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MULTIPLE TURNPIKES FOR AN OPTIMAL CONTROL PROBLEM IN A COMPACT METRIC SPACE

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Abstract. In this paper, we study the structure of approximate solutions of a discrete-time optimal control in a compact metric which possesses multiple turnpikes.

Keywords. Approximate solutions; Optimal control problem, trajectory-control pair; Turnpike.

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1. Introduction

The study of the existence and the structure of solutions of variational problems, optimal control problems and dynamic games defined on infinite intervals and on sufficiently large intervals has been a rapidly growing area of research [2, 3, 8, 9, 14, 17, 19, 20, 22, 33, 35, 36, 41, 54, 56, 57, 58, 59] which has various applications in engineering [14, 54], in models of economic growth [1, 13, 14, 18, 23, 31, 32, 34, 38, 40, 46, 50, 52, 54], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [4, 53], in model predictive control [16, 25] and in the theory of thermodynamical equilibrium for materials [15, 39, 43, 44, 45]. Discrete-time problems optimal control problems were considered in [2, 5, 6, 12, 21, 29], finite-dimensional continuous-time problems were analyzed in [8, 10, 11, 38, 42], infinite-dimensional optimal control was studied in [14, 26, 27, 47, 48, 51, 58] while solutions of dynamic games were discussed in [7, 24, 28, 30, 37, 49].

In this paper we study the turnpike phenomenon for discrete-time optimal control problems in compact metric spaces. To have the turnpike property means, roughly speaking, that the approximate solutions of the problems are determined mainly by the objective function and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints.

The turnpike property was discovered by P. Samuelson in 1948 when he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). It is well known in the economic literature, where it

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was studied for various models of economic growth. Usually for these models a turnpike is a singleton.

Now it is well-known that the turnpike property is a general phenomenon which holds for large classes of variational and optimal control problems. In our research, using the Baire category (generic) approach, it was shown that the turnpike property holds for a generic (typical) variational problem [54] and for a generic optimal control problem [55].

In this paper we are interested in individual (non-generic) turnpike results for an optimal control problem with a finite number of turnpikes. In this case the description of the structure of approximate optimal solutions is obtained. For this optimal control problem we extend the results obtained in [60] for its particular case arising in model of economic dynamics.

2. Preliminaries

Assume that (X,ρ_X), (Y,ρ_Y) are compact metric spaces, $X\times Y$ is equipped with the product metric $\rho_{X\times Y}$, where

$$\rho_{X\times Y}((x_1,y_1),(x_2,y_2)) = \rho_X(x_1,x_2)+\rho_Y(y_1,y_2),\ (x_i,y_i)\in X\times Y,\ i=1,2,$$

$\mathscr{U}: X\to 2^X$, $\mathrm{graph}(\mathscr{U}) = \{(x,y):\ x\in X,\ y\in\mathscr{U}(x)\}$ is a nonempty and closed set,

$$F:\mathrm{graph}(F)\to 2^X\setminus\{\emptyset\},$$

$$\mathscr{M} = \mathrm{graph}(\mathscr{F})\{(x,y,z):\ x\in X,\ y\in\mathscr{U}(x),\ z\in F(x,y)\} \tag{2.1}$$

is a closed set. For each $x\in X$, each $y\in Y$ and each $r>0$ set $B(x,r)=\{z\in X:\ \rho_X(x,z)\le r\}$ and $B(y,r)=\{z\in Y:\ \rho_Y(z,y)\le r\}$. Here we use for simplicity the same notation for balls in the both spaces X and Y.

We suppose that the sum over empty set is zero and that the infimum over empty set is infinity. For each $x\in X$ and each set $A\subset X$, put $\rho_X(x,A)=\inf\{\rho_X(x,a):\ a\in A\}$ and for each $x\in Y$ and each set $A\subset Y$, put $\rho_Y(x,A)=\inf\{\rho_Y(x,a):\ a\in A\}$. For each $(x_1,x_2)\in X\times Y$ and each set $A\subset X\times Y$, put

$$\rho_{X\times Y}((x_1,x_2),A) = \inf\{\rho_{X\times Y}((x_1,x_2),(a_1,a_2)):\ (a_1,a_2)\in A\}.$$

Denote by $\mathrm{Card}(A)$ the cardinality of a set A. Assume l is a natural number and $x_i^*\in X$, $i=1,\dots,l$ satisfy

$$x_{i_1}\neq x_{i_2} \tag{2.2}$$

for each $i_1,i_2\in\{1,\dots,l\}$ such that $i_1\neq i_2$ and that for each $i\in\{1,\dots,l\}$ there exists $u_i^*\in Y$ satisfying

$$(x_i^*,u_i^*,x_i^*)\in\mathscr{M}. \tag{2.3}$$

Assume that for each $i_1,i_2\in\{1,\dots,l\}$ such that $i_1\neq i_2$ we have

$$x_{i_2}\notin F(x_{i_1},u_{i_1}). \tag{2.4}$$

Let $T_2>T_1$ be integers. A sequence $(\{x_t\}_{t=T_1}^{T_2},\{u_t\}_{t=T_1}^{T_2-1})$ is called a trajectory-control pair if

$$(x_t,u_t,x_{t+1})\in\mathscr{M} \tag{2.5}$$

for all $t = T_1,\dots,T_2-1$. A sequence $(\{x_t\}_{t=T_1}^{\infty},\{u_t\}_{t=T_1}^{\infty})$ is called a trajectory-control pair if (2.5) holds for all integers $t \geq T_1$. As usual in optimal control theory we consider the minimization of some objective functions over sets of trajectory-control pairs. Note that in Chapter 2 we consider a partial case of the optimal control problems considered here for which

$$u_t = x_{t+1},\ t = T_1,\dots,T_2-1.$$

Proposition 2.1. *Let $i_1,i_2 \in \{1,\dots,l\}$ satisfy $i_1 \neq i_2$. Then there exists $\gamma > 0$ such that for each $(x,u) \in X \times Y$ satisfying $\rho_{X\times Y}(((x^*_{i_1},u^*_{i_1}),(x,u)) \leq \gamma$, $\rho_X(x^*_{i_2},F(x,u)) \geq \gamma$ holds.*

Proof. Assume the contrary. Then for each integer $k \geq 1$ there exists $(x_k,u_k) \in X \times Y$ such that $\rho_X(x^*_{i_1},x_k)$, $\rho_Y(u^*_{i_1},u_k) \leq 1/k$ and $\rho_X(x^*_{i_2},F(x_k,u_k)) \leq k^{-1}$. This implies that for each integer $k \geq 1$ there exists $z_k \in F(x_k,u_k)$ such that $\rho_X(x^*_{i_2},z_k) \leq 2/k$. Since the graph of F is closed we obtain $x_{i_2} \in F(x^*_{i_1},u^*_{i_1})$, a contradiction. The contradiction we have reached completes the proof of Proposition 2.1. □

Proposition 2.1 implies the following result.

Proposition 2.2. *There exists $\gamma > 0$ such that for each $i_1,i_2 \in \{1,\dots,l\}$ satisfying $i_1 \neq i_2$ and each $(x,u) \in X \times Y$ satisfying $\rho_{X\times Y}(((x^*_{i_1},u^*_{i_1}),(x,u)) \leq \gamma$, $\rho_X((x^*_{i_2},F(x,u)) \geq \gamma$ holds.*

Assume that $L : \mathrm{graph}(U) \to [0,\infty)$ is a lower semicontinuous function such that

$$\{(x^*_i,u^*_i) :\ i \in \{1,\dots,l\}\} = \{(x,y) \in \mathrm{graph}(\mathscr{U}) :\ L(x,y) = 0\}, \tag{2.6}$$

$\bar{a} > 0$, $\mu \in R^1$ and that $\pi : X \to R^1$ is a bounded function satisfying

$$|\pi(x)| \leq \bar{a},\ x \in X. \tag{2.7}$$

Define for all $(x,u,z) \in \mathscr{M}$,

$$v(x,u,z) = L(x,u) + \mu + \pi(x) - \pi(z) \tag{2.8}$$

which is our objective function. Infinite horizon optimal control problems with such objective functions are often considered in the literature [14, 25, 56]. They correspond to certain models of economic dynamics.

The next result easily follows from (2.8).

Proposition 2.3. *Let $T_2 > T_1$ be integers and $(\{x_t\}_{t=T_1}^{T_2},\{u_t\}_{t=T_1}^{T_2-1})$ be a trajectory-control pair. Then*

$$\sum_{t=T_1}^{T_2-1} v(x_t,u_t,x_{t+1}) = \sum_{t=T_1}^{T_2-1} L(x_t,u_t) + \mu(T_2-T_1) + \pi(x_{T_1}) - \pi(x_{T_2}).$$

Proposition 2.4. *Assume that $(\{x_t\}_{t=0}^{\infty},\{u_t\}_{t=0}^{\infty})$ is a trajectory-control pair. Then either the sequence $\{\sum_{t=0}^{T-1} v(x_t,u_t,x_{t+1}) - \mu\}_{T=1}^{\infty}$ is bounded or it tends to infinity. It is bounded if and only is $\sum_{t=0}^{\infty} L(x_t,u_t) < \infty$ and in this case the sequence $\{(x_t,u_t)\}_{t=0}^{\infty}$ converges to one of the points of the set $\{(x^*_i,u^*_i) :\ i = 1,\dots,l\}$.*

Proof. Assume that $\sum_{t=0}^{\infty} L(x_t,u_t) < \infty$. In order to to prove the theorem it is sufficient to show that the sequence $\{(x_t,u_t)\}_{t=0}^{\infty}$ converges to one of the points of the set $\{(x^*_i,u^*_i) :\ i = 1,\dots,l\}$. Clearly, $\lim_{t\to\infty} L(x_t,u_t) = 0$. This implies that the following property holds:

(a) all limit points of the sequence $\{(x_t,u_t)\}_{t=0}^{\infty}$ belongs to the set $\{(x^*_i,u^*_i) :\ i = 1,\dots,l\}$.

Let $\varepsilon \in (0,1)$ be such that

$$\rho(x^*_{i_1}, x^*_{i_2}) > 2\varepsilon \tag{2.9}$$

for each pair of integer $i_1, i_2 \in \{1,\dots,l\}$ satisfying $i_1 \neq i_2$. Proposition 2.2 implies that there exists $\varepsilon_0 \in (0, \varepsilon/4)$ such that the following property holds:

(b) for each $i_1, i_2 \in \{1,\dots,l\}$ satisfying $i_1 \neq i_2$ and each $(x,u) \in X \times Y$ satisfying

$$\rho_{X\times Y}((x^*_{i_1}, u^*_{i_1}), (x,u)) \le \varepsilon_0,$$

the inequality $\rho_X(x^*_{i_2}, F(x,u)) \ge \varepsilon_0$ holds.

