts $\bar{\Phi}(u^*)=\bar{\Phi}(v^*)$. Hence, $u^*=v^*$. □

We can also demonstrate some results for MWTD by further relaxing the invexity conditions.

Theorem 3.7. *(WDT) Let u^* be feasible for problem P and (v^*,C), where $C=(C_\iota)$,$\iota=1,2,.....n$, is feasible for problem M. If $\bar{\Phi}$ is pseudo-invex and $\bar{\Psi}_\iota,\ \iota=1,2...,n$. are quasi-invex, then $\bar{\Phi}(u^*)\geq\bar{\Phi}(v^*)$.*

Proof. Since u^* and (v^*,C) are feasible for problem P and problem M, respectively, then $C_\iota\bar{\Psi}_\iota(u^*)\leq C_\iota\bar{\Psi}_\iota(v^*)$ for all $C_\iota\geq 0$. From the quasi-invexity of $C_\iota\bar{\Psi}_\iota$, we have $\langle C_\iota t_\iota,\tau(u^*,v^*)\rangle\leq 0$ for all $t_\iota\in\partial^\tau\bar{\Psi}_\iota(v^*)$. We can write $\langle\sum_{\iota=1}^{n}C_\iota t_\iota,\tau(u^*,v^*)\rangle\leq 0$ for all $t_\iota\in\partial^\tau\bar{\Psi}_\iota(v^*)$. From (3.12), we see that there exists $t\in\partial^\tau\bar{\Phi}(v^*),\ t_\iota\in\partial^\tau\bar{\Psi}_\iota(v^*)$, and $\bar{t}\in N_B^\tau(v^*),\ \iota=1,2,....n$, such that $0=t+\sum_{\iota=1}^{l}C_\iota t_\iota+\bar{t}$. We can further write $\langle t,\tau(u^*,v^*)\rangle\geq 0$ for all $t\in\partial^\tau\bar{\Phi}(v^*)$. From pseudo-invexity of $\bar{\Phi}$, we find $\bar{\Phi}(u^*)\geq\bar{\Phi}(v^*)$. □

Theorem 3.8. *(SDT) If u^* is an optimal solution for problem P with a suitable constraint qualification. If the above theorem holds for problems P and M, then there exists $C=(C_\iota),\iota=1,2,.....n$ such that (u^*,C) is an optimal solution of problem M and respective objective values are equal.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.5. □

Theorem 3.9. *(SCDT) Let $\bar{\Phi}$ be pseudo-invex and $\bar{\Psi}_\iota,\ \iota = 1,2,....n$, be quasi-invex functions with respect to a common τ. Let u^* be optimal for problem P and (v^*,C) be optimal for problem M. If a constraints qualification is satisfied for problem P and $\bar{\Phi}$ is strictly pseudo-invex for problem P at v^*, then $u^* = v^*$.*

Proof. Let $u^* \neq v^*$. By Theorem 3.8, we can write $\bar{\Phi}(u^*) = \bar{\Phi}(v^*)$. Since $\bar{\Psi}_\iota(u^*) \leq 0$ and $C_\iota \geq 0$, then $C_\iota\bar{\Psi}_\iota(u^*) \leq C_\iota\bar{\Psi}_\iota(v^*)$. The quasi-invexity of $\bar{\Psi}_\iota$ yields $\langle C_\iota t_\iota, \tau(u^*,v^*)\rangle \leq 0$ for all $t_\iota \in \partial^\tau\bar{\Psi}_\iota(v^*)$. We can write

$$\left\langle \sum_{\iota=1}^{n} C_\iota t_\iota, \tau(u^*,v^*) \right\rangle \leq 0\ \forall t_\iota \in \partial^\tau\bar{\Psi}_\iota(v^*). \tag{3.19}$$

From (3.12), there exists $t \in \partial^\tau\bar{\Phi}(v^*),\ t_\iota \in \partial^\tau\bar{\Psi}_\iota(v^*)$ and $\bar{t} \in N_B^\tau(v^*),\ \iota = 1,2,....n$, such that $0 = t + \sum_{\iota=1}^{l} C_\iota t_\iota + \bar{t}$. From (3.19) and the equation above, we can write $\langle t, \tau(u^*,v^*)\rangle \geq 0$ for all $t \in \partial^\tau\bar{\Phi}(v^*)$. From strict pseudo-invexity of $\bar{\Phi}$, we find $\bar{\Phi}(u^*) > \bar{\Phi}(v^*)$. This contradicts $\bar{\Phi}(u^*) = \bar{\Phi}(v^*)$. Hence, $u^* = v^*$. □

Example 3.10. Consider a primal problem:

P inf $\Phi^*(v)$ subject to $v \in B$, where $B := \{v \in \mathbb{R} \mid \Psi_\iota^*(v) \leq 0, \iota = 1,2,3\}$.

