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SPATIAL AUTOREGRESSION MODEL: DEPENDENT ERRORS

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Dedicated to Professor Ram Mohapatra

Abstract. Asymptotic results are given for testing $H_0 : \alpha = \beta = 1$ for the doubly geometric model $Z_{ij} = \alpha Z_{i-1,j} + \beta Z_{i,j-1} - \alpha\beta Z_{i-1,j-1} + \delta_{ij}$, where δ_{ij} is a dependent error structure. The Gauss-Newton estimation procedure is used.

Keywords. Dependent errors; Spatial model; Unit root testing.

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1. Introduction and Preliminaries

Consider the spatial model $\{Z_{ij}\}$ on $\mathbb{N}^2$ satisfying

$$Z_{ij} = \alpha Z_{i-1,j} + \beta Z_{i,j-1} - \alpha\beta Z_{i-1,j-1} + \delta_{ij}, \qquad 1 \le i, j \le n. \tag{1.1}$$

Model (1.1) was introduced by [7, 8] when $\delta_{ij} = \varepsilon_{ij}$, where $\{\varepsilon_{ij}\}$ are independent and identically distributed random variables with mean zero and finite fourth moment, and was used to analyze field data. Asymptotic properties of estimators of the parameters in model (1.1) have been attained in the stationary case and can be found in the work of [2, 3, 4] and the references contained therein. Moreover, they illustrate the feasibility of data obeying model (1.1) and having nonstationary errors. Motivated by these works and following [9, 10], the form of our error structure is

$$\delta_{ij} = \varepsilon_{ij} - \theta_n \varepsilon_{i-1,j}.$$

As documented in [9, 10], this model can closely approximate economic data when θ_n is close to one. Further, [3] fit a spatial model having moving average parameters in both directions to the yield of barley data. One of the estimated parameters is near one and much larger than the others indicating the possibility of a large parameter in only one direction.

All of our work is in the spatial setting. An alternate approach is given in [1] using the periodogram ordinate in the spectral domain. In contrast, the dependent error structure adopted here induces unique scaling behavior necessary for convergence.

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Throughout this work, the assumptions listed below are made in model (1.1). The existence of initial estimators listed in (A5) is needed in the Gauss-Newton procedure and is proved in Lemma 3.1.

(A1) $\alpha = \beta = 1$.
(A2) Z_{ij}, ε_{ij}, and all derived processes are 0 when $i \leq 0$ or $j \leq 0$.
(A3) $\delta_{ij} = \varepsilon_{ij} - \theta_n \varepsilon_{i-1,j}, 1 \leq i, j \leq n$, where ε_{ij} are i.i.d. with mean zero, variance σ^2, and finite fourth moments.
(A4) $\theta_n \to 1$ and $n^{\rho}(1-\theta_n) \to 1$ for some $0 < \rho < \frac{1}{8}$.
(A5) $n^{1-2\rho}(\overline{\alpha}_n - \alpha) = O_p(1)$ and $n^{3/2}(\overline{\beta}_n - \beta) = O_p(1)$, for some initial estimators $\overline{\alpha}_n$ and $\overline{\beta}_n$.

The authors' primary result is given below and proved in Section 3.

Theorem 1.1. *Under the assumptions of model (1.1) and (A1)-(A5), the Gauss-Newton estimators* $\hat{\theta}_n' = (\hat{\alpha}_n, \hat{\beta}_n)$ *of* $\theta_0' = (1,1)$ *satisfy* $(n^{1-2\rho}(\hat{\alpha}_n - 1), n^{3/2}(\hat{\beta}_n - 1)) \xrightarrow{D} (-2, \mathcal{N}(0,3))$.

Using model (1.1), we define

$$X_{ij} = Z_{ij} - \beta Z_{i,j-1} \text{ and } Y_{ij} = Z_{ij} - \alpha Z_{i-1,j}. \tag{1.2}$$

Then $X_{ij} = \alpha(Z_{i-1,j} - \beta Z_{i-1,j-1}) + \delta_{ij} = \alpha X_{i-1,j} + \delta_{ij}$ and likewise, $Y_{ij} = \beta Y_{i,j-1} + \delta_{ij}$. Using (A2),

$$X_{ij} = \sum_{k=1}^{i} \alpha^{i-k} \delta_{kj} \text{ and } Y_{ij} = \sum_{\ell=1}^{j} \beta^{j-\ell} \delta_{i\ell}$$

and, by (A1),

$$X_{i-1,j} = \sum_{k=1}^{i-1} \delta_{kj} = \sum_{k=1}^{i-1} (\varepsilon_{kj} - \theta_n \varepsilon_{k-1,j}) \text{ and } Y_{i,j-1} = \sum_{\ell=1}^{j-1} \delta_{i\ell} = \sum_{\ell=1}^{j-1} (\varepsilon_{i\ell} - \theta_n \varepsilon_{i-1,\ell}).$$

The primary components in finding an asymptotic test involve determining the scaling factors needed to get convergence of $\sum_{i,j=1}^{n} X_{i-1,j}\delta_{ij}$ and $\sum_{i,j=1}^{n} Y_{i,j-1}\delta_{ij}$ and is shown in Lemma 2.3 (ii) and Lemma 2.7. These two results are combined to give Corollary 2.8. Corollary 2.8 and Lemma 3.2 are the main results needed to prove Theorem 1.1.

The paper is organized as follows. The scaling required for the component estimators to converge is acquired in Section 2. The existence and convergence of $\hat{\theta}_n$ are shown in Section 3. Section 4 provides a hypothesis test and numerical results from a simulation. Concluding remarks are made in Section 5.

2. Component Estimators

The following order in probability results are used in subsequent proofs. These are straightforward but some are tedious to verify.

Lemma 2.1. *Given model (1.1) and assumptions (A1)-(A5), the following statements are true*

(i) $\frac{1}{n^{3-2\rho}} \sum_{i,j=1}^{n} X_{i-1,j}^2 = \frac{\sigma^2}{2} + o_p(1)$.
(ii) $\frac{1}{n^3} \sum_{i,j=1}^{n} Y_{i,j-1}^2 = \sigma^2 + o_p(1)$.
(iii) $\sum_{i,j=1}^{n} Z_{i-1,j-1}^2 = O_p(n^4)$.
(iv) $\sum_{i,j=1}^{n} X_{i-1,j} Y_{i,j-1} = O_p(n^2)$.

(v) $\sum_{i,j=1}^{n} X_{i-1,j}Z_{i-1,j-1} = O_p(n^{3-2\rho})$.
(vi) $\sum_{i,j=1}^{n} Y_{i,j-1}Z_{i-1,j-1} = O_p(n^{7/2})$.
(vii) $\sum_{i,j=1}^{n} Z_{i-1,j-1}\delta_{ij} = O_p(n^{2-\rho})$.

When properly normalized the sequences $\sum_{i,j=1}^{n} X_{i-1,j}\delta_{ij}$ and $\sum_{i,j=1}^{n} Y_{i,j-1}\delta_{ij}$ have a central role in proving Theorem 1.1. Convergence of these sequences is given in this section. It will be assumed, without further mention, that (A1)-(A4) are satisfied.

Lemma 2.2. $\frac{1}{n^{3-2\rho}}\sum_{i,j=1}^{n} X_{i-1,j}^2 = \frac{\sigma^2}{2} + o_p(1)$ *and* $\frac{1}{n^{3-2\rho}}\sum_{i,j=1}^{n}\mathbb{E}[X_{i-1,j}^2] \to \frac{\sigma^2}{2}$.