Property (a) implies that there exists a natural number n_0 such that for each integer $n \ge n_0$,

$$\min\{\rho_{X\times Y}((x_n,u_n),(x^*_i,u^*_i)) :\ i = 1,\dots,l\} < \varepsilon/4. \tag{2.10}$$

Let $n \ge n_0$ be an integer. By (2.10), there exists $i_n \in \{1,\dots,l\}$ such that

$$\rho_{X\times Y}((x_n,u_n),(x^*_{i_n},u^*_{i_n})) < \varepsilon_0/4. \tag{2.11}$$

We we show that such $i_n \in \{1,\dots,l\}$ satisfying (2.11) is unique. Assume that $j \in \{1,\dots,l\}$ and

$$\rho_{X\times Y}((x_n,u_n),(x^*_j,u^*_j)) < \varepsilon_0/4.$$

Together with (2.11) this implies that $\rho_{X\times Y}((x^*_j,u^*_j),(x^*_{i_n},u^*_{i_n})) < \varepsilon_0/2$ and in view of (2.9), $i_n = j$. Thus for each integer $n \ge n_0$ there exists a unique $i_n \in \{1,\dots,l\}$ such that (2.11) holds. Let $n \ge n_0$ be an integer. Equation (2.11) implies that

$$\begin{aligned}
&\rho_{X\times Y}((x_n,u_n),(x^*_{i_n},u^*_{i_n})) < \varepsilon_0/4,\\
&\rho_{X\times Y}((x_{n+1},u_{n+1}),(x^*_{i_{n+1}},u^*_{i_{n+1}})) < \varepsilon_0/4,\\
&\rho_X(x_{n+1},x^*_{i_{n+1}}) < \varepsilon_0/4.
\end{aligned} \tag{2.12}$$

Let $i_n \neq i_{n+1}$. Property (b) and (2.11) imply that $\rho_X(x^*_{i_{n+1}}, F(x_n,u_n)) \ge \varepsilon_0/2$ and $\rho_X(x_{n+1}, x^*_{i_{n+1}}) \ge \varepsilon_0$. This contradicts (2.12). Therefore $i_n = i_{n+1}$ for each integer $n \ge n_0$. This implies that $i_n = i_{n_0}$ for each integer $n \ge n_0$ and that for each integer $n \ge n_0$, $\rho_X(x_n, x^*_{i_{n_0}})$, $\rho_Y(u_n, u^*_{i_{n_0}}) < \varepsilon_0/4$. This implies that any limit point (z,u) of the sequence $\{(x_i,u_i)\}_{i=0}^\infty$ satisfies

$$\rho_{X\times Y}((z,u),(x^*_{i_{n_0}},x^*_{i_{n_0}})) \le \varepsilon_0/4.$$

Together with property (a) and (2.9), this implies that $\lim_{n\to\infty}(x_n,u_n) = (x^*_{i_{n_0}}, u^*_{i_{n_0}})$. Proposition 2.4 is proved. □

Let M be a natural number. Denote by $\mathscr{A}(M)$ the set of all $x \in X$ for which there exists a trajectory-control pair $(\{x_t\}_{t=0}^T, \{u_t\}_{t=0}^{T-1})$ such that $x_0 = x$, $T \in \{1,\dots,M\}$,

$$\sum_{t=0}^{T-1} v(x_t,u_t,x_{t+1}) \le M,\ x_T \in \{x^*_i :\ i = 1,\dots,l\}.$$

For each integer $T > 0$ and each $x \in X$ set

$$U(T,x) = \inf\{\sum_{t=0}^{T-1} v(x_t,u_t,x_{t+1}) :\ (\{x_t\}_{t=0}^T, \{u_t\}_{t=0}^{T-1}) \text{ is a trajectory and } x_0 = x\}.$$

Proposition 2.5. *Let M be a natural number and $x \in \mathscr{A}(M)$. Then, for each integer $T > M$, $U(T,x) \le M + \mu T + |\mu| M$.*

Proof. By the definition of $\mathscr{A}(M)$, there exists a trajectory-control pair $(\{x_t\}_{t=0}^{S}, \{u_t\}_{t=0}^{S-1})$ such that $x_0 = x$, $S \in \{1,\dots,M\}$ and $\sum_{t=0}^{S-1} v(x_t,u_t,x_{t+1}) \le M$, $x_S = x_i^p$, where $p \in \{1,\dots,l\}$. For each integer $t > S$, set $u_t = u_p^*$, $x_{t+1} = x_p^*$. Clearly, $(\{x_t\}_{t=0}^{\infty}, \{u_t\}_{t=0}^{\infty})$ is a trajectory-control pair. It is not difficult to see that for each integer $T > M$,

$$U(T,x) \le \sum_{t=0}^{T-1} v(x_t,u_t,x_{t+1}) = \sum_{t=0}^{S-1} v(x_t,u_t,x_{t+1}) + \sum_{t=S}^{T-1} v(x_t,u_t,x_{t+1})$$
$$\le M + (T-S)\mu \le M + \mu T + |\mu| M.$$

Proposition 2.5 is proved. □

3. A First Weak Turnpike Result

In this section, we prove our first turnpike result. It shows that for approximate solutions $(\{x_t\}_{t=0}^{T}, \{u_t\}_{t=0}^{T})$ of our optimal control problem on an interval $[0,T]$, where T is sufficiently large, cardinality of all points $t \in \{0,\dots,T\}$ such that (x_t,u_t) does not belong to an ε-neighborhood of the set $\{(x_i^*,u_i^*) : i = 1,\dots,l\}$ does not exceed a constant Q which depends on ε and does not depend on T. In the literature this property is known as the weak turnpike property.

Theorem 3.1. *Let M_0 be a natural number, $M_1 > 0$ and $\varepsilon \in (0,1)$. Then there exists a natural number Q such that for each natural number $T > Q$ and each trajectory-control pair $(\{x_t\}_{t=0}^{T}, \{u_t\}_{t=0}^{T-1})$ which satisfies $x_0 \in \mathscr{A}(M_0)$ and $\sum_{t=0}^{T-1} v(x_t,u_t,x_{t+1}) \le U(T,x_0) + M_1$, there exists a finite strictly increasing sequence of nonnegative integers $t_1 < \cdots < t_k$ where $k \le Q$ is a natural number such that $t_1 = 0$, $t_k = T$ and for each integer $p \in \{1,\dots,k\} \setminus \{k\}$ satisfying $t_{p+1} - t_p \ge 2$ there exists $i_p \in \{1,\dots,l\}$ such that*

$$\rho_X(x_t, x_{i_p}^*),\ \rho_Y(u_t, u_{i_p}^*) \le \varepsilon,\ t = t_p + 1,\dots,t_{p+1} - 1.$$

Proof. We may assume without loss of generality that

$$\rho(x_{i_1}^*, x_{i_2}^*) > 4\varepsilon \tag{3.1}$$

for each pair of integers $i_1, i_2 \in \{1,\dots,l_2\}$ satisfying $i_1 \ne i_2$. Proposition 2.2 implies that there exists $\varepsilon_0 \in (0, \varepsilon/4)$ such that the following property holds:

(a) for each $i_1, i_2 \in \{1,\dots,l\}$ satisfying $i_1 \ne i_2$ and each $(x,u) \in X \times Y$ satisfying

$$\rho_{X\times Y}((x_{i_1}^*, u_{i_1}^*), (x,u)) \le \varepsilon_0,$$

the inequality $\rho_X(x_{i_2}^*, F(x,u)) \ge \varepsilon_0$ holds. There exists $\delta \in (0,\varepsilon_0)$ such that the following property holds:

(b) for each $(x,y) \in \Omega$ satisfying $L(x,y) \le \delta$, we have $\rho_{X\times Y}((x,u), \{(x_i^*,u_i^*) : i = 1,\dots,l\}) < \varepsilon/4$. Choose a natural number

$$Q > \delta^{-1}(M_1 + 2\bar{a} + M_0(|\mu| + 1)) + 2. \tag{3.2}$$

Assume that $T > Q$ is a natural number and that a trajectory-control pair $(\{x_t\}_{t=0}^{T}, \{u_t\}_{t=0}^{T-1})$ satisfies

$$x_0 \in \mathscr{A}(M_0) \tag{3.3}$$

and

$$\sum_{t=0}^{T-1} v(x_t,u_t,x_{t+1}) \le U(x_0,T) + M_1. \tag{3.4}$$

Proposition 2.5 and equations (3.3) and (3.4) imply that

$$\sum_{t=0}^{T-1} v(x_t,u_t,x_{t+1}) \le M_1 + M_0(|\mu|+1) + \mu T. \tag{3.5}$$

By (2.7) and (2.8),

$$\sum_{t=0}^{T-1} v(x_t,u_t,x_{t+1}) = \sum_{t=0}^{T-1} (\mu + L(x_t,u_t) + \pi(x_t) - \pi(x_{t+1}))$$

$$\ge T\mu + \sum_{t=0}^{T-1} L(x_t,u_t) + \pi(x_0) - \pi(x_T) \ge T\mu + \sum_{t=0}^{T-1} L(x_t,u_t) - 2\bar{a}. \tag{3.6}$$

It follows from (3.2), (3.5) and (3.6) implies that

$$\sum_{t=0}^{T-1} L(x_t,u_t) + T\mu - 2\bar{a} \le M_1 + \mu T + (|\mu|+1)M_0,$$

$$M_1 + M_0(|\mu|+1) + 2\bar{a} \ge \sum_{t=0}^{T-1} L(x_t,u_t)$$

$$\ge \delta \mathrm{Card}(\{t \in \{0,\dots,T-1\} : L(x_t,u_t) \ge \delta\})$$

and

$$\mathrm{Card}(\{t \in \{0,\dots,T-1\} : L(x_t,x_{t+1}) \ge \delta\})$$
$$\le \delta^{-1}(M_1 + (|\mu|+1)M_0 + 2\bar{a}) < Q - 2. \tag{3.7}$$

Inequality (3.7) implies that there exists a finite strictly increasing sequence of nonnegative integers $t_1 < \dots < t_k$ where $k \le Q$ is a natural number such that

$$t_1 = 0,\ t_k = T \tag{3.8}$$

and that for each integer $p \in \{1,\dots,k\} \setminus \{k\}$ satisfying $t_{p+1} - t_p \ge 2$ we have

$$L(x_t,u_t) \le \delta,\ t = t_p + 1,\dots,t_{p+1} - 1. \tag{3.9}$$

Assume that $p \in \{1,\dots,k\} \setminus \{k\}$ satisfies $t_{p+1} - t_p \ge 2$. Let $t \in \{t_p + 1,\dots,t_{p+1} - 1\}$. Property (b) and (3.9) imply that there exists $i_t \in \{1,\dots,l\}$ such that

$$\rho_{X\times Y}((x_t,u_t),(x^*_{i_t},u^*_{i_t})) \le \varepsilon_0/2. \tag{3.10}$$

We show that i_t is a unique element of $\{1,\dots,l\}$ such that (3.10) holds. Let $j \in \{1,\dots,l\}$ satisfy

$$\rho_{X\times Y}((x_t,u_t),(x^*_j,u^*_j)) \le \varepsilon_0/2.$$

Together with (3.10), this implies that $\rho_X(x^*_j,x^*_{i_t}) \le \varepsilon_0 < \varepsilon$. In view of (3.1), we have $j = i_t$. Thus i_t is a unique element of $\{1,\dots,l\}$ such that (3.10) holds.