WTD problem W:

$$\text{Maximize } \Phi^*(v) + \sum_{\iota=1}^{3} \Psi_\iota^*(v)$$

$$\text{Subject to } 0 \in \partial^\tau\Phi^*(v) + \sum_{\iota=1}^{3} C_\iota\partial^\tau\Psi_\iota^*(v) + N_B^\tau(v),$$

Mond-Weir dual problem M:

$$\text{Maximize } \Phi^*(v)$$

$$\text{Subject to } 0 \in \partial^\tau\Phi^*(v) + \sum_{\iota=1}^{3} C_\iota\partial^\tau\Psi_\iota^*(v) + N_B^\tau(v),$$

$$\sum_{\iota=1}^{3} C_\iota\Psi_\iota^*(v) \geq 0,$$

where $C_\iota \geq 0$ and not all $C_\iota = 0$, $\iota = 1,2,....n$ & $v \in [-3,3]$. We consider $\Phi^*(v) =\mid v-1 \mid +2$ and

$$\Psi_1^*(v) = \begin{cases} v-1 & v \leq 2, \\ 2v-3 & v \geq 2, \end{cases}$$

$\Psi_2^*(v) =\mid v \mid -1, \Psi_3^*(v) =\mid v \mid -2$, & $\tau(u,v) = u - v$. Here Φ^*, Ψ_ι^* satisfies Assumption A at point 1 and $\partial^\tau\Phi^*(1) = [-1,1]$. For $\iota = 1$, $\partial^\tau\Psi_1^*(1) = 1$; For $\iota = 2$, $\partial^\tau\Psi_2^*(1) = 1$; For $\iota = 3$, $\partial^\tau\Psi_3^*(1) = 1$; and $N_B^\tau(1) = [0,\infty)$. Now, $\partial^\tau\Phi^*(1) + \sum_{\iota=1}^{3} C_\iota\partial^\tau\Psi_\iota^*(1) + N_B^\tau(1) = [-1,1] + C_1.1 + C_2.1 + C_3.1 + [0,\infty) = [-1,\infty)$, for all $C_\iota \geq 0$ and $0 \in [-1,\infty)$. We can see that 1 is feasible for problem P and also feasible for problems W and M. We can also see that WDT and SDT conditions for problems W and M are also satisfied at 1. Since $\Phi^*(v)$ is strictly invex function at 1, the results for SCDT are also satisfied at 1 for problems W and M. Moreover, we can conclude that 1 is the optimal solution for problems P, W, and M.

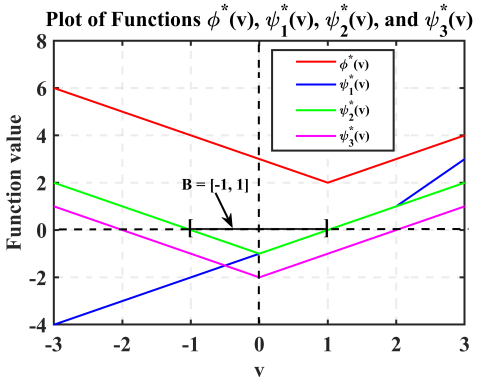


FIGURE 1. graphical representation of problem P showing the $\Phi^*(v)$ and $\Psi_\iota^*(v)$, *where* $\iota = 1,2,3$ are invex function with respect to same $\tau(u,v) = u - v$. & feasible set $B = [-1,1]$.

4. CONCLUSION

We considered a nonsmooth constrained optimization problem and defined WTD (W) and MWTD (M) problems. Further, we derived WDT, SDT, and SCDT for considered WTD and MWTD problems under generalized invexity by using τ-subdifferential. Moreover, we verified our results with some examples. For a particular case, τ-subdifferential works as convex subdifferetial. In future, this research could be helpful to solve mixed-type dual problems and the τ-subdifferential could be useful to solve nonsmooth nonconvex optimization problems.

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