Proof. By (1.2), one has

$$X_{i-1,j} = \sum_{k=1}^{i-1}\varepsilon_{kj} - \theta_n\sum_{k=2}^{i-1}\varepsilon_{k-1,j} = \sum_{k=1}^{i-1}\varepsilon_{kj} - \theta_n\sum_{k=1}^{i-2}\varepsilon_{kj} = (1-\theta_n)\sum_{k=1}^{i-2}\varepsilon_{kj} + \varepsilon_{i-1,j}$$

and thus

$$X_{i-1,j}^2 = (1-\theta_n)^2\left(\sum_{k=1}^{i-2}\varepsilon_{kj}\right)^2 + 2(1-\theta_n)\varepsilon_{i-1,j}\sum_{k=1}^{i-2}\varepsilon_{kj} + \varepsilon_{i-1,j}^2. \tag{2.1}$$

Then $\frac{1}{n^{3-2\rho}}\sum_{i,j=1}^{n} X_{i-1,j}^2$ is given by

$$\frac{(n^\rho(1-\theta_n))^2}{n^3}\sum_{i,j=1}^{n}\left(\sum_{k=1}^{i-2}\varepsilon_{kj}\right)^2 + \frac{2n^\rho(1-\theta_n)}{n^{3-\rho}}\sum_{i,j=1}^{n}\left(\varepsilon_{i-1,j}\sum_{k=1}^{i-2}\varepsilon_{kj}\right) + \frac{1}{n^{3-2\rho}}\sum_{i,j=1}^{n}\varepsilon_{i-1,j}^2.$$

It follows from Lemma 2.1(i) [5] that $\frac{1}{n^3}\sum_{i,j=1}^{n}(\sum_{k=1}^{i-2}\varepsilon_{kj})^2 = \frac{\sigma^2}{2} + o_p(1)$. Moreover, since $n^\rho(1-\theta_n) \to 1$, $\frac{(n^\rho(1-\theta_n))^2}{n^3}\sum_{i,j=1}^{n}(\sum_{i,j=1}^{i-2}\varepsilon_{kj})^2 = \frac{\sigma^2}{2} + o_p(1)$. Observe that

$$\begin{aligned}
\operatorname{Var}\sum_{i,j=1}^{n}\left(\varepsilon_{i-1,j}\sum_{k=1}^{i-2}\varepsilon_{kj}\right) &= \sum_{i,i',j=1}^{n}\operatorname{Cov}\left(\varepsilon_{i-1,j}\sum_{k=1}^{i-2}\varepsilon_{kj}, \varepsilon_{i'-1,j}\sum_{\ell=1}^{i'-2}\varepsilon_{\ell j}\right)\\
&= n\sum_{i=1}^{n}\operatorname{Var}\left(\varepsilon_{i-1,j}\sum_{k=1}^{i-2}\varepsilon_{kj}\right)\\
&= n\sum_{i=3}^{n}(i-2)\sigma^4\\
&= O(n^3).
\end{aligned}$$

Then $\sum_{i,j=1}^{n}(\varepsilon_{i-1,j}\sum_{k=1}^{i-2}\varepsilon_{kj}) = O_p(n^{3/2})$ since it has mean zero. Further, $\sum_{i,j=1}^{n}\varepsilon_{i-1,j}^2 = O_p(n^2)$ and it follows that $\frac{1}{n^{3-2\rho}}\sum_{i,j=1}^{n} X_{i-1,j}^2 = \frac{\sigma^2}{2} + o_p(1)$. Employing (2.1), $\mathbb{E}[X_{i-1,j}^2] = (1-\theta_n)^2\operatorname{Var}(\sum_{k=1}^{i-2}\varepsilon_{kj}) + \sigma^2$ and thus

$$\begin{aligned}
\frac{1}{n^{3-2\rho}}\sum_{i,j=1}^{n}\mathbb{E}[X_{i-1,j}^2] &= \frac{n^{2\rho}(1-\theta_n)^2}{n^3}\sum_{i,j=1}^{n}(i-2)\sigma^2 + \frac{n^2\sigma^2}{n^{3-2\rho}}\\
&= \frac{(n^\rho(1-\theta_n))^2}{n^3}\frac{(n-2)(n-1)n\sigma^2}{2} + o(1) \to \frac{\sigma^2}{2}.
\end{aligned}$$

Hence $\frac{1}{n^{3-2\rho}}\sum_{i,j=1}^{n}\mathbb{E}[X_{i-1,j}^2] \to \frac{\sigma^2}{2}$ as $n \to \infty$. □

Observe that if $n^\rho(1-\theta_n) \to \lambda$ for some $\lambda \in (0,\infty)$, then the limiting expression in Lemma 2.2 is scaled by λ^2 and consequently the limiting distribution of the α component in Theorem 1.1 would be $\frac{-2}{\lambda^2}$ rather than -2. Hence, this normalization to unity is adopted without loss of generality.

Lemma 2.3. *The sequence $\{X_{ij}\}$ satisfies the following:*

(i) $\sum_{i,j=1}^n X_{i-2,j}\varepsilon_{ij} = O_p(n^{3/2-\rho})$, *and*

(ii) $\frac{1}{n^2}\sum_{i,j=1}^n X_{i-1,j}\delta_{ij} \xrightarrow{P} -\sigma^2$.

Proof. *(i)* As shown in Lemma 2.2, $X_{i-2,j} = (1-\theta_n)\sum_{k=1}^{i-3}\varepsilon_{kj} + \varepsilon_{i-2,j}$ and thus $\mathrm{Var}(X_{i-2,j}) = (1-\theta_n)^2(i-3)\sigma^2 + \sigma^2$. Then

$$\begin{aligned}
\mathrm{Var}\left(\sum_{i,j=1}^n X_{i-2,j}\varepsilon_{ij}\right) &= n\mathrm{Var}\left(\sum_{i=1}^n X_{i-2,j}\varepsilon_{ij}\right) \\
&= n\sum_{i=3}^n[(1-\theta_n)^2(i-3)\sigma^2+\sigma^2]\sigma^2 \\
&= n\sigma^4\left[\frac{(1-\theta_n)^2(n-3)(n-2)}{2} + (n-2)\right] \\
&= \sigma^4\frac{(n^\rho(1-\theta_n)^2 n(n-2)(n-3)}{2n^{2\rho}} + n(n-2)\sigma^4 \\
&= O(n^{3-2\rho})
\end{aligned}$$

and thus $\sum_{i,j=1}^n X_{i-2,j}\varepsilon_{ij} = O_p(n^{3/2-\rho})$. Similarly, $\sum_{i,j=1}^n X_{i-2,j}\varepsilon_{i-1,j} = O_p(n^{3/2-\rho})$.