Assume that $t,\ t+1 \in \{t_p + 1,\dots,t_{p+1} - 1\}$. It follows from (3.10) that

$$\rho_{X\times Y}((x_t,u_t),(x^*_{i_t},u^*_{i_t})) < \varepsilon_0/2$$

and that

$$\rho_{X\times Y}((x_{t+1},u_{t+1}),(x^*_{i_{t+1}},u^*_{i_{t+1}})) < \varepsilon_0/2. \tag{3.11}$$

We show that $i_t = i_{t+1}$. Assume that $i_t \neq i_{t+1}$. Property (a) and (3.10) imply $\rho_X(x^*_{i_{t+1}}, F(x_t,u_t)) \geq \varepsilon_0$ and $\rho_X(x^*_{i_{t+1}}, x_{t+1}) \geq \varepsilon_0$. This contradicts (3.11). The contradiction we have reached proves that $i_t = i_{t+1}$ for each $t \in \{t_p+1,\dots,t_{p+1}-1\} \setminus \{t_{p+1}-1\}$,

$$i_i = i_{t_p+1},\ t = t_p+1,\dots,t_{p+1}-1$$

and

$$\rho_{X\times Y}((x_t,u_t),(x^*_{t_p+1},u^*_{t_p+1})) < \varepsilon,\ t \in \{t_p+1,\dots,t_{p+1}-1\}.$$

Theorem 3.1 is proved. □

4. A General Problem

Now we begin to study a general optimal control problem which contains as a particular case the model considered in the previous sections. We will not assumed any more that the objective function v has a special form which was assumed before. On the other hand the mapping F (here G) is single-valued.

Assume that (X,ρ_X), (Y,ρ_Y) be compact metric spaces, $X \times Y$ is equipped with the product metric $\rho_{X\times Y}$, where

$$\rho_{X\times Y}((x_1,y_1),(x_2,y_2)) = \rho_X(x_1,x_2) + \rho_Y(y_1,y_2),\ (x_i,y_i) \in X\times Y,\ i = 1,2,$$

$\mathscr{U} : X \to 2^Y$ is a point to set mapping with a graph

$$\mathscr{M} = \mathrm{graph}(\mathscr{U} = \{(x,y) :\ x \in X,\ y \in \mathscr{U}(x)\}$$

which is is a nonempty and closed set.

Assume that $G : \mathscr{M} \to X,\ f : \mathscr{M} \to R^1$ the graph graph(G) is a closed set and the function f is bounded from below and lower semicontinuos. Let $T_2 > T_1$ be integers. We denote by $X(T_1,T_2,\mathscr{M},G)$ the set of all finite sequences $(\{x_t\}_{t=T_1}^{T_2}, \{u_t\}_{t=T_1}^{T_2-1})$ which are called as a trajectory-control pair such that for each $t \in \{T_1,\dots,T_2-1\}$,

$$x_t \in X,\ ,u_t \in \mathscr{U}(x_t),\ x_{t+1} = G(x_t,u_t).$$

For simplicity, we use the notation $X(T_1,T_2) = X(T_1,T_2,\mathscr{M},G)$. Let T_1 be an integer. Denote by $X(T_1,\infty,\mathscr{M},G)$ (or $X(T_1,\infty)$, if the pair $(\mathscr{M},G)$ is understood) the set of all sequences $(\{x_t\}_{t=T_1}^{\infty}, \{u_t\}_{t=T_1}^{\infty})$ which are called as a trajectory-control pair such that for each integer $T_2 > T_1$, we have $(\{x_t\}_{t=T_1}^{T_2}, \{u_t\}_{t=T_1}^{T_2-1}) \in X(T_1,T_2,\mathscr{M},G)$. Let $0 \leq T_1 < T_2$ be integers. A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$ ($\{x_t\}_{t=T_1}^{\infty} \subset X$ respectively) is called a trajectory if there exists a sequence $\{u_t\}_{t=T_1}^{T_2-1} \subset Y$ ($\{u_t\}_{t=T_1}^{\infty} \subset Y$ respectively) such that

$$(\{x_t\}_{t=T_1}^{T_2}, \{u_t\}_{t=T_1}^{T_2-1}) \in X(T_1,T_2,\mathscr{M},G),\ ((\{x_t\}_{t=T_1}^{\infty}, \{u_t\}_{T_1}^{\infty}) \in X(T_1,\infty,\mathscr{M},G)$$

respectively).

For each pair of integers $T_2 > T_1$ and each $y,z \in X$ define

$$U(T_1,T_2,y,z) = \inf\{\sum_{t=T_1}^{T_2-1} f(x_t,u_t) :$$

$$(\{x_t\}_{t=T_1}^{T_2}, \{u_t\}_{t=T_1}^{T_2-1}) \in X(T_1,T_2),\ x_{T_1} = y,\ x_{T_2} = z\} \tag{4.1}$$

and

$$U(T_1,T_2,y)=\inf\{\sum_{t=T_1}^{T_2-1} f(x_t,u_t):$$

$$(\{x_t\}_{t=T_1}^{T_2},\{u_t\}_{t=T_1}^{T_2-1})\in X(T_1,T_2),\ x_{T_1}=y\}. \tag{4.2}$$

Assume l is a natural number and $x_i^*\in X$, $i=1,\dots,l$ satisfy

$$x_{i_1}\neq x_{i_2} \tag{4.3}$$

for each $i_1,i_2\in\{1,\dots,l\}$ such that $i_1\neq i_2$ and that for each $i_1\in\{1,\dots,l\}$ there exists $u_i^*\in Y$ satisfying

$$(x_i^*,u_i^*)\in\mathscr{M},\ x_i^*=G(x_i^*,u_i^*) \tag{4.4}$$

and f is continuous at (x_i^*,u_i^*) on $\mathscr{M}$.

Assume that the following property holds:

(a) for each $\varepsilon>0$ there exists $\delta>0$ such that for each $i\in\{1,\dots,l\}$ and each $x\in B(x_i^*,\delta)$ there exist $u\in Y$ satisfying

$$(x,u)\in\mathscr{M},\ \rho_Y(u,u_i^*)\le\varepsilon,\ G(x,u)=x_i^*$$

and $v\in B(u_i^*,\varepsilon)$ such that $(x_i^*,v)\in\mathscr{M}$, $G(x_i^*,v)=x$.

Proposition 4.1. *Let $i_1,i_2\in\{1,\dots,l\}$ satisfy $i_1\neq i_2$. Then there exists $\gamma>0$ such that for each $(x,u)\in\mathscr{M}$ satisfying $\rho_{X\times Y}((x_{i_1}^*,u_{i_1}^*),(x,u))\le\gamma$, the inequality $\rho_X(x_{i_2}^*,G(x,u))\ge\gamma$ holds.*

Proof. Assume the contrary. Then, for each integer $k\ge1$ there exists $(x_k,u_k)\in X\times Y$ such that $u_k\in\mathscr{U}(x_k)$, $\rho_X((x_{i_1}^*,x_k),\ \rho_Y(u_{i_1}^*,u_k))\le 1/k$, and $\rho_X(x_{i_2}^*,G(x_k,u_k))\le k^{-1}$. Since the graph of G is closed, we obtain $x_{i_2}=G(x_i^*,u_i^*)$, a contradiction. The contradiction we have reached completes the proof of Proposition 4.1. □

Assume that $\mu=f(x_1^*,u_1^*)$, $i=1,\dots,l$. We suppose that the following assumption holds.

(A1) There exists $\bar c>0$ such that for each pair of integers $T_2>T_1$ and each trajectory-control pair $(\{x_t\}_{t=T_1}^{T_2},\{u_t\}_{t=T_1}^{T_2-1})\in X(T_1,T_2)$, $\sum_{t=T_1}^{T_2-1} f(x_t,u_t)\ge (T_2-T_1)\mu-\bar c$ holds.

Assumption (A1) implies the following result.

Proposition 4.2. *Assume that $(\{x_t\}_{t=0}^\infty,\{u_t\}_{t=0}^\infty)\in X(0,\infty)$. Then either $\{\sum_{t=0}^{T-1} f(x_t,u_t)-T\mu\}_{T=1}^\infty$ is bounded or $\lim_{T\to\infty}\sum_{t=0}^{T-1}(f(x_t,u_t)-\mu)=\infty$.*

Proof. By (A1), we see that $\{\sum_{t=0}^{T-1} f(x_t,u_t)-T\mu\}_{T=1}^\infty$ is bounded from below. Assume that it is not bounded. Let $M>0$. Then there exists a natural number T_M such that $\sum_{t=0}^{T_M-1} f(x_t,u_t)-T_M\mu>M+\bar c$. It follows from this inequality and assumption (A1) that for each integer $T>T_M$,

$$\sum_{t=0}^{T-1} f(x_t,u_t)-T\mu\ge\sum_{t=0}^{T_M-1} f(x_t,u_t)-T_M\mu+\sum_{t=T_M}^{T-1} f(x_t,u_t)-(T-T_M)\mu\ge M+\bar c-\bar c=M.$$

Since M is any positive number, this implies $\lim_{T\to\infty}\sum_{t=0}^{T-1}(f(x_t,u_t)-\mu)=\infty$. Proposition 4.2 is proved. □

A trajectory-control pair $(\{x_t\}_{t=0}^\infty,\{u_t\}_{t=0}^\infty)\in X(0,\infty)$ is called good if $\{\sum_{t=0}^{T-1} f(x_t,u_t)-\mu\}_{T=1}^\infty$ is bounded.

We suppose that the following asymptotic turnpike property ((ATP), for short) holds.

(ATP) For each good trajectory-control pair $(\{x_t\}_{t=0}^{\infty}, \{u_t\}_{t=0}^{\infty}) \in X(0,\infty)$, we have

$$\lim_{t\to\infty} \rho_{X\times Y}((x_t,u_t), \{(x_i^*,u_i^*):\ i=1,\dots,l\}) = 0.$$

Note that (ATP) holds for models of economic dynamics [56] and for optimal control problems considered in the previous section.

Theorem 4.3. *Assume that a trajectory-control pair $(\{x_t\}_{t=0}^{\infty}, \{u_t\}_{t=0}^{\infty}) \in X(0,\infty)$ is good. Then there exists $i \in \{1,\dots,l\}$ such that $\lim_{t\to\infty} \rho_{X\times Y}((x_t,u_u),(x_i^*,u_i^*)) = 0$.*

Proof. Let $\varepsilon \in (0,1)$. Proposition 4.1 implies that there exists $\varepsilon_0 \in (0,\varepsilon/4)$ such that the following property holds:

(i) for each $i_1, i_2 \in \{1,\dots,l\}$ satisfying $i_1 \neq i_2$, we have $\rho_X(x_{i_1}^*, u_{i_1}^*) > 2\varepsilon_0$ and for each $(x,u) \in \mathscr{M}$ satisfying $\rho_{X\times Y}((x_{i_1}^*,u_{i_1}^*),(x,u)) \le \varepsilon_0$, the inequality $\rho_X(x_{i_2}^*, G(x,u)) \ge \varepsilon_0$ holds.