(ii) Observe that $X_{i-1,j} = X_{i-2,j} + \delta_{i-1,j}$ and thus $\sum_{i,j=1}^n X_{i-1,j}\delta_{ij} = \sum_{i,j=1}^n X_{i-2,j}\delta_{ij} + \sum_{i,j=1}^n \delta_{i-1,j}\delta_{ij}$. Further, $\sum_{i,j=1}^n \delta_{i-1,j}\delta_{ij} = \sum_{i,j=1}^n(\varepsilon_{i-1,j} - \theta_n\varepsilon_{i-2,j})(\varepsilon_{ij} - \theta_n\varepsilon_{i-1,j})$. One easily checks that $\sum_{i,j=1}^n \varepsilon_{ij}\varepsilon_{i+s,j} = O_p(n)$ when $s \neq 0$ and thus $\frac{1}{n^2}\sum_{i,j=1}^n \delta_{i-1,j}\delta_{ij} \xrightarrow{\mathrm{P}} -\sigma^2$. Hence $\frac{1}{n^2}\sum_{i,j=1}^n X_{i-1,j}\delta_{ij} = \frac{1}{n^2}\sum_{i,j=1}^n X_{i-2,j}\delta_{ij} + \frac{1}{n^2}\sum_{i,j=1}^n \delta_{i-1,j}\delta_{ij} \xrightarrow{\mathrm{P}} 0 - \sigma^2 = -\sigma^2$. □

It is interesting to compare Lemma 2.3 with Lemma 2.4 below. Lemma 2.4 is not needed here but can be proved using Brown's martingale central limit theorem [6].

Lemma 2.4. $\frac{1}{n^{3/2-\rho}}\sum_{i,j=1}^n X_{i-2,j}\delta_{ij} \xrightarrow{D} \mathcal{N}(0, \frac{\sigma^4}{2})$.

Next, convergence of the component $\frac{1}{n^{3/2}}\sum_{i,j=1}^n Y_{i,j-1}\delta_{ij}$ is considered. Brown's martingale central limit theorem [6] is used to show convergence. Let $\mathscr{F}_{n_j} = \mathscr{F}_j$ denote the smallest σ-field such that each $\varepsilon_{k\ell}, 1 \le \ell \le j \le n$, and all $k \ge 1$ is measurable. Define $\mathscr{F}_{n_0}$ to be the trivial σ-field, $n \ge 1$. Denote $W_{n_j} = \frac{1}{n^{3/2}}\sum_{i=1}^n Y_{i,j-1}\delta_{ij}$. Observe that $\mathbb{E}[W_{n_j}|\mathscr{F}_{j-1}] = \frac{1}{n^{3/2}}\sum_{i=1}^n \mathbb{E}[Y_{i,j-1}\delta_{ij}|\mathscr{F}_{j-1}] = \frac{1}{n^{3/2}}\sum_{i=1}^n Y_{i,j-1}\mathbb{E}[\delta_{ij}] = 0$ and hence $\{W_{n_j}, \mathscr{F}_{n_j}, 1 \le j \le n, n \ge 1\}$ is a mean zero martingale difference array.

Lemma 2.5. *Fix integers s and t. Then*

(i) $\frac{1}{n^3}\sum_{i,j=1}^n \sum_{\ell,\ell'=1}^{j-1}\varepsilon_{i-s,\ell}\varepsilon_{i-t,\ell'} \xrightarrow{P} \begin{cases} \frac{\sigma^2}{2}, & \text{if } s=t, \\ 0, & \text{if } s \neq t \end{cases}$.

(ii) $\frac{1}{n^3}\sum_{i,j=1}^n Y_{i,j-1}^2 = \sigma^2 + o_p(1)$.

Proof. (i) First, assume that $s = t$. Then

$$\frac{1}{n^3}\sum_{i,j=1}^{n}\sum_{\ell,\ell'=1}^{j-1}\varepsilon_{i-s,\ell}\varepsilon_{i-s,\ell'} = \frac{1}{n^3}\sum_{i,j=1}^{n}\left[\sum_{\ell=1}^{j-1}\varepsilon_{i-s,\ell}\right]^2 \xrightarrow{\mathrm{P}} \frac{\sigma^2}{2}$$

according to Lemma 3.1 in [5]. Next, suppose that $s \neq t$ and let $A_n = \sum_{i,j=1}^{n}\sum_{\ell,\ell'=1}^{j-1}\varepsilon_{i-s,\ell}\varepsilon_{i-t,\ell'}$. Then

$$\begin{aligned}
\mathrm{Var}A_n &= \sum_{i,i'=1}^{n}\sum_{j,j'=1}^{n}\sum_{\ell,\ell'=1}^{j-1}\sum_{m,m'=1}^{j'-1}\mathrm{Cov}(\varepsilon_{i-s,\ell}\varepsilon_{i-t,\ell'},\varepsilon_{i'-s,m}\varepsilon_{i'-t,m'}) \\
&= \sum_{i=1}^{n}\sum_{j,j'=1}^{n}\sum_{\ell,\ell'=1}^{j-1}\sum_{m,m'=1}^{j'-1}\mathrm{Cov}(\varepsilon_{i-s,\ell}\varepsilon_{i-t,\ell'},\varepsilon_{i-s,m}\varepsilon_{i-t,m'}) \\
&= \sum_{i=1}^{n}\sum_{j,j'=1}^{n}\sum_{\ell,\ell'=1}^{(j\wedge j')-1}\mathrm{Cov}(\varepsilon_{i-s,\ell}\varepsilon_{i-t,\ell'},\varepsilon_{i-s,\ell}\varepsilon_{i-t,\ell'}) \\
&\leq \sum_{i=1}^{n}\sum_{j,j'=1}^{n}\sum_{\ell,\ell'=1}^{(j\wedge j')-1}\mathbb{E}[\varepsilon_{11}^4] \\
&\leq \sum_{i=1}^{n}\sum_{j,j'=1}^{n}(j\wedge j')^2\mathbb{E}[\varepsilon_{11}^4] \\
&\leq 2\mathbb{E}[\varepsilon_{11}^4]\sum_{i=1}^{n}\sum_{j'=1}^{n}\sum_{j=1}^{j'}j^2 = O(n^5).
\end{aligned}$$

Hence $A_n = O_p(n^{5/2})$ and $\frac{1}{n^3}A_n \xrightarrow{\mathrm{P}} 0$.

(ii) Observe that $Y_{i,j-1}^2 = \left(\sum_{\ell=1}^{j-1}\delta_{i\ell}\right)^2 = \sum_{\ell,\ell'=1}^{j-1}(\varepsilon_{i\ell}-\theta_n\varepsilon_{i-1,\ell})(\varepsilon_{i\ell'}-\theta_n\varepsilon_{i-1,\ell'})$ and thus

$$\begin{aligned}
\frac{1}{n^3}\sum_{i,j=1}^{n}Y_{i,j-1}^2 &= \frac{1}{n^3}\sum_{i,j=1}^{n}\sum_{\ell,\ell'=1}^{j-1}(\varepsilon_{i\ell}\varepsilon_{i\ell'}-\theta_n\varepsilon_{i-1,\ell}\varepsilon_{i\ell'}-\theta_n\varepsilon_{i\ell}\varepsilon_{i-1,\ell'}+\theta_n^2\varepsilon_{i-1,\ell}\varepsilon_{i-1,\ell'}) \\
&\xrightarrow{\mathrm{P}} \frac{\sigma^2}{2}+0+0+\frac{\sigma^2}{2} = \sigma^2
\end{aligned}$$

according to (i).

□

Since $\delta_{ij}\delta_{i'j} = (\varepsilon_{ij}-\theta_n\varepsilon_{i-1,j})(\varepsilon_{i'j}-\theta_n\varepsilon_{i'-1,j})$, it follows that

$$\mathbb{E}[\delta_{ij}\delta_{i'j}] = \begin{cases} -\theta_n\sigma^2, & i' = i\pm 1 \\ (1+\theta_n^2)\sigma^2, & i' = i \\ 0, & |i-i'|>1 \end{cases}.$$

Lemma 2.6. *Let $s^2 = 3\sigma^4$. Then*

(i) $s_n^2 = \sum_{j=1}^{n}\mathbb{E}[W_{nj}^2] \to s^2$,

(ii) $V_n^2 = \sum_{j=1}^{n}\mathbb{E}[W_{nj}^2|\mathscr{F}_{j-1}] \xrightarrow{P} s^2$, *and*

(iii) $\mathbb{E}\left|\sum_{i=1}^{n}Y_{i,j-1}\delta_{ij}\right|^4 = O(n^4)$.

Proof. (i) Note that $\mathbb{E}[W_{nj}^2] = \frac{1}{n^3}\sum_{i,i'=1}^{n}\mathbb{E}[Y_{i,j-1}Y_{i',j-1}]\mathbb{E}[\delta_{ij}\delta_{i'j}]$. Consider the following cases for $\mathbb{E}[Y_{i,j-1}Y_{i',j-1}]$:

(a) $i' = i$: $\mathbb{E}[Y_{i,j-1}^2] = \mathbb{E}[\sum_{\ell=1}^{j-1}\delta_{i\ell}]^2 = (j-1)\mathbb{E}[\delta_{i\ell}^2] = (j-1)\sigma^2(1+\theta_n^2)$.

(b) $i' = i+1$: $\mathbb{E}[Y_{i,j-1}Y_{i+1,j-1}] = \mathbb{E}[\sum_{\ell=1}^{j-1}\delta_{i\ell}\sum_{\ell'=1}^{j-1}\delta_{i+1,\ell'}]$
$= \sum_{\ell=1}^{j-1}\mathbb{E}[\delta_{i\ell}\delta_{i+1,\ell}] = \sum_{\ell=1}^{j-1}(-\theta_n\sigma^2) = -(j-1)\sigma^2\theta_n$.

(c) $i' = i-1$: $\mathbb{E}[Y_{i,j-1}Y_{i-1,j-1}] = -(j-1)\sigma^2\theta_n$.

Combining these facts, one has

$$\begin{aligned} n^3\mathbb{E}[W_{nj}^2] &= \sum_{i=1}^{n}\mathbb{E}[Y_{i,j-1}^2]\mathbb{E}[\delta_{ij}^2] + \sum_{i=1}^{n-1}\mathbb{E}[Y_{i,j-1}Y_{i+1,j-1}]\mathbb{E}[\delta_{ij}\delta_{i+1,j}] \\ &\quad + \sum_{i=2}^{n}\mathbb{E}[Y_{i,j-1}Y_{i-1,j-1}]\mathbb{E}[\delta_{ij}\delta_{i-1,j}] \\ &= \sigma^4[n(j-1)(1+\theta_n^2)^2 + 2(n-1)(j-1)\theta_n^2]. \end{aligned}$$

Then $s_n^2 = \sum_{j=1}^{n}\mathbb{E}[W_{nj}^2] = \frac{\sigma^4}{n^3}\left[\frac{n^2(n-1)}{2}(1+\theta_n^2)^2 + 2\frac{(n-1)^2n}{2}\theta_n^2\right] \to \sigma^4(2+1) = 3\sigma^4 = s^2$.

(ii) Observe that

$$\begin{aligned} n^3\mathbb{E}[W_{nj}^2|\mathscr{F}_{j-1}] &= \sum_{i,i'=1}^{n}\mathbb{E}[Y_{i,j-1}Y_{i',j-1}\delta_{ij}\delta_{i'j}|\mathscr{F}_{j-1}] \\ &= \sum_{i,i'=1}^{n}Y_{i,j-1}Y_{i',j-1}\mathbb{E}[\delta_{ij}\delta_{i'j}] \\ &= \sum_{i=1}^{n}Y_{i,j-1}^2\mathbb{E}[\delta_{ij}^2] + \sum_{i=1}^{n}Y_{i,j-1}Y_{i+1,j-1}\mathbb{E}[\delta_{ij}\delta_{i+1,j}] + \sum_{i=2}^{n}Y_{i,j-1}Y_{i-1,j-1}\mathbb{E}[\delta_{ij}\delta_{i-1,j}]. \end{aligned}$$

Employing Lemma 2.5 (ii), $\frac{1}{n^3}\sum_{i,j=1}^{n}Y_{i,j-1}^2 \xrightarrow{\mathrm{P}} \sigma^2$. Further,

$$Y_{i,j-1}Y_{i+1,j-1} = \sum_{\ell=1}^{j-1}\delta_{i\ell}\sum_{\ell'=1}^{j-1}\delta_{i+1,\ell'} = \sum_{\ell=1}^{j-1}(\varepsilon_{i\ell}-\theta_n\varepsilon_{i-1,\ell})\sum_{\ell'=1}^{j-1}(\varepsilon_{i+1,\ell'}-\theta_n\varepsilon_{i\ell'}).$$

It follows that

$$\begin{aligned} \frac{1}{n^3}\sum_{i,j=1}^{n}Y_{i,j-1}Y_{i+1,j-1} &= \frac{1}{n^3}\sum_{i,j=1}^{n}\sum_{\ell,\ell'=1}^{j-1}[\varepsilon_{i\ell}\varepsilon_{i+1,\ell'} - \theta_n\varepsilon_{i-1,\ell}\varepsilon_{i+1,\ell'} - \theta_n\varepsilon_{i\ell}\varepsilon_{i\ell'} + \theta_n^2\varepsilon_{i-1,\ell}\varepsilon_{i\ell'}] \\ &\xrightarrow{\mathrm{P}} 0 + 0 - \frac{\sigma^2}{2} + 0 = -\frac{\sigma^2}{2} \end{aligned}$$

by Lemma 2.5 (i). Likewise, $\frac{1}{n^3}\sum_{i,j=1}^{n}Y_{i,j-1}Y_{i-1,j-1} \xrightarrow{\mathrm{P}} -\frac{\sigma^2}{2}$. Therefore

$$
\begin{aligned}
V_n^2 &= \sum_{j=1}^{n} \mathbb{E}[W_{n_j}^2 | \mathscr{F}_{j-1}] \\
&= \frac{1}{n^3} \sum_{i,j=1}^{n} Y_{i,j-1}^2 \sigma^2 (1+\theta_n^2) - \frac{1}{n^3} \sum_{i,j=1}^{n} Y_{i,j-1} Y_{i+1,j-1} \sigma^2 \theta_n - \frac{1}{n^3} \sum_{i,j=1}^{n} Y_{i,j-1} Y_{i-1,j-1} \sigma^2 \theta_n \\
&\xrightarrow{\mathrm{P}} 2\sigma^4 + \frac{\sigma^4}{2} + \frac{\sigma^4}{2} \\
&= 3\sigma^4.
\end{aligned}
$$

(iii) Note that

$$
\begin{aligned}
\mathbb{E}[n^{3/2} W_{n_j}]^4 &= \mathbb{E} \left| \sum_{i=1}^{n} Y_{i,j-1} \delta_{ij} \right|^4 \\
&= \sum_{i_1,i_2,i_3,i_4=1}^{n} \mathbb{E}[Y_{i_1,j-1} Y_{i_2,j-1} Y_{i_3,j-1} Y_{i_4,j-1} \delta_{i_1 j} \delta_{i_2 j} \delta_{i_3 j} \delta_{i_4 j}] \\
&= \sum_{i_1,i_2,i_3,i_4=1}^{n} \mathbb{E}[Y_{i_1,j-1} Y_{i_2,j-1} Y_{i_3,j-1} Y_{i_4,j-1}] \mathbb{E}[\delta_{i_1 j} \delta_{i_2 j} \delta_{i_3 j} \delta_{i_4 j}].
\end{aligned}
$$