By (A2), there exists a natural number n_0 such that for each integer $n \ge n_0$,

$$\rho_{X\times Y}((x_n,u_n), \{(x_i^*,u_i^*):\ i=1,\dots,l\}) < \varepsilon_0/4. \tag{4.5}$$

Let $n \ge n_0$ be an integer. By (4.5), there exists $i_n \in \{1,\dots,l\}$ such that

$$\rho_{X\times Y}((x_n,u_n),(x_{i_n}^*,u_{i_n}^*)) < \varepsilon/4. \tag{4.6}$$

We we show that such $i_n \in \{1,\dots,l\}$ satisfying (4.6) is unique. Assume that $j \in \{1,\dots,l\}$ and

$$\rho_{X\times Y}((x_n,u_n),(x_j^*,u_j^*)) < \varepsilon_0/4.$$

Together with (4.6), this implies $\rho_X(x_j, x_{i_n}^*) < \varepsilon_0/2$. In view of property (i), $i_n = j$. Thus, for each integer $n \ge n_0$, there exists a unique integer $i_n \in \{1,\dots,l\}$ such that (4.6) holds.

Let $n \ge n_0$ be an integer. Then (4.6) holds and

$$\rho_{X\times Y}((x_{n+1},u_{n+1}),(x_{i_{n+1}}^*,u_{i_{n+1}}^*)) < \varepsilon_0/4. \tag{4.7}$$

We show that $i_n = i_{n+1}$. Assume that $i_n \neq i_{n+1}$. Property (i) and (4.6) imply $\rho_{X\times Y}(G(x_n,u_n), x_{i_{n+1}}^*) \ge \varepsilon_0$ and $\rho_X(x_{n+1}, x_{i_{n+1}}^*) > \varepsilon_0$. This contradicts (4.7). The contradiction we have reached proves that $i_n = i_{n+1}$ for each integer $n \ge n_0$. This implies that any limit point (x,u) of the sequence $(\{x_i\}_{i=0}^{\infty}, \{u_i\}_{i=0}^{\infty})$ satisfies $\rho_{X\times Y}((x,u),(x_{i_{n_0}}^*,u_{i_{n_0}}^*)) \le \varepsilon_0$ and combined with (A2) and property (i) this implies that $(x,u) = (x_{i_{n_0}}^*, u_{i_{n_0}}^*)$. Therefore, $\lim_{n\to\infty}(x_n,u_n) = (x_{i_{n_0}}^*, u_{i_{n_0}}^*)$. Theorem 4.3 is proved. □

5. Auxiliary Results

The following two results follow from the lower semicontinuity of the function f and the closedness of the

$$\operatorname{graph}(G) := \{(x,u,G(x,u)):\ (x,u) \in \mathscr{M}\}.$$

Proposition 5.1. *Let $T_2 > T_1$ be integers and let $x,y \in X$ satisfy $U(T_1,T_2,x,y) < \infty$. Then there exists $(\{x_t\}_{t=T_1}^{T_2}, \{u_t\}_{t=T_1}^{T_2-1}) \in X(T_1,T_2)$ such that $x_{T_1} = x$, $x_{T_2} = y$ and $\sum_{t=T_1}^{T_2-1} f(x_t,u_t) = U(T_1,T_2,x,y)$.*

Proposition 5.2. *Let $T_2 > T_1$ be integers and let $x \in X$ satisfy $U(T_1,T_2,x) < \infty$. Then there exists $(\{x_t\}_{t=T_1}^{T_2}, \{u_t\}_{t=T_1}^{T_2-1}) \in X(T_1,T_2)$ such that $x_{T_1} = x$ and $\sum_{t=T_1}^{T_2-1} f(x_t,u_t) = U(T_1,T_2,x)$.*

Proposition 5.3. *Let T be a natural number and $j \in \{1, \dots, l\}$. Then*

$$U(0,T,x_j^*,x_j^*) = T\mu \tag{5.1}$$

and $(\{x_t\}_{t=0}^T, \{u_t\}_{t=0}^{T-1}) \in X(0,T)$ satisfies $x_0 = x_j^*$, $x_T = x_j^*$, and $\sum_{t=0}^{T-1} f(x_t,u_t) = T\mu$ if and only if $x_t = x_j^*$, $u_t = u_j^*$, $t = 0, \dots, T$.

Proof. Assume that $(\{x_t\}_{t=0}^T, \{u_t\}_{t=0}^{T-1}) \in X(0,T)$ satisfies $x_0 = x_j^*$, $x_T = x_j^*$. Clearly, there exists $(\{x_t\}_{t=0}^\infty, \{u_t\}_{t=0}^\infty) \in X(0,\infty)$ such that for all integers $t \geq 0$,

$$x_{t+T} = x_t, \; u_{t+T} = u_t. \tag{5.2}$$

Clearly, if $\sum_{t=0}^{T-1} f(x_t,x_{t+1}) < T\mu$, then $\liminf_{S\to\infty}(\sum_{t=0}^{S-1} f(x_t,u_t) - S\mu) = -\infty$ and this contradicts Proposition 3.8. Thus $\sum_{t=0}^{T-1} f(x_t,u_t) \geq T\mu$. This inequality implies (5.1).

Assume that

$$\sum_{t=0}^{T-1} f(x_t,u_t) = T\mu. \tag{5.3}$$

It follows from (5.3) and Proposition 4.2 that $(\{x_t\}_{t=0}^\infty, \{u_t\}_{t=0}^\infty)$ is a good trajectory-control pair and in view of Theorem 4.3, it converges to $(x_i,^*, u_i^*)$ where $i \in \{1, \dots, l\}$. By (5.2), $i = j$. Proposition 5.3 is proved. □

6. The first basic lemma

Lemma 6.1. *Let $\varepsilon \in (0,1)$. Then there exists a number $\delta > 0$ such that for each natural number $T \geq 2$, each $p \in \{1, \dots, l\}$ and each $(\{x_t\}_{t=0}^T, \{u_t\}_{t=0}^{T-1}) \in X(0,T)$ which satisfies*

$$\rho_X(x_0,x_p^*), \; \rho_X(x_T,x_p^*) \leq \delta,$$

$\sum_{t=0}^{T-1} f(x_t,u_t) \leq U(0,T,x_0,x_T) + \delta$, the inequalities, $\rho_X(x_t,x_p^*)$, $\rho_Y(u_t,u_p^*) \leq \varepsilon$ hold for all integers $t = 0, \dots, T-1$.

Proof. Let $k \geq 1$ be an integer. Since f is continuous at (x_i^*, u_i^*), $i = 1, \dots, l$ there exists a number

$$\varepsilon_k \in (0, 4^{-k-1}) \tag{6.1}$$

such that, for each integer $p \in \{1, \dots, l\}$ and each $(x,u) \in \mathscr{M}$ satisfying $\rho_{X\times Y}((x,u),(x_p^*,u_p^*)) \leq 4\varepsilon_k$, we have

$$|f(x,u) - f(x_p^*,u_p^*)| \leq 4^{-k-1}. \tag{6.2}$$

Property (a) implies that here exists

$$\delta_k \in (0, 2^{-1}\varepsilon_k)] \tag{6.3}$$

such that the following property holds:

(i) for each $p \in \{1, \dots, l\}$ and each $x \in B(x_p^*, \delta_k)$, there exist $u, v \in B(u_p^*, \varepsilon_k)$ such that $(x,u) \in \mathscr{M}$, $G(x,u) = x_p^*$ and $(x_p^*, v) \in \mathscr{M}$, $G(x_p^*, v) = x$.

Assume that the lemma does not hold. Then for each integer $k \geq 2$ there exist a natural number $T_k \geq 2$, an integer $p_k \in \{1, \dots, l\}$ and a trajectory-control pair

$$(\{x_t^{(k)}\}_{t=0}^{T_k}, \{u_t^{(k)}\}_{t=0}^{T_k-1}) \in X(0,T_k)$$

such that

$$\rho_X(x_0^{(k)}, x_{p_k}^*), \; \rho_X(x_{T_k}^{(k)}, x_{p_k}^*) \leq \delta_k, \tag{6.4}$$

$$\sum_{t=0}^{T_k-1} f(x_t^{(k)}, u_t^{(k)}) \le U(0, T_k, x_0^{(k)}, x_{T_k}^{(k)}) + \delta_k, \tag{6.5}$$

$$\max\{\rho_X(x_t^{(k)}, x_{p_k}^*) + \rho_Y(u_t^{(k)}, u_{p_k}^*) :\ t = 0, \dots, T_k\} > \varepsilon. \tag{6.6}$$

Extracting a subsequence and re-indexing if necessary we may assume without loss of generality that

$$p_k = p_1,\ k = 1, 2, \dots. \tag{6.7}$$

For simplicity, set $p_1 = p$. Let $k \ge 1$ be an integer. Define

$$\begin{gathered} y_0 = x_0^{(k)},\ y_{T_k} = x_{T_k}^{(k)}, \\ u_t = u_p^*,\ t \in \{1, \dots, T_k - 1\} \setminus \{T_k - 1\}, \\ y_t = x_p^*,\ t \in \{1, \dots, T_k - 1\}. \end{gathered} \tag{6.8}$$

By property (i) and (6.4), there exist

$$u_0 \in B(u_p^*, \varepsilon_k),\ u_{T_k-1} \in B(u_p^*, \varepsilon_k) \tag{6.9}$$

such that

$$\begin{gathered} (y_0, u_0) \in \mathscr{M},\ G(y_0, u_0) = x_p^*, \\ (x_p^*, u_{T_k-1}) \in \mathscr{M},\ G(x_p^*, u_{T_k-1}) = x_{T_k}^{(k)}. \end{gathered} \tag{6.10}$$

Clearly, $(\{y_t\}_{t=0}^{T_k}, \{u_t\}_{t=0}^{T_k-1}) \in X(0, T_k)$. Equations (6.5) and (6.8) imply that

$$\sum_{t=0}^{T_k-1} f(x_t^{(k)}, u_t^{(k)}) \le \sum_{t=0}^{T_k-1} f(y_t, u_t) + \delta_k = f(x_0^{(k)}, u_0) + f(x_p^*, u_{T_k-1}) + f(x_p^*, u_p^*)(T_k - 2) + \delta_k. \tag{6.11}$$

It follows from (6.2), (6.4), (6.9) and (6.10) that

$$|f(x_0^{(k)}, u_0) - f(x_p^*, u_p^*)|,\ |f(x_p^*, u_{T_k-1}) - f(x_p^*, u_p^*)| \le 4^{-k-1}. \tag{6.12}$$

By (6.11) and (6.12),

$$\sum_{t=0}^{T_k-1} f(x_t^{(k)}, u_t^{(k)}) \le T_k f(x_p^*, u_p^*) + \delta_k + 2 \cdot 4^{-k-1}. \tag{6.13}$$

Set

$$\tilde{x}_0^{(k)} = x_p^*,\ \tilde{x}_t^{(k)} = x_{t-1}^{(k)},\ t = 1, \dots, T_k + 1,\ \tilde{x}_{T_k+2}^{(k)} = x_p^* \tag{6.14}$$

and

$$\tilde{u}_t^{(k)} = u_{t-1}^{(k)},\ t = 1, \dots, T_k. \tag{6.15}$$

Property (a) and equations (6.4), (6.14), (6.15) imply that there exist

$$\tilde{u}_0^{(k)} \in B(u_p^*, \varepsilon_k) \tag{6.16}$$

such that

$$(x_p^*, \tilde{u}_0^{(k)}) \in \mathscr{M},\ G(x_p^*, \tilde{u}_0^{(k)}) = x_0^{(k)} \tag{6.17}$$

and

$$\tilde{u}_{T_k+1}^{(k)} \in B(u_p^*, \varepsilon_k) \tag{6.18}$$

such that

$$(x_{T_k}^{(k)}, \tilde{u}_{T_k+1}^{(k)}) \in \mathscr{M},\ G(x_{T_k}^{(k)}, \tilde{u}_{T_k+1}^{(k)}) = x_p^*. \tag{6.19}$$