The term $\sum_{i_1,i_2=1,i_1\neq i_2}^{n} \mathbb{E}[Y_{i_1,j-1}^2 Y_{i_2,j-1}^2] \mathbb{E}[\delta_{i_1 j}^2 \delta_{i_2 j}^2]$ is the primary component contributing to the order of $\mathbb{E}[n^{3/2} W_{n_j}]^4$. Using Cauchy's inequality,

$$
\sum_{i_1,i_2=1}^{n} \mathbb{E}[Y_{i_1,j-1}^2 Y_{i_2,j-1}^2] \mathbb{E}[\delta_{i_1 j}^2 \delta_{i_2 j}^2] \leq \sum_{i_1,i_2=1}^{n} (\mathbb{E}[Y_{i_1,j-1}^4] \mathbb{E}[Y_{i_2,j-1}^4] \mathbb{E}[\delta_{i_1 j}^4] \mathbb{E}[\delta_{i_2 j}^4])^{\frac{1}{2}}
$$

Observe that

$$
\mathbb{E}[\delta_{ij}^4] = \mathbb{E}[\varepsilon_{ij} - \theta_n \varepsilon_{i-1,j}]^4 \leq 2^3 (\mathbb{E}[\varepsilon_{ij}^4] + \theta_n^4 \mathbb{E}[\varepsilon_{i-1,j}^4]) = O(1).
$$

Further,

$$
\mathbb{E}[Y_{i,j-1}^4] = \mathbb{E} \left[\sum_{\ell=1}^{j-1} (\varepsilon_{i\ell} - \theta_n \varepsilon_{i-1,\ell}) \right]^4 \leq 2^3 \left(\mathbb{E} \left[\sum_{\ell=1}^{j-1} \varepsilon_{i\ell} \right]^4 + \theta_n^4 \mathbb{E} \left[\sum_{\ell=1}^{j-1} \varepsilon_{i-1,\ell} \right]^4 \right).
$$

Moreover,

$$
\mathbb{E} \left[\sum_{\ell=1}^{j-1} \varepsilon_{i\ell} \right]^4 = \sum_{\ell=1}^{j-1} \mathbb{E}[\varepsilon_{i\ell}^4] + \sum_{\ell\neq\ell'=1}^{j-1} \mathbb{E}[\varepsilon_{i\ell}^2 \varepsilon_{i\ell'}^2] = O((j-1)^2)
$$

and thus $\mathbb{E}[Y_{i,j-1}^4] = O((j-1)^2), 1 \leq i, j \leq n$. It follows that for all $1 \leq j \leq n$, $\mathbb{E}[n^{3/2} W_{n_j}]^4 = O(n^4)$.

□

Lemma 2.7. $\frac{1}{n^{3/2}} \sum_{i,j=1}^{n} Y_{i,j-1} \delta_{ij} \xrightarrow{D} \mathscr{N}(0, 3\sigma^4)$.

Proof. Brown's martingale central limit theorem [6] for arrays is used in the proof. Recall that $\{W_{n_j}, \mathscr{F}_{n_j}, 1 \leq j \leq n, n \geq 1\}$ is a mean zero martingale difference array and by Lemma 2.6, $\frac{V_n^2}{s_n^2} \xrightarrow{\mathrm{P}} 1, s_n^2 \to s^2 = 3\sigma^4$ and, for all $1 \leq j \leq n$, $\mathbb{E}[\sqrt{n} W_{n_j}]^4 = \frac{1}{n^4} \mathbb{E} |\sum_{i=1}^{n} Y_{i,j-1} \delta_{ij}|^4 = O(1)$. Hence

$\{nW_{n_j}^2 : 1 \le j \le n, n \ge 1\}$ is uniformly integrable. Fix $b > 0$, then for any given $\rho > 0$, there exists N such that for all $n \ge N$, $\mathbb{E}[nW_{n_j}^2 \mathbf{1}_{\{\sqrt{n}|W_{n_j}|>\sqrt{n}b\}}] < \rho$ for $1 \le j \le n$. Therefore for all $n \ge N$, $\sum_{j=1}^n \mathbb{E}[W_{n_j}^2 \mathbf{1}_{\{|W_{n_j}|>b\}}] < \rho$. Hence $\frac{1}{n^{3/2}} \sum_{i,j=1}^n Y_{i,j-1}\delta_{ij} \xrightarrow{\mathrm{D}} \mathcal{N}(0, 3\sigma^4)$. □

The next result follows from Lemma 2.3(ii) and Lemma 2.7.

Corollary 2.8. $\left(\frac{1}{n^2}\sum_{i,j=1}^n X_{i-1,j}\delta_{ij}, \frac{1}{n^{3/2}}\sum_{i,j=1}^n Y_{i,j-1}\delta_{ij}\right) \xrightarrow{D} (-\sigma^2, \mathcal{N}(0, 3\sigma^4))$.

3. Gauss-Newton Estimators

Recall that $X_{ij} = \alpha X_{i-1,j} + \delta_{ij}$ and $Y_{ij} = \beta Y_{i,j-1} + \delta_{ij}$. Initial estimators $\{\overline{\alpha}_n\}$ and $\{\overline{\beta}_n\}$ are used in the Gauss-Newton estimator of $\theta = \theta_0 = (\alpha, \beta)$.

Lemma 3.1. *There exist initial estimators obeying* $n^{1-2\rho}(\overline{\alpha}_n - \alpha) = O_p(1)$ *and* $n^{3/2}(\overline{\beta}_n - \beta) = O_p(1)$.

Proof. Since $X_{ij} = \alpha X_{i-1,j} + \delta_{ij}$, the ordinary least squares estimator of α satisfies $n^{1-2\rho}(\overline{\alpha}_n - \alpha) = \frac{1}{n^2}\sum_{i,j=1}^n X_{i-1,j}\delta_{ij} \div \frac{1}{n^{3-2\rho}}\sum_{i,j=1}^n X_{i-1,j}^2 \xrightarrow{\mathrm{P}} -\frac{\sigma^2}{\sigma^2/2} = O_p(1)$ by employing Lemma 2.2 and Lemma 2.3 (ii). Likewise, $n^{3/2}(\overline{\beta}_n - \beta) = \frac{1}{n^{3/2}}\sum_{i,j=1}^n Y_{i,j-1}\delta_{ij} \div \frac{1}{n^3}\sum_{i,j=1}^n Y_{i,j-1}^2 \xrightarrow{\mathrm{D}} \frac{\mathcal{N}(0,3\sigma^4)}{\sigma^2} = O_p(1)$ by Lemma 2.5 (ii) and Lemma 2.7. □