Equations (6.14), (6.15), (6.17) and (6.19) imply that

$$\{(\tilde{x}_t^{(k)}\}_{t=0}^{T_k+2}, (\tilde{u}_t^{(k)}\}_{t=0}^{T_k+1}) \in X(0, T_k+2).$$

By (6.14) and (6.15), we have

$$\sum_{t=0}^{T_k+1} f(\tilde{x}_t^{(k)}, \tilde{u}_t^{(k)}) = f(\tilde{x}_0^{(k)}, \tilde{u}_0^{(k)}) + \sum_{t=1}^{T_k} f(\tilde{x}_t^{(k)}, \tilde{u}_t^{(k)}) + f(\tilde{x}_{T_k+1}^{(k)}, \tilde{u}_{T_k+1}^{(k)})$$

$$= f(\tilde{x}_0^{(k)}, \tilde{u}_0^{(k)}) + \sum_{t=0}^{T_k-1} f(x_t^{(k)}, u_t^{(k)}) + f(x_p^*, \tilde{u}_{T_k+1}^{(k)}). \tag{6.20}$$

It follows from (6.2), (6.14), (6.16) and (6.18) that

$$|f(\tilde{x}_0^{(k)}, \tilde{u}_0) - f(x_p^*, u_p^*)| \le 4^{-l-1},\ |f(\tilde{x}_{T_k+1}^{(k)}, \tilde{u}_{T_k+1}) - f(x_p^*, u_p^*)| \le 4^{-k-1}.$$

Together with (6.13) and (6.20) this implies that

$$\sum_{t=0}^{T_k+1} f(\tilde{x}_t^{(k)}, \tilde{u}_t^{(k)}) \le T_k\mu + \delta_k + 2\cdot 4^{-k-1} + 2\mu + 2\cdot 4^{-k-1}$$

$$\le (T_k+2)\mu + 5\cdot 4^{-k-1}. \tag{6.21}$$

Now we construct a concatenation of all the trajectories $(\{\tilde{x}_t^{(k)}\}_{t=0}^{T_k+2}, \{\tilde{u}_t^{(k)}\}_{t=0}^{T_k+1}) \in X(0, T_k+2)$, $k = 1, 2, \ldots$. Set $S_1 = T_1 + 2$ and for each integer $k \ge 2$ set $S_{k+1} = S_k + T_k + 2$. Put

$$x_t = \tilde{x}_t^{(1)},\ t = 0, \ldots, T_1 + 2,$$

$$u_t = \tilde{u}_t^{(1)},\ t = 0, \ldots, T_1 + 1$$

and for each integer $k \ge 2$, define

$$x_{t+S_k} = \tilde{x}_t^{(k)},\ t = 1, \ldots, T_{k+1} + 2,$$

$$u_{t+S_k} = \tilde{u}_t^{(k)},\ t = 0, \ldots, T_{k+1} + 1. \tag{6.22}$$

By the definition above, (6.11) and (6.21), $(\{x_t\}_{t=0}^{\infty}, \{u_t\}_{t=0}^{\infty}) \in X(0, \infty)$ and for each natural number k,

$$\sum_{t=0}^{S_k-1} f(x_t, u_t) = \sum_{p=1}^{k} \sum_{t=0}^{T_p+1} f(\tilde{x}_t^{(p)}, \tilde{u}_t^{(p)}) \le \sum_{p=1}^{k} ((T_p+2)\mu + 5\cdot 4^{-k-1}) \le S_k\mu + 2.$$

Thus $(\{x_t\}_{t=0}^{\infty}, \{u_t\}_{t=0}^{\infty})$ is a good trajectory-control pair satisfying $\liminf_{t\to\infty} \rho_X(x_t, x_p^*) = 0$. Together with Theorem 4.3, this relation implies that $\lim_{t\to\infty}(x_t, u_t) = (x_p^*, u_p^*)$. On the other hand, it follows from our construction, (6.16) and (6.22) that $\limsup_{t\to\infty}(\rho_X(x_t, x_p^*) + \rho_Y(u_t, u_p^*)) \ge \varepsilon$. The contradiction we have reached proves Lemma 6.1. □

7. The Second Basic Lemma

Lemma 7.1. *Let $\varepsilon \in (0,1)$ $M > 0$ and L_0 be a natural number. Then there exists an integer $L_1 > L_0$ such that for each $(\{x_t\}_{t=0}^{L_1}, \{u_t\}_{t=0}^{L_1-1}) \in X(0,L_1)$ which satisfies $\sum_{t=0}^{T-1} f(x_t,u_{t+1}) \leq L_1\mu + M$, there are $\tau \in \{0,\dots,L_1-L_0\}$ and $p \in \{1,\dots,l\}$ such that*

$$\rho_X(x_t,x_p^*) \leq \varepsilon,\ t = \tau,\dots,\tau+L_0,$$

$$\rho_Y(u_t,u_p^*) \leq \varepsilon,\ t = \tau,\dots,\tau+L_0-1.$$

Proof. Assume the contrary. Then for each integer $T > L_0$ there exists $(\{x_t^{(T)}\}_{t=0}^{T}, \{u_t^{(T)}\}_{t=0}^{T-1}) \in X(0,T)$ such that

$$\sum_{t=0}^{T-1} f(x_t^{(T)},u_t^{(T)}) \leq T\mu + M \tag{7.1}$$

and for each $\tau \in \{0,\dots,T-L_0\}$ and each $p \in \{1,\dots,l\}$ we have

$$\max\{\max\{\rho_X(x_t^{(T)},x_p^*) : t \in \{\tau,\dots,\tau+L_0\}\},$$
$$\max\{\rho_Y(u_t^{(T)},u_p^*) : t \in \{\tau,\dots,\tau+L_0-1\}\}\} > \varepsilon. \tag{7.2}$$

Let S be a natural number. By (A1) and (7.1), for each integer $T > L_0$ satisfying $T \geq S$,

$$\sum_{t=0}^{S-1} f(x_t^{(T)},u_t^{(T)}) = \sum_{t=0}^{T-1} f(x_t^{(T)},u_t^{(T)}) - \sum\{f(x_t^{(T)},u_t^{(T)}) : t \in \{S,\dots,T\}\setminus\{T\}\}$$
$$\leq T\mu + M - (\mu(T-S)-\bar{c}) \leq S\mu + M + \bar{c}.$$

Extracting subsequences and using the diagonalization process we obtain that there exists a strictly increasing sequence of natural numbers $\{T_k\}_{k=1}^{\infty}$ such that for each nonnegative integer t there exists a limit

$$(x_t,u_t) = \lim_{k\to\infty}(x_t^{(T_k)},u_t^{(T_k)}). \tag{7.3}$$

Since the graph of the map G is closed we have $(\{x_t\}_{t=0}^{\infty}, \{u_t\}_{t=0}^{\infty}) \in X(0,\infty)$ and by (7.2) for each integer $S \geq 1$,

$$\sum_{t=0}^{S-1} f(x_t,u_t) \leq \liminf_{k\to\infty} \sum_{t=0}^{S-1} f(x_t^{(k)},u_t^{(k)}) \leq S\mu + M + \bar{c}.$$

Thus $(\{x_t\}_{t=0}^{\infty}, \{u_t\}_{t=0}^{\infty})$ is a good trajectory-control pair. Theorem 4.3 implies that there exists $p \in \{1,\dots,l\}$ such that $\lim_{t\to\infty}(x_t,u_t) = (x_p^*,u_p^*)$ in $X \times Y$. Thus there exists a natural number τ such that for each integer $t \geq \tau$,

$$\rho_X(x_t,x_p^*) \leq \varepsilon/4,\ \rho_Y(u_t,u_p^*) \leq \varepsilon/4. \tag{7.4}$$

It follows from (7.3) that there exists a natural number k such that $T_k > \tau + 2L_0 + 2$ and

$$\rho(x_t,x_t^{(k)}) \leq \varepsilon/4,\ t = 0,\dots,\tau+2L_0+2, \tag{7.5}$$

$$\rho(u_t,u_t^{(k)}) \leq \varepsilon/4,\ t = 0,\dots,\tau+2L_0+2. \tag{7.6}$$

By (7.4)-(7.6), for each $t \in \{\tau,\dots,\tau+L_0\}$,

$$\rho_X(x_t^{(T_k)},x_p^*) \leq \rho_X(x_t^{(T_k)},x_t) + \rho(x_t,x_p^*) \leq \varepsilon/2,$$

$$\rho_Y(u_t^{(T_k)},u_p^*) \leq \rho_Y(u_t^{(T_k)},u_t) + \rho(u_t,u_p^*) \leq \varepsilon/2.$$

This contradicts (7.2). The contradiction we have reached proves Lemma 7.1. □

8. The Second Turnpike Result

We continue to study our general optimal control problem.

Now we state and prove our main turnpike result which give the full description of the structure of approximate optimal trajectory-control pairs $(\{x_t\}_{t=0}^{T}, \{u_t\}_{t=0}^{T-1}) \in X(0,T)$ where T is sufficiently large. It is shown that there are mutually disjoint subintervals E_i, $i = 1,\dots,q$ of $[0,T]$ where $q \le l$ and an injective mapping $p : \{1,\dots,q\} \to \{1,\dots,l\}$ such that the cardinality of the complement $\{0,\dots,T\} \setminus \cup_{i=1}^{q} E_i$ does not exceed a constant which does not depend on T and for each $i \in \{1,\dots,q\}$ the set $\{(x_t,u_t) : t \in E_i\}$ is contained in a small neighborhood of $(x^*_{p(i)}, u^*_{p(i)})$.

Theorem 8.1. *Let L_0 be a natural number, $M > 0$ and $\varepsilon \in (0,1)$. Then there exist a natural number $L_1 > L_0$ and $\delta \in (0,\varepsilon)$ such that for each natural number $T > 2L_1 + 8$ and each trajectory-control pair $(\{x_t\}_{t=0}^{T}, \{u_t\}_{t=0}^{T-1}) \in X(0,T)$ which satisfies*

$$\sum_{t=0}^{T-1} f(x_t,u_t) \le \min\{U(0,T,x_0,x_T) + \delta,\ T\mu + M\}$$

there exist a natural number $Q \le l$ and nonnegative integers $S_{i,1} < S_{i,2}$, $i = 1,\dots,Q$ from the interval $[0,T]$ such that

$$S_{i,2} < S_{i+1,1},\ i \in \{1,\dots,Q\} \setminus \{Q\},$$

$$0 \le S_{1,1} \le L_1 - L_0,$$

$$S_{i,2} - S_{i,1} \ge L_0,\ i \in \{1,\dots,Q\},$$

$$S_{i+1,1} - S_{i,2} \le L_1,\ i \in \{1,\dots,Q\} \setminus \{Q\},$$

$$S_{Q,2} \ge T - L_1 - 1$$

and that for each $i \in \{1,\dots,Q\}$ there exists $\tilde{p}_i \in \{1,\dots,l\}$ such that

$$\rho_X(x_{S_{i,1}}, x^*_{\tilde{p}_i}),\ \rho_X(x_{S_{i,2}}, x^*_{\tilde{p}_i}) \le \delta,$$

$$\rho_X(x_t, x^*_{\tilde{p}_i}) \le \varepsilon,\ t = S_{i,1},\dots,S_{i,2}$$

$$\rho_Y(u_t, u^*_{\tilde{p}_i}) \le \varepsilon,\ t = S_{i,1},\dots,S_{i,2} - 1$$

and that $\tilde{p}_{i_1} \ne \tilde{p}_{i_2}$ for each $i_1 . i_2 \in \{1,\dots,Q\}$ satisfying $i_1 \ne i_2$.