Lemma 3.1 shows the existence of initial estimators obeying assumption (A5) listed in Section 1. Let $\overline{\theta}_n' = (\overline{\alpha}_n, \overline{\beta}_n)$ denote any initial estimator of $\theta_0' = (\alpha, \beta)$ which obeys (A5). Define $f_{ij}(a,b) = aZ_{i-1,j} + bZ_{i,j-1} - abZ_{i-1,j-1}$ and $F_{ij}'(a,b) = \left(\frac{\partial f_{ij}(a,b)}{\partial a}, \frac{\partial f_{ij}(a,b)}{\partial b}\right)$. A Taylor expansion of $f_{ij}(\theta_0)$ about $\overline{\theta}_n$ in model (1.1) is

$$Z_{ij} = f_{ij}(\overline{\theta}_n) + F_{ij}'(\overline{\theta}_n)(\theta_0 - \overline{\theta}_n) + R_{ij}(\overline{\theta}_n) + \delta_{ij}, \tag{3.1}$$

where the remainder term is $R_{ij}(\overline{\theta}_n) = -(\alpha - \overline{\alpha}_n)(\beta - \overline{\beta}_n)Z_{i-1,j-1}$. Denote

$$\hat{\Delta}_n = \left[\sum_{i,j=1}^n F_{ij}(\overline{\theta}_n)F_{ij}'(\overline{\theta}_n)\right]^{-1}\left[\sum_{i,j=1}^n F_{ij}(\overline{\theta}_n)(Z_{ij} - f_{ij}(\overline{\theta}_n))\right]. \tag{3.2}$$

The "Gauss-Newton estimator" of θ_0 is

$$\hat{\theta}_n = \overline{\theta}_n + \hat{\Delta}_n. \tag{3.3}$$

Substituting (3.1) into (3.2) and simplifying gives

$$\hat{\theta}_n - \theta_0 = \left[\sum_{i,j=1}^n F_{ij}(\overline{\theta}_n)F_{ij}'(\overline{\theta}_n)\right]^{-1}\left[\sum_{i,j=1}^n F_{ij}(\overline{\theta}_n)(R_{ij}(\overline{\theta}_n) + \delta_{ij})\right]. \tag{3.4}$$

Observe that

$$F_{ij}(\overline{\theta}) = \begin{bmatrix} X_{i-1,j} + (\beta - \overline{\beta}_n)Z_{i-1,j-1} \\ Y_{i,j-1} + (\alpha - \overline{\alpha}_n)Z_{i-1,j-1} \end{bmatrix}$$

and denote $G_n(\overline{\theta}_n) = \sum_{i,j=1}^n F_{ij}(\overline{\theta}_n)F_{ij}'(\overline{\theta}_n), A_n = \mathrm{diag}(\frac{1}{n^2}, \frac{1}{n^{3/2}}), B_n = \mathrm{diag}(\frac{1}{n^{1-2\rho}}, \frac{1}{n^{3/2}})$, and $A = \mathrm{diag}(\frac{\sigma^2}{2}, \sigma^2)$.

The next result is used in finding the limit of $\hat{\theta}_n - \theta_0$ given in (3.4).

Lemma 3.2. *Assume that conditions (A1)-(A5) are satisfied. Then*

(i) $A_n \sum_{i,j=1}^{n} F_{ij}(\overline{\theta}_n) R_{ij}(\overline{\theta}_n) \xrightarrow{P} \underline{0}$.

(ii) $A_n G_n(\overline{\theta}_n) B_n \xrightarrow{P} A$.

Proof. (i) Note that

$$A_n \sum_{i,j=1}^{n} F_{ij}(\overline{\theta}_n) R_{ij}(\overline{\theta}_n) = -\begin{bmatrix} \frac{1}{n^2} \sum\limits_{i,j=1}^{n} (X_{i-1,j} + (\beta - \overline{\beta}_n) Z_{i-1,j-1}) Z_{i-1,j-1} \\ \frac{1}{n^{3/2}} \sum\limits_{i,j=1}^{n} (Y_{i,j-1} + (\alpha - \overline{\alpha}_n) Z_{i-1,j-1}) Z_{i-1,j-1} \end{bmatrix} (\alpha - \overline{\alpha}_n)(\beta - \overline{\beta}_n).$$

The order properties listed in Lemma 2.1 along with assumption (A5) show that

$$A_n \sum_{i,j=1}^{n} F(\overline{\theta}_n) R_{ij}(\overline{\theta}_n) \xrightarrow{\mathrm{P}} \underline{0}.$$

(ii) Denote $G_n(\overline{\theta}_n) = \begin{bmatrix} a_n & b_n \\ b_n & c_n \end{bmatrix}$. Then $A_n G_n(\overline{\theta}_n) B_n = \begin{bmatrix} \frac{a_n}{n^{3-2\rho}} & \frac{b_n}{n^{7/2}} \\ \frac{b_n}{n^{5/2-2\rho}} & \frac{c_n}{n^3} \end{bmatrix}$. Using Lemma 2.1 and (A5), one has

$$\begin{aligned} \frac{a_n}{n^{3-2\rho}} &= \frac{1}{n^{3-2\rho}} \sum_{i,j=1}^{n} (X_{i-1,j} + (\beta - \overline{\beta}_n) Z_{i-1,j-1})^2 \\ &= \frac{1}{n^{3-2\rho}} \sum_{i,j=1}^{n} \left[X_{i-1,j}^2 + 2(\beta - \overline{\beta}_n) X_{i-1,j} Z_{i-1,j-1} + (\beta - \overline{\beta}_n)^2 Z_{i-1,j-1}^2 \right] \xrightarrow{\mathrm{P}} \frac{\sigma^2}{2}, \end{aligned}$$

$$\begin{aligned} \frac{b_n}{n^{5/2-2\rho}} &= \frac{1}{n^{5/2-2\rho}} \sum_{i,j=1}^{n} (X_{i-1,j} + (\beta - \overline{\beta}_n) Z_{i-1,j-1})(Y_{i,j-1} + (\alpha - \overline{\alpha}_n) Z_{i-1,j-1}) \\ &= \frac{1}{n^{5/2-2\rho}} \sum_{i,j=1}^{n} \Big[X_{i-1,j} Y_{i,j-1} + (\beta - \overline{\beta}_n) Z_{i-1,j-1} Y_{i,j-1} + (\alpha - \overline{\alpha}_n) X_{i-1,j} Z_{i-1,j-1} \\ &\qquad + (\beta - \overline{\beta}_n)(\alpha - \overline{\alpha}_n) Z_{i-1,j-1}^2 \Big] \xrightarrow{\mathrm{P}} 0, \end{aligned}$$

$$\begin{aligned} \frac{c_n}{n^3} &= \frac{1}{n^3} \sum_{i,j=1}^{n} (Y_{i,j-1} + (\alpha - \overline{\alpha}_n) Z_{i-1,j-1})^2 \\ &= \frac{1}{n^3} \sum_{i,j=1}^{n} \left[Y_{i,j-1}^2 + 2(\alpha - \overline{\alpha}_n) Y_{i,j-1} Z_{i-1,j-1} + (\alpha - \overline{\alpha}_n)^2 Z_{i-1,j-1}^2 \right] \xrightarrow{\mathrm{P}} \sigma^2. \end{aligned}$$

Hence $A_n G_n(\overline{\theta}_n) B_n \xrightarrow{\mathrm{P}} A$.

□

We now prove Theorem 1.1.