Proof. Lemma 6.1 implies that there exists a number $\delta \in (0,\varepsilon)$ such that the following property holds:

(i) for each pair of integers T_2, T_1 satisfying $T_2 \ge T_1 + 2$, each $p \in \{1,\dots,l\}$ and each trajectory-control pair $(\{x_t\}_{t=T_1}^{T_2}, \{u_t\}_{t=T_1}^{T_2-1}) \in X(T_1,T_2)$ which satisfies

$$\rho_X(x_{T_1}, x^*_p),\ \rho_X(x_{T_2}, x^*_p) \le \delta,\ \sum_{t=T_1}^{T_2-1} f(x_t,u_t) \le U(T_1,T_2,x_{T_1},x_{T_2}) + \delta,$$

the inequalities $\rho_X(x_t, x^*_p),\ \rho_Y(u_t, u^*_p) \le \varepsilon$ holds for all integers $t = T_1,\dots,T_2 - 1$. Lemma 7.1 implies that there exists an integer $L_1 > L_0$ such that the following property holds:

(ii) for each trajectory-control pair $(\{x_t\}_{t=0}^{L_1}, \{u_t\}_{t=0}^{L_1-1}) \in X(0,L_1)$ which satisfies

$$\sum_{t=0}^{L_1-1} f(x_t,u_t) \le L_1\mu + M + 2\bar{c}$$

there are $\tau \in \{0,\dots,L_1-L_0\}$ and $p \in \{1,\dots,l\}$ such that

$$\rho_X(x_t,x_p^*) \le \delta,\ t=\tau,\dots,\tau+L_0$$

and

$$\rho_Y(u_t,u_p^*) \le \delta,\ t=\tau,\dots,\tau+L_0-1.$$

Assume that $T > 2L_1+8$ is a natural number and that $(\{x_t\}_{t=0}^{T}, \{u_t\}_{t=0}^{T-1}) \in X(0,T)$ satisfies

$$\sum_{t=0}^{T-1} f(x_t,u_t) \le T\mu + M \tag{8.1}$$

and

$$\sum_{t=0}^{T-1} f(x_t,u_t) \le U(0,T,x_0,x_T)+\delta. \tag{8.2}$$

Assumption (A1) and (8.1) imply that for each pair of integers $S_1,S_2 \in [0,T]$ satisfying $S_1 < S_2$ we have

$$\begin{aligned}\sum_{t=S_1}^{S_2-1} f(x_t,u_t) &= \sum_{t=0}^{T-1} f(x_t,u_t) - \sum\{f(x_t,u_t):\ t\in\{0,\dots,S_1\}\setminus\{S_1\}\} \\ &\quad - \sum\{f(x_t,u_t):\ t\in\{S_2,\dots,T\}\setminus\{T\}\} \\ &\le T\mu + M - (\mu S_1-\bar{c}) - (\mu(T-S_2)-\bar{c}) = \mu(S_2-S_1)+M+2\bar{c}. \end{aligned}\tag{8.3}$$

Property (ii) and (8.34) imply that the following property holds:

(iii) for each integer $S \in \{0,\dots,T-L_1\}$ there are $\tau \in \{S,\dots,S+L_1-L_0\}$ and $p\in\{1,\dots,l\}$ such that

$$\rho_X(x_t,x_p^*) \le \delta,\ \rho_Y(u_t,u_p^*) \le \delta,\ t=\tau,\dots,\tau+L_0.$$

Now we apply property (iii) by induction. Property (iii) implies that there exist

$$\tau_1 \in \{0,\dots,L_1-L_0\},\ p_1 \in \{1,\dots,l\}$$

such that

$$\rho_X(x_t,x_{p_1}^*) \le \delta,\ t=\tau_1,\dots,\tau_1+L_0, \tag{8.4}$$

$$\rho_Y(u_t,u_{p_1}^*) \le \delta,\ t=\tau_1,\dots,\tau_1+L_0, \tag{8.5}$$

If $\tau_1+1+L_0+L_1 > T$, then our construction is completed. Assume that

$$\tau_1+1+L_0+L_1 \le T. \tag{8.6}$$

By property (iii) and (8.6), there exist

$$\tau_2 \in \{\tau_1+1+L_0,\dots,\tau_1+1+L_1\},\ p_2\in\{1,\dots,l\}$$

such that

$$\rho_X(x_t,x_{p_2}^*) \le \delta,\ t=\tau_2,\dots,\tau_2+L_0,$$

$$\rho_Y(u_t,u_{p_2}^*) \le \delta,\ t=\tau_2,\dots,\tau_2+L_0-1.$$

Assume that $k \geq 2$ is an integer and we already defined a finite strictly increasing sequence of nonnegative integers $\tau_1, \dots, \tau_k$ from the interval $[0,T]$ and $p_1, \dots, p_k \in \{1, \dots, l\}$ such that

$$0 \leq \tau_1 \leq L_1 - L_0, \tag{8.7}$$

(8.4) and (8.5) hold and that for each integer $j \in \{2, \dots, k\}$,

$$\tau_j \in \{\tau_{j-1} + 1 + L_0, \dots, \tau_{j-1} + 1 + L_1\}, \tag{8.8}$$

$$\rho_X(x_t, x^*_{p_j}) \leq \delta,\ t = \tau_j, \dots, \tau_j + L_0, \tag{8.9}$$

$$\rho_Y(u_t, u^*_{p_j}) \leq \delta,\ t = \tau_j, \dots, \tau_j + L_0 - 1, \tag{8.10}$$

$$\tau_k + L_0 \leq T. \tag{8.11}$$

(Note that for $k = 2$ our assumption holds.) If $\tau_k + 1 + L_0 + L_1 > T$, then our construction is completed. Assume that

$$\tau_k + 1 + L_0 + L_1 \leq T. \tag{8.12}$$

By (8.12) and property (iii) applied with $S = \tau_k + 1 + L_0$, there exist

$$\tau_{k+1} \in \{\tau_k + 1 + L_0, \dots, \tau_k + 1 + L_1\},\ p_{k+1} \in \{1, \dots, l\}$$

such that

$$\rho_X(x_t, x^*_{p_{k+1}}) \leq \delta,\ t = \tau_{k+1}, \dots, \tau_{k+1} + L_0,$$
$$\rho_Y(u_t, u^*_{p_{k+1}}) \leq \delta,\ t = \tau_{k+1}, \dots, \tau_{k+1} + L_0 - 1$$

and our assumption holds for $k+1$ too. Clearly, our construction is completed in a finite number of steps which is denoted by q. Therefore there exist integers $\tau_1, \dots, \tau_q \in [0,T]$ and

$$p_1, \dots, p_q \in \{1, \dots, l\}$$

such that (8.4), (8.7) hold, for each $j \in \{2, \dots, q\}$ relation (8.8)-(8.10) are true and

$$\tau_q + L_0 \leq T < \tau_q + L_0 + L_1 + 1. \tag{8.13}$$

In view of (8.7) and (8.13), we have $q \geq 2$.

Now we construct another finite sequence which is a subsequence of the sequence $\{\tau_i\}_{i=1}^q$.

Set

$$S_{1,1} = \tau_1,\ \tilde{p}_1 = p_1. \tag{8.14}$$

If, for each $i \in \{1, \dots, q\} \setminus \{1\}$, we have $p_i \neq p_1$, then $S_{1,2} = \tau_1 + L_0$, $S_{2,1} = \tau_2$. If $p_i = p_1$ for some $i \in \{1, \dots, q\} \setminus \{1\}$, then we define

$$t(1) = \max\{i \in \{1, \dots, q\} \setminus \{1\} :\ p_i = p_1\},$$

$$S_{1,2} = \tau_{t(1)} + L_0,$$

and if $t(1) < q$, then $S_{2,1} = \tau_{t(1)+1}$. Then for each integer i satisfying $q \geq i > t(1)$, we have $p_i \neq p_1$,

$$\rho_X(x_{S_{1,1}}, x^*_{p_1}),\ \rho_Y(x_{S_{1,2}}, x^*_{p_1}) \leq \delta.$$

In view (8.2), we have

$$U(S_{1,1}, S_{1,2}, x_{S_{1,1}}, x_{S_{1,2}}) + \delta \geq \sum_{t=S_{1,1}}^{S_{1,2}-1} f(x_t, u_t).$$

Together with property (i), this implies $\rho_X(x_t, x^*_{p_1}),\ \rho_Y(u_t, u^*_{p_1}) \leq \varepsilon,\ t = S_{1,1}, \dots, S_{1,2} - 1$.

Assume that $k < q$ is a natural number. We defined, for each $i \in \{1,\dots,k\}$,

$$S_{i,1} \in \{\tau_1,\dots,\tau_q\},\ S_{i,2} \in \{\tau_1 + L_0,\dots,\tau_q + L_0\} \tag{8.15}$$

and an integer $\tilde{p}_i$ such that

$$S_{i,1} + L_0 \le S_{i,2}, \tag{8.16}$$

$$\text{if } i < k, \text{then } S_{i,2} < S_{i+1,1} < S_{i,2} + L_1, \tag{8.17}$$

$$\tilde{p}_i \in \{p_1,\dots,p_q\},$$

$$\rho_X(x_{S_{i,1}}, x^*_{\tilde{p}_i}),\ \rho_X(x_{S_{i,2}}, x^*_{\tilde{p}_i}) \le \delta, \tag{8.18}$$

$$\rho_X(x_t, x^*_{\tilde{p}_i}),\ \rho_Y(u_t, u^*_{\tilde{p}_i}) \le \varepsilon,\ t = S_{i,1},\dots,S_{i,2} - 1 \tag{8.19}$$

and if $j \in \{1,\dots,q\}$ satisfies $\tau_j > S_{i,2}$ we have

$$\tilde{p}_i \neq p_j. \tag{8.20}$$

(Clearly, our assumption holds for $k = 1$.)