Proof. As shown in (3.4), the Gauss-Newton estimator $\hat{\theta}_n' = (\hat{\alpha}_n, \hat{\beta}_n)$ of $\theta_0' = (\alpha, \beta) = (1,1)$ under $H_0 : \alpha = \beta = 1$ satisfies $\hat{\theta}_n - \theta_0 = G_n(\overline{\theta}_n)^{-1} \sum_{i,j=1}^n F_{ij}(\overline{\theta}_n)(R_{ij}(\overline{\theta}_n) + \delta_{ij})$. Then

$$B_n^{-1}(\hat{\theta}_n - \theta_0) = (A_n G_n(\overline{\theta}_n) B_n)^{-1} A_n \sum_{i,j=1}^n F_{ij}(\overline{\theta}_n)(R_{ij}(\overline{\theta}_n) + \delta_{ij}).$$

According to Lemma 3.2, $A_n G_n(\overline{\theta}_n) B_n \xrightarrow{\mathrm{P}} A$ and $A_n \sum_{i,j=1}^n F_{ij}(\overline{\theta}_n) R_{ij}(\overline{\theta}_n) \xrightarrow{\mathrm{P}} \underline{0}$. Observe that

$$A_n \sum_{i,j=1}^n F_{ij}(\overline{\theta}_n)\delta_{ij} = \begin{bmatrix} \frac{1}{n^2} \sum_{i,j=1}^n (X_{i-1,j} + (\beta - \overline{\beta}_n) Z_{i-1,j-1}) \delta_{ij} \\ \frac{1}{n^{3/2}} \sum_{i,j=1}^n (Y_{i-1,j} + (\alpha - \overline{\alpha}_n) Z_{i-1,j-1}) \delta_{ij} \end{bmatrix}$$

and by Lemma 2.1 and (A5), $\frac{\beta - \overline{\beta}_n}{n^2} \sum_{i,j=1}^n Z_{i-1,j-1}\delta_{ij} \xrightarrow{\mathrm{P}} 0$ and $\frac{\alpha - \overline{\alpha}_n}{n^{3/2}} \sum_{i,j=1}^n Z_{i-1,j-1}\delta_{ij} \xrightarrow{\mathrm{P}} 0$. Further, by Corollary 2.8, $\left(\frac{1}{n^2} \sum_{i,j=1}^n X_{i-1,j}\delta_{ij}, \frac{1}{n^{3/2}} \sum_{i,j=1}^n Y_{i,j-1}\delta_{ij} \right) \xrightarrow{\mathrm{D}} (-\sigma^2, \mathscr{N}(0, 3\sigma^4))$. Hence $B_n^{-1}(\hat{\theta}_n - \theta_0) \xrightarrow{\mathrm{D}} A_n^{-1} \begin{bmatrix} -\sigma^2 \\ \mathscr{N}(0, 3\sigma^4) \end{bmatrix}$ and thus

$$\begin{bmatrix} n^{1-2\rho}(\hat{\alpha}_n - 1) \\ n^{3/2}(\hat{\beta}_n - 1) \end{bmatrix} \xrightarrow{\mathrm{D}} \begin{bmatrix} \frac{2}{\sigma^2} & 0 \\ 0 & \frac{1}{\sigma^2} \end{bmatrix} \begin{bmatrix} -\sigma^2 \\ \sigma^2 \mathscr{N}(0,3) \end{bmatrix} = \begin{bmatrix} -2 \\ \mathscr{N}(0,3) \end{bmatrix}.$$

□

4. Numerical Simulation

Theorem 1.1 is used to give an asymptotic test for testing $H_0 : \alpha = \beta = 1$. The asymptotic power is found at the sequence of local alternatives $\alpha_n = e^{c/n}, \beta_n = e^{d/n}$, for fixed values of c and d. In this case, the alternative model is

$$Z_{ij}(n) = \alpha_n Z_{i-1,j} + \beta_n Z_{i,j-1} - \alpha_n \beta_n Z_{i-1,j-1} + \delta_{ij}, \qquad 1 \le i, j \le n, \tag{4.1}$$

and the alternative hypothesis is $H_A : \alpha = \alpha_n = e^{c/n}, \beta = \beta_n = e^{d/n}$, where $c \neq 0, d \neq 0$. Modifying the proof of Theorem 1.1, the corresponding asymptotic result for model (4.1) is given below.

Theorem 4.1. *Given model* (4.1) *and assumptions (A2)–(A5), the Gauss-Newton estimator* $\hat{\theta}_n' = (\hat{\alpha}_n, \hat{\beta}_n)$ *of* (α_n, β_n) *obeys*

$$\left(n^{1-2\rho}(\hat{\alpha}_n - \alpha_n), n^{3/2}(\hat{\beta}_n - \beta_n) \right) \xrightarrow{D} (s, \mathscr{N}(0,t)),$$

where $s = -4c^2/(e^{2c} - 2c - 1)$ *and* $t = 6d^2/(e^{2d} - 2d - 1)$.

A size γ test ϕ_n is chosen, where $0 < \gamma < 1$. Find $b > 0$ such that $\mathbb{P}\{|\mathscr{N}(0,3)| \ge b\} = \gamma$ and denote $A = (-a, a) \times (-b, b)$, where $a > 0$. Define $\phi_n(\omega) = 1$ when

$$\left(n^{1-2\rho}(\hat{\alpha}_n(\omega) - 1) + 2, n^{3/2}(\hat{\beta}_n(\omega) - 1) \right) \in A^c$$

and $\phi_n(\omega) = 0$ otherwise. Since $n^{1-2\rho}(\hat{\alpha}_n - 1) + 2 \xrightarrow{\mathrm{P}} 0$, it follows that ϕ_n is an asymptotically size γ test. Theorem 4.1 is used to find the asymptotic power of ϕ_n at the sequence of alternatives

(α_n, β_n). Observe that under H_A, Theorem 4.1 implies that $n^{1-2\rho}(\hat{\alpha}_n - 1) + 2 = n^{1-2\rho}(\hat{\alpha}_n - \alpha_n) + n^{1-2\rho}(\alpha_n - 1) + 2 \xrightarrow{\mathrm{P}} \frac{-4c^2}{e^{2c}-2c-1} + 0 + 2$, which converges to 0 as $c \to 0$. Likewise,

$$|n^{3/2}(\hat{\beta}_n - 1)| = |n^{3/2}(\hat{\beta}_n - \beta_n) + n^{3/2}(\beta_n - 1)| \xrightarrow{\mathrm{D}} |\mathscr{N}(0,t) \pm \infty| = \infty.$$

It follows that the asymptotic power of ϕ_n at the sequence of alternatives (α_n, β_n) is one.

Some comments are made concerning the different scaling for $\hat{\alpha}_n$ and $\hat{\beta}_n$. Observe that the rate of convergence of $\hat{\alpha}_n$ depends on the parameter ρ. According to Lemma 2.4, we see that $\frac{1}{n^{3/2-\rho}} \sum_{i,j=1}^n X_{i-2,j}\delta_{ij} \to \mathscr{N}(0, \sigma^4/2)$ and one might expect this to extend to $\sum_{i,j=1}^n X_{i-1,j}\delta_{ij}$. However, $\sum_{i,j=1}^n X_{i-1,j}\delta_{ij} = \sum_{i,j=1}^n X_{i-2,j}\delta_{ij} + \sum_{i,j=1}^n \delta_{i-1,j}\delta_{ij}$, where

$$\begin{aligned}\sum_{i,j=1}^n \delta_{i-1,j}\delta_{ij} &= \sum_{i,j=1}^n [\varepsilon_{i-1,j}\varepsilon_{ij} - \theta_n\varepsilon_{i-2,j}\varepsilon_{ij} - \theta_n\varepsilon_{i-1,j}^2 + \theta_n^2\varepsilon_{i-2,j}\varepsilon_{i-1,j}] \\ &= O_p(n) - \theta_n \sum_{i,j=1}^n \varepsilon_{i-1,j}^2.\end{aligned}$$

Hence $\frac{1}{n^2}$ is needed in order to get convergence of the last term and this leads to a degenerate estimator of α. Since δ_{ij} is a constrained moving average in the one direction, this problem does not exist for the corresponding term $\sum_{i,j=1}^n Y_{i,j-1}\delta_{ij}$ and this leads to different scaling terms for $\hat{\alpha}_n$ and $\hat{\beta}_n$.