If $S_{k,2} = \tau_q + L_0$, then the construction is completed. Assume that $S_{k,2} < \tau_q + L_0$. Then there exists $j \in \{1,\dots,q\}$ such that $S_{k,2} = \tau_{j-1} + L_0$. Set $S_{k+1,1} = \tau_j$. In view of (8.8), we have $S_{k+1,1} - S_{k,2} \le L_1$. If for each integer $i > j$ satisfying $i \le q$, we have $p_i \neq p_j$, then set $S_{k+1,2} = \tau_j + L_0$, $\tilde{p}_{k+1} = p_j$. If there exists a natural number i satisfying $j < i \le q$ and $p_i = p_j$, then we set $t_k = \max\{i \in \{1,\dots,q\} :\ k < i,\ p_i = p_j\}$ and $S_{k+1,2} = \tau_{t(k)} + L_0$. In view of (8.2), (8.9), (810) and property (i), we have

$$\rho_X(x_t, x^*_{\tilde{p}_{k+1}}),\ \rho_Y(u_t, u^*_{\tilde{p}_{k+1}}) \le \varepsilon,\ t = S_{k+1,1},\dots,S_{k+1,2} - 1.$$

Now it is not difficult to see that our assumption holds for $k+1$ too. Clearly, our construction is completed in a finite number of steps which is denoted by Q. Therefore our assumption above holds for $k = Q$, Since $\tilde{p}_i \in \{1,\dots,l\}$, $i = 1,\dots,l$ satisfy $\tilde{p}_{i_1} \neq \tilde{p}_{i_2}$ for each $i_1.i_2 \in \{1,\dots,Q\}$ satisfying $i_1 \neq i_2$, we have $Q \le l$. Since our assumption above holds for $k = Q$ we have that (8.14) is true, (8.15)-(8.20) hold for each $i \in \{1,\dots,Q\}$, for each $j \in \{1,\dots,Q\}$ satisfying $j > i$, $\tilde{p}_j \neq \tilde{p}_i$, $S_{Q,2} = \tau_q + L_0$, for each $i \in \{1,\dots,Q\}$,

$$\rho_X(x_t, x^*_{\tilde{p}_i}),\ \rho_Y(u_t, u^*_{\tilde{p}_i}) \le \varepsilon,\ t = S_{i,1},\dots,S_{i,2} - 1.$$

Theorem 8.1 is proved. □

9. Extensions of Theorem 8.1

Let M be a natural number and $p \in \{1,\dots\}$. Denote by $\mathscr{A}(p,M)$ the set of all $x \in X$ for which there exist a natural number $T \le M$ and a trajectory-control pair $(\{x_t\}_{t=0}^T, \{u_t\}_{t=0}^{T-1}) \in X(0,T)$ such that $x_0 = x$, $\sum_{t=0}^{T-1} f(x_t,u_t) \le M$, $x_T = x^*_p$ and by $\bar{\mathscr{A}}(p,M)$ the set of all $x \in X$ for which there exist a natural number $T \le M$ and a trajectory-control pair $(\{x_t\}_{t=0}^T, \{u_t\}_{t=0}^{T-1}) \in X(0,T)$ such that

$$\sum_{t=0}^{T-1} f(x_t,u_t) \le M,\ x_0 = x^*_p,\ x_T = x.$$

Proposition 9.1. *1. Let $L \geq 1$ be an integer, $p \in \{1,\dots,l\}$ and $T \geq 2L$ be an integer, $y \in \mathscr{A}(p,L)$ and $z \in \bar{\mathscr{A}}(p,M)$. Then*

$$U(0,T,y,z) \leq \mu(T) + 2L(|\mu|+1).$$

2. Let $L \geq 1$ be an integer, $p \in \{1,\dots,l\}$ and $T \geq L$ be an integer and $y \in \mathscr{A}(p,L)$. Then

$$U(0,T,y) \leq \mu(T) + L(|\mu|+1).$$

Proof. Let us prove assertion 1. By definition, there exists $(\{x_t\}_{t=0}^T, \{u_t\}_{t=0}^{T-1}) \in X(0,T)$ and integers $L_1, L_2 \in [1,L]$ such that $x_0 = y$, $x_{L_1} = x_p^*$, $\sum_{t=0}^{L_1-1} f(x_t,u_t) \leq L$, $x_T = z$, $x_{T-L_2} = x_p^*$, $\sum_{t=T-L_2}^{T-1} f(x_t,u_t) \leq L$ and

$$x_t = x_p^*,\ t = L_1,\dots,T-L_2,$$
$$u_t = u_p^*,\ t = L_1,\dots,T-L_2-1.$$

The relations above imply that

$$\begin{aligned} U(0,T,y,z) &\leq \sum_{t=0}^{T-1} f(x_t,u_t) \leq 2L + \{f(x_t,u_t) :\ t \in \{L_1,\dots,T-L_2-1\}\} \\ &\leq (T-L_1-L_2)\mu + 2L \leq \mu T + 2L(|\mu|+1). \end{aligned}$$

Assertion 1 is proved.

Let us prove assertion 2. By definition, there exists $(\{x_t\}_{t=0}^T, \{u_t\}_{t=0}^{T-1}) \in X(0,T)$ and an integer $L_1 \in [1,L]$ such that $x_0 = y$, $x_{L_1} = x_p^*$, $\sum_{t=0}^{L_1-1} f(x_t,u_t) \leq L$, and

$$x_t = x_p^*,\ t = L_1,\dots,T,$$
$$u_t = u_p^*,\ t = L_1,\dots,T-1.$$

Clearly,

$$U(0,T,y) \leq \sum_{t=0}^{T-1} f(x_t,u_t) \leq L + (T-L_1)\mu \leq \mu T + L(|\mu|+1).$$

Assertion 2 is proved. This completes the proof of Proposition 9.1. □

Theorem 8.1 and Proposition 9.1 imply the following result.

Theorem 9.2. *Let L_0, M be natural numbers and $\varepsilon \in (0,1)$. Then there exist a natural number $L_1 > L_0$ and $\delta \in (0,\varepsilon)$ such that for each natural number $T > 2L_1 + 8$ and each trajectory-control pair $(\{x_t\}_{t=0}^T, \{u_t\}_{t=0}^{T-1}) \in X(0,T)$ which satisfies at least one of the conditions below:*

$$\sum_{t=0}^{T-1} f(x_t,u_t) \leq U(0,T,x_0,x_T) + \delta,\ x_0 \in \mathscr{A}(p,M),\ x_T \in \bar{\mathscr{A}}(p,M);$$

$$\sum_{t=0}^{T-1} f(x_t,u_t) \leq U(0,T,x_0) + \delta,\ x_0 \in \mathscr{A}(p,M)$$

where $p \in \{1,\dots,l\}$ there exist a natural number $Q \leq l$ and nonnegative integers $S_{i,1} < S_{i,2}$, $i = 1,\dots,Q$ from the interval $[0,T]$ such that

$$S_{i,2} < S_{i+1,1},\ i \in \{1,\dots,Q\} \setminus \{Q\},$$
$$0 \leq S_{1,1} \leq L_1 - L_0,$$
$$S_{i,2} - S_{i,1} \geq L_0,\ i \in \{1,\dots,Q\},$$
$$S_{i+1,1} - S_{i,2} \leq L_1,\ i \in \{1,\dots,Q\} \setminus \{Q\},$$

$$S_{Q,2} \geq T - L_1 - 1$$

and that for each $i \in \{1,\dots,Q\}$ *there exists* $\tilde{p}_i \in \{1,\dots,l\}$ *such that*

$$\rho_X(x_{S_{i,1}}, x^*_{\tilde{p}_i}),\ \rho_X(x_{S_{i,2}}, x^*_{\tilde{p}_i}) \leq \delta,$$

$$\rho_X(x_t, x^*_{\tilde{p}_i}),\ \rho_Y(y_t, u^*_{\tilde{p}_i}) \leq \varepsilon,\ t = S_{i,1},\dots,S_{i,2}-1$$

and that $\tilde{p}_{i_1} \neq \tilde{p}_{i_2}$ *for each* $i_1.i_2 \in \{1,\dots,Q\}$ *satisfying* $i_1 \neq i_2$.

10. The Third Turnpike Result

Theorem 8.1 describes the structure of M-approximate solutions $(\{x_t\}_{t=0}^T, \{u_t\}_{t=0}^{T-1})$ of our optimal control problems on an interval $[0,T]$ where T is sufficiently large natural number and $M > 0$ is sufficiently small. Our next result is related to the case when M is fixed but not necessarily small. It is shown that there are mutually disjoint subintervals E_i, $i = 1,\dots,q$ of $\{0,\dots,T\}$ where q does not exceed a constant which does not depend on T and a mapping $p : \{1,\dots,q\} \to \{1,\dots,l\}$ (not necessarily injective) such that the cardinality of the complement $\{0,\dots,T\} \setminus \cup_{i=1}^q E_i$ does not exceed a constant which does not depend on T and for each $i \in \{1,\dots,q\}$ the set $\{(x_t,u_t) :\ t \in E_i\}$ is contained in a small neighborhood of $(x^*_{p(i)}, u^*_{p(i)})$.

Theorem 10.1. *Let* L_0, M_0 *be a natural numbers and* $\varepsilon \in (0,1)$. *Then there exist natural numbers* $L_1 > L_0$, q_1 *and* $\delta \in (0,\varepsilon)$ *such that for each natural number* $T > L_1$ *and each trajectory-control pair* $(\{x_t\}_{t=0}^T, \{u_t\}_{t=0}^{T-1}) \in X(0,T)$, *which satisfies* $\sum_{t=0}^{T-1} f(x_t,u_t) \leq T\mu + M_0$, *there exist a natural number* $q \leq q_1$ *and integers* $a_i, b_i \in [0,T]$, $i = 1,\dots,q$ *such that*

$$a_i < b_i,\ i = 1,\dots,q,$$

$$b_i < a_{i+1},\ i \in \{1,\dots,q\} \setminus \{q\},$$

$$Card(\{0,\dots,T\} \setminus \cup_{j=1}^q \{a_j,\dots,b_j\}\} \leq (2L_1+11)q_1$$

and for each $j \in \{1,\dots,q\}$ *satisfying* $b_j - a_j > 2L_1 + 10$ *there exist a natural number* $Q^{(j)} \leq l$ *and and nonnegative integers* $S^{(j)}_{i,1}, S^{(j)}_{i,2} \in [a_j, b_j]$, $i = 1,\dots,Q^{(j)}$ *such that*

$$S^{(j)}_{i,1} \leq S^{(j)}_{i,2},\ i = 1,\dots,Q^{(j)},$$

$$S^{(j)}_{i,2} < S^{(j)}_{i+1,1} \leq S^{(j)}_{i,2} + L_1,\ i \in \{1,\dots,Q^{(j)}\} \setminus \{Q^{(j)}\},$$

$$a_j \leq S^{(j)}_{1,1} \leq a_j + L_1 - L_0,$$

$$S^{(j)}_{i,2} - S^{(j)}_{i,1} \geq L_0,\ i \in \{1,\dots,Q^{(j)}\},$$

$$S_{Q^{(j)},2} \geq b_j - L_1 - 1$$

and that for each $i \in \{1,\dots,Q^{(j)}\}$ *there exists* $\tilde{p}_i \in \{1,\dots,l\}$ *such that*

$$\rho_X(x_t, x^*_{\tilde{p}_i}) \leq \varepsilon,\ t = S^{(j)}_{i,1},\dots,S^{(j)}_{i,2}$$

$$\rho_Y(ux_t, u^*_{\tilde{p}_i}) \leq \varepsilon,\ t = S^{(j)}_{i,1},\dots,S^{(j)}_{i,2} - 1$$

and that $\tilde{p}_{i_1} \neq \tilde{p}_{i_2}$ *for each* $i_1.i_2 \in \{1,\dots,Q^{(j)}\}$ *satisfying* $i_1 \neq i_2$.