Finally, the results of a targeted simulation study estimating the parameters α_n and β_n and illustrating the power and size of the asymptotic test ϕ_n are reported in the tables below under moderate sample sizes $n = 5, 10, 20, 40$. Each simulation consists of $K = 1{,}000$ Monte Carlo replications under H_A, with the values (c,d) fixed at $c = 5, d = \pm 5$. The empirical size was calculated in an additional simulation under H_0.

The Gauss-Newton estimators $\hat{\theta}_n' = (\hat{\alpha}_n, \hat{\beta}_n)$ are computed using (3.3) and the ε_{ij} are generated from $\mathscr{N}(0,1)$. Further, $\theta_n = 1 - \frac{1}{n^\rho}$ with $\rho = \frac{1}{10}$ and $\rho = \frac{1}{20}$ selected, along with an asymptotically size $\gamma = 0.05$ test ϕ_n mentioned below Theorem 4.1. Since $\hat{\alpha}_n$ is degenerate, $A = (-a, a) \times (-b, b)$ is a 95% confidence region for each $a > 0$. Here $a = 2$ was selected and the empirical size is listed in Tables 1 and 4. Due to the slow convergence of $\hat{\alpha}_n$ at the alternative values, the empirical power was found to be one and hence omitted from the tables.

Although the nuisance parameter σ^2 does not appear in the limiting distribution of the Gauss-Newton estimator, it does affect finite sample estimation. Only $\sigma^2 = 1$ was used to form the error structure δ_{ij} in our simulation study.

TABLE 1. Empirical size under $H_0 : c = d = 0, \rho = \frac{1}{10}$

n	$\hat{\alpha}_n - \alpha_n$	$\hat{\beta}_n - \beta_n$	STD($\hat{\alpha}_n$)	STD($\hat{\beta}_n$)	Size
5	-0.1178	-0.0467	0.1878	0.1738	0.32
10	-0.0738	-0.0153	0.0667	0.0536	0.17
20	-0.0483	-0.0032	0.0269	0.0189	0.06
40	-0.0325	-0.0009	0.0110	0.0062	0.04

TABLE 2. Empirical Estimates at $H_A : c = d = 5, \rho = \frac{1}{10}$

n	$\hat{\alpha}_n - \alpha_n$	$\hat{\beta}_n - \beta_n$	STD($\hat{\alpha}_n$)	STD($\hat{\beta}_n$)
5	-0.0049	-0.0001	0.0255	0.0260
10	-0.0005	-0.0001	0.0041	0.0042
20	-0.0001	0.0000	0.0011	0.0010
40	-0.0001	-0.0000	0.0003	0.0003

TABLE 3. Empirical Estimates at $H_A : c = 5, d = -5, \rho = \frac{1}{10}$

n	$\hat{\alpha}_n - \alpha_n$	$\hat{\beta}_n - \beta_n$	STD($\hat{\alpha}_n$)	STD($\hat{\beta}_n$)
5	-0.0017	-0.0289	0.0278	0.2209
10	-0.0007	-0.0050	0.0045	0.0868
20	-0.0003	-0.0042	0.0011	0.0360
40	-0.0001	-0.0017	0.0003	0.0133

TABLE 4. Empirical size under $H_0 : c = d = 0, \rho = \frac{1}{20}$

n	$\hat{\alpha}_n - \alpha_n$	$\hat{\beta}_n - \beta_n$	STD($\hat{\alpha}_n$)	STD($\hat{\beta}_n$)	Size
5	-0.0884	-0.0470	0.1866	0.1680	0.41
10	-0.0412	-0.0179	0.0618	0.0555	0.31
20	-0.0217	-0.0025	0.0213	0.0163	0.16
40	-0.0134	-0.0012	0.0080	0.0059	0.07

TABLE 5. Empirical Estimates at $H_A : c = d = 5, \rho = \frac{1}{20}$

n	$\hat{\alpha}_n - \alpha_n$	$\hat{\beta}_n - \beta_n$	STD($\hat{\alpha}_n$)	STD($\hat{\beta}_n$)
5	-0.0025	-0.0013	0.0260	0.0261
10	-0.0003	-0.0000	0.0043	0.0040
20	-0.0000	-0.0000	0.0010	0.0011
40	-0.0000	-0.0000	0.0003	0.0003

TABLE 6. Empirical Estimates at $H_A : c = 5, d = -5, \rho = \frac{1}{20}$

n	$\hat{\alpha}_n - \alpha_n$	$\hat{\beta}_n - \beta_n$	STD($\hat{\alpha}_n$)	STD($\hat{\beta}_n$)
5	-0.0020	-0.0188	0.0296	0.2182
10	-0.0002	-0.0059	0.0043	0.0863
20	-0.0001	-0.0061	0.0011	0.0345
40	-0.0000	-0.0009	0.0003	0.0126

Overall, the simulation results indicate that the estimators are consistent with errors and standard deviations decreasing as the sample size increases, and the test exhibits good size in moderate samples but noticeable over-rejection when n is small. Comparing the two values of ρ, over-rejection under the null is more pronounced for the smaller ρ, although the estimators exhibit faster convergence.

5. CONCLUSION

Our analysis demonstrates that Gauss–Newton estimators in spatial autoregressive models with one-directional near-unit dependence exhibit unique scaling behavior, which is markedly different from the classical i.i.d. case as in e.g., [5]. This highlights the importance of carefully accounting for persistent spatial correlations when conducting inference, even in relatively simple models. While we focus on a particular constrained moving average error structure, the framework could extend to other types of spatial dependence, and previous work has shown that models of this form provide useful approximations for field data [7, 8]. The referee mentions an extension to the more general error structure of the form $\delta_{ij} = \varepsilon_{ij} - \theta_{n1}\varepsilon_{i-1,j} - \theta_{n2}\varepsilon_{i,j-1}$, $1 \leq i, j \leq n$, where $\theta_{n1}, \theta_{n2} \to 1$ as $n \to \infty$. Our preliminary work shows that in

this case, $\frac{1}{n^3}\sum_{i,j=1}^{n} X_{i-1,j}^2 \xrightarrow{\mathrm{P}} \frac{\sigma^2}{2}$ and $\frac{1}{n^2}\sum_{i,j=1}^{n} X_{i-1,j}\delta_{ij} \xrightarrow{\mathrm{P}} -\sigma^2$. Likewise, due to symmetry, $\frac{1}{n^3}\sum_{i,j=1}^{n} Y_{i,j-1}^2 \xrightarrow{\mathrm{P}} \frac{\sigma^2}{2}$ and $\frac{1}{n^2}\sum_{i,j=1}^{n} Y_{i,j-1}\delta_{ij} \xrightarrow{\mathrm{P}} -\sigma^2$. This suggests that perhaps the Gauss-Newton estimators are both degenerate; however, further details need to be undertaken.

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