Proof. Let $\delta \in (0,\varepsilon)$ and an integer $L_1 > L_0$ be as guaranteed by Theorem 8.1 with $M = M_0 + 2\bar{c}$ and let a natural number

$$q_1 > 2 + \delta^{-1}(M_0 + \bar{c}). \tag{10.1}$$

Assume that an integer $T > L_1$ and that a trajectory-control pair $(\{x_t\}_{t=0}^{T}, \{u_t\}_{t=0}^{T-1}) \in X(0,T)$ satisfies

$$\sum_{t=0}^{T-1} f(x_t,u_t) \le T\mu + M_0. \tag{10.2}$$

Let integers τ_1, τ_2 satisfy $0 \le \tau_1 < \tau_2 \le T$. Assumption (A1) and (10.2) imply that

$$\begin{aligned}\sum_{t=\tau_1}^{\tau_2} f(x_t,u_t) &= \sum_{t=0}^{T-1} f(x_t,u_t) - \{f(x_t,u_t) :\ t \in \{0,\dots,\tau_1\} \setminus \{\tau_1\}\} \\ &\quad -\{f(x_t,u_t) :\ t \in \{\tau_2,\dots,T\} \setminus \{T\}\} \\ &\le T\mu + M_0 - (\tau_1\mu - \bar{c}) - ((T-\tau_2)\mu - \bar{c}) = (\tau_2 - \tau_1)\mu + M_0 + 2\bar{c}.\end{aligned}$$

Thus we showed that the following property holds:

(i) for each pair of integers τ_1, τ_2 with $0 \le \tau_1 < \tau_2 \le T$, we have

$$\sum_{t=\tau_1}^{\tau_2-1} f(x_t,u_t) \le (\tau_2 - \tau_1)\mu + M_0 + 2\bar{c}.$$

Set $t_0 = 0$. If $f(x_0,u_0) > U(0,1,x_0,x_1) + \delta$, then set $t_1 = 1$. Assume that

$$f(x_0,u_0) \le U(0,1,x_0,x_1) + \delta.$$

If $\sum_{t=0}^{T-1} f(x_t,u_t) \le U(0,T,x_0,x_T) + \delta$, then we set $t_1 = T$ and our construction is completed and if $\sum_{t=0}^{T-1} f(x_t,u_t) > U(0,T,x_0,x_T) + \delta$, then we define

$$t_1 = \min\{p \text{ is a natural number such that } \sum_{t=0}^{p-1} f(x_t,u_t) > U(0,p,x_0,x_p) + \delta\}.$$

Let $k \ge 1$ be an integer and we defined a finite strictly increasing sequence of integers

$$t_1 < \dots, < t_k$$

belonging to the interval $[0,T]$ such that for each integer $j \in \{0,\dots,k-1\}$ the following conditions hold: if

$$f(x_{t_j},u_{t_j}) > U(t_j,t_j+1,x_{t_j},x_{t_j+1}) + \delta, \tag{10.3}$$

then $t_{j+1} = t_j + 1$;

if

$$\sum_{t=t_j}^{T-1} f(x_t,u_t) \le U(t_j,T,x_{t_j},x_T) + \delta, \tag{10.4}$$

then $k = j+1,\ t_{j+1} = T$;

if

$$\sum_{t=t_j}^{T-1} f(x_t,u_t) > U(t_j,T,x_{t_j},x_T) + \delta, \tag{10.5}$$

then

$$t_{j+1} = \min\{p > t_j \text{ is a natural number such that } \sum_{t=t_j}^{p-1} f(x_t,u_t) > U(t_j,p,x_{t_j},x_p) + \delta\}. \tag{10.6}$$

(Clearly, our assumption holds for $k = 1$.) If $t_k = T$, then our construction is completed. Assume that $t_k < T$. If $f(x_{t_k}, u_{t_k}) > U(t_k, t_k+1, x_{t_k}, x_{t_k+1}) + \delta$, then $t_{k+1} = t_k + 1$. Assume that

$$f(x_{t_k}, u_{t_k}) \le U(t_k, t_k+1, x_{t_k}, x_{t_k+1}) + \delta.$$

If $\sum_{t=t_k}^{T-1} f(x_t, u_t) \le U(t_k, T, x_{t_k}, x_T) + \delta$, then $t_{k+1} = T$ and our construction is completed. Assume that $\sum_{t=t_k}^{T-1} f(x_t, x_{t+1}) > U(t_k, T, x_{t_k}, x_T) + \delta$. Then we define

$$t_{k+1} = \min\{p > t_k \text{ is a natural number such that } \sum_{t=t_k}^{p-1} f(x_t, u_t) > U(t_k, p, x_{t_k}, x_p) + \delta\}.$$

Clearly, our assumption made for k holds for $k+1$ too. It is easy to see that our construction is completed after a finite number of steps which is denoted by Q. In view of the construction, $t_Q = T$ and the assumption made above holds for $k = Q$. It is easy to see that there exists a trajectory-control pair $(\{y_t\}_{t=0}^{T}, \{v_t\}_{t=0}^{T-1}) \in X(0,T)$ such that

$$y_{t_i} = x_{t_i},\ i = 0, \dots, Q \tag{10.7}$$

and that for each $i \in \{0, \dots, Q-1\}$,

$$\sum_{t=t_i}^{t_{i+1}-1} f(y_t, v_{t+1}) = U(t_i, t_{i+1}, x_{t_i}, x_{t_{i+1}}). \tag{10.8}$$

Assumption (A1) and equations (10.2), (10.3), (10.7) and (10.8) imply that

$$T\mu + M_0 - (T\mu - \bar{c}) \ge \sum_{t=0}^{T-1} f(x_t, u_t) - \sum_{t=0}^{T-1} f(y_t, v_t) \ge \delta(Q-2)$$

and

$$Q \le 2 + \delta^{-1}(M_0 + \bar{c}). \tag{10.9}$$

Set

$$E = \{t_i :\ i \in \{0, \dots, Q-1\} :\ t_{i+1} - t_i \ge 2L_1 + 10\}. \tag{10.10}$$

Clearly,

$$E = \{t_{i_1}, \dots, t_{i_q}\} \text{ where } i_1, \dots, i_q \in \{0, \dots, Q-1\}, \tag{10.11}$$

$$t_{i_s} < t_{i_{s+1}},\ s \in \{1, \dots, q\} \setminus \{q\}. \tag{10.12}$$

Set

$$a_s = t_{i_s},\ b_s = t_{i_s+1} - 1,\ s = 1, \dots, q. \tag{10.13}$$

It follows from (10.1) and (10.9)-(10.13) that $q \le Q \le 2 + \delta^{-1}(M_0 + \bar{c}) < q_1$. Let $s \in \{1, \dots, q\}$. It follows from (10.6), (10.10) and (10.13) that

$$b_s - a_s = t_{i_s+1} - t_{i_s} - 1 \ge 2L_1 + 8 \tag{10.14}$$

and

$$\sum_{t=a_s}^{b_s-1} f(x_t, u_t) \le U(a_s, b_s, x_{a_s}, x_{b_s}) + \delta. \tag{10.15}$$

Equations (10.1) and (10.1) imply that

$$\mathrm{Card}(\{0, \dots, T\} \setminus \cup_{j=1}^{q} \{a_j, \dots, b_j\}) \le (2L_1 + 11)Q \le q_1(2L_1 + 11).$$

In view of equations (10.14), (10.15), for each $j \in \{1,\dots,q\}$, the assertion of the theorem follows from the choice of L_1, δ and Theorem 8.1 applied with $M = M_0 + 2\bar{c}$. Theorem 10.1 is proved. □

Theorem 10.1 and Proposition 9.1 imply the following result.

Theorem 10.2. *Let L_0, M be a natural numbers and $\varepsilon \in (0,1)$. Then there exist natural numbers $L_1 > L_0$, q_1 and $\delta \in (0,\varepsilon)$ such that for each natural number $T > L_1$ and trajectory-control pair*

$$(\{x_t\}_{t=0}^{T}, \{u_t\}_{t=0}^{T-1}) \in X(0,T)$$

which satisfies at least one of the following conditions:

$$\sum_{t=0}^{T-1} f(x_t, u_{t+1}) \le U(0,T,x_0,x_T) + M,$$

and

$$x_0 \in \mathscr{A}(p,M),\ x_T \in \bar{\mathscr{A}}(p,M),$$

where, $p \in \{1,\dots,l\}$, $\sum_{t=0}^{T-1} f(x_t,u_t) \le U(0,T,x_0) + \delta$, and $x_0 \in \mathscr{A}(p,M)$, where $p \in \{1,\dots,l\}$, there exist a natural number $q \le q_1$ and integers $a_i, b_i \in [0,T]$, $i = 1,\dots,q$ such that

$$a_i \le b_i,\ i = 1,\dots,q,$$

$$b_i < a_{i+1},\ i \in \{1,\dots,q\} \setminus \{q\},$$

$$Card(\{0,\dots,T\} \setminus \cup_{j=1}^{q} \{a_j, b_j\}\} \le (2L_1 + 11)q_1$$

and for each $j \in \{1,\dots,q\}$ satisfying $b_j - a_j > 2L_1 + 10$, there exist a natural number $Q^{(j)} \le l$ and and nonnegative integers $S_{i,1}^{(j)}, S_{i,2}^{(j)} \in [a_j, b_j]$, $i = 1,\dots,Q^{(j)}$ such that

$$S_{i,1}^{(j)} \le S_{i,2}^{(j)},\ i = 1,\dots,Q^{(j)},$$

$$S_{i,2}^{(j)} < S_{i+1,1}^{(j)},\ i \in \{1,\dots,Q^{(j)}\} \setminus \{Q^{(j)}\},$$

$$a_j \le S_{1,1}^{(j)} \le a_j + L_1 - L_0,$$

$$S_{i,2}^{(j)} - S_{i,1}^{(j)} \ge L_0,\ i \in \{1,\dots,Q^{(j)}\},$$

$$S_{i+1,1}^{(j)} - S_{i,2}^{(j)} \le L_1,\ i \in \{1,\dots,Q^{(j)}\} \setminus \{Q^{(j)}\},$$

$$S_{Q^{(j)},2} \ge b_j - L_1 - 1$$

and that for each $i \in \{1,\dots,Q^{(j)}\}$, there exists $\tilde{p}_i \in \{1,\dots,l\}$ such that

$$\rho_X(x_t, x^*_{\tilde{p}_i}) \le \varepsilon,\ t = S_{i,1}^{(j)},\dots,S_{i,2}^{(j)},\ \rho_Y(u_t, u^*_{\tilde{p}_i}) \le \varepsilon,\ t = S_{i,1}^{(j)},\dots,S_{i,2}^{(j)} - 1$$

and that $\tilde{p}_{i_1} \ne \tilde{p}_{i_2}$ for each $i_1.i_2 \in \{1,\dots,Q^{(j)}\}$ satisfying $i_1 \ne i_2$.

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