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COMMONS, CONVOLUTED CRITERIA AND MONOTONE MARGINS

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Abstract. Common property, renewable resources are often inefficiently managed or used. For improvements, this paper argues that auction and trade of short-term individual quotas can comply with long-term, principal planning of aggregate take-outs. Arguments hinge on convoluted criteria and monotone margins.

Keywords. Common property; Competitive equilibria; Convoluted criteria; Monotone margins.

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1. INTRODUCTION

Considered here is governance and use of common-property renewable resources [2, 14].[1] In that optic, main arguments revolve around best values

$$\mathscr{V}_t(x_t) := \sup\left\{ \sum_{\tau=t}^{+\infty} v_\tau(x_\tau, x_{\tau+1}) \;\middle|\; (x_\tau, x_{\tau+1}) \in \mathscr{T},\ x_t \text{ already given} \right\}, \tag{1.1}$$

generated by a discrete-time profile $t \mapsto x_t$, representing resource stocks, recorded as vectors in some real Euclidean space $\mathbb{X}$.[2]

Benveniste and Scheinkman [1] considered classical differentiability of $\mathscr{V}_0(\cdot)$ at x_0. This paper adds to their important study. As minor and technical addition, it allows for generalized derivatives - that is, for less smoothness of underlying data. More substantial enlargements are motivated by the following questions: At whatever time $t \in \{0, 1, ..\}$ and system state $x_t \in \mathbb{X}$, *who, if any, owns stock* x_t*? Who, if any, spells out criteria* $v_\tau(\cdot,\cdot), \tau \geq t$*? And who might - to reasonable approximation - solve problem* $\mathscr{V}_t(x_t)$?

As to commons, the answer to each question is often rather simple yet problematic: There is no single owner, and nobody can fully take up any of the said tasks. To wit: *The stocks remain - or ought remain - common property throughout. Whatever the value* $v_\tau(x_\tau, x_{\tau+1})$*, it's*

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[1] Fish stocks and standing forests are chief instances.
[2] The convention $\sup \varnothing = -\infty$ applies throughout.

hardly identifiable. And, whoever the principal or planner, he must contend with some surrogate version $V_t(\cdot)$ of $\mathscr{V}_t(\cdot)$.

In short, problem (1.1) involves many agents, none well informed, and some troubled by distorted data. To simplify - and to shy away from concerns about time consistency [7, 8, 11] - posit $v_\tau = \delta^\tau v$ with discount factor $\delta \in (0,1)$. Thereby each instance (1.1) takes the unifying, present-value form

$$\mathscr{V}(x_0) := \sup\left\{ \sum_{t=0}^{\infty} \delta^t v(x_t, x_{t+1}) \mid (x_t, x_{t+1}) \in \mathscr{T},\ x_0 \text{ already given} \right\}. \tag{1.2}$$

Technical assumptions: *The* ***technology*** $\mathscr{T} \subseteq X \times X \subset \mathbb{X} \times \mathbb{X}$ *and the* ***hypograph***

$$\{(r,x,x_{+1}) \in \mathbb{R} \times \mathbb{X} \times \mathbb{X} \mid r \le v(x,x_{+1})\}$$

are both closed convex. The single-period function $v : \mathbb{X} \times \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$ *is finite-valued on some open subset of* $\mathscr{T}$.[3] *The* ***viability domain*** $X \subset \mathbb{X}_+$ *is compact convex.*[4]

These assumptions - following [1], except differentiability - reflect concerns with tractable analysis. They are specialized here, *first*, to account for renewable resources; *second*, to accommodate diverse users - each somewhat egocentric, myopic or non-cooperative.

Assumptions as to resource growth. The finite-dimensional space $\mathbb{X}$ has customary order $\le$. State $x \in X \subset \mathbb{X}$ denotes a viable stock (bundle) left as *savings* in situ - just after some total quota $q \in \mathbb{Q} := \mathbb{X}$ has been taken out. *Gross growth* function $x \in X \mapsto G(x) \in X$ is concave (jointly and separately in each component). Let $x_{+1} \in X$ denote the subsequent savings. Clearly, single-period take-out $q \in \mathbb{Q}_+$ cannot exceed the aggregate allowance $G(x) - x_{+1}$. And naturally, the latter is fully taken. Thus, in (1.1),

$$(x, x_{+1}) \in \mathscr{T} \iff x, x_{+1} \in X \text{ and } x_{+1} \le G(x). \tag{1.3}$$

Assumptions as to agents' benefits and resource use. Accommodated throughout, is a fixed finite ensemble $\mathscr{I}$ of autonomous, legitimate resource-users, at least two. At any time $t = 0, 1...$, upon facing stock $x = x_t \in X$, member $i \in \mathscr{I}$ gets short-term, *monetary benefit* $\mathscr{B}_i(x, q_i)$ by taking out his exclusive part $q_i \in \mathbb{Q}_+$ of total quota $q_{\mathscr{I}}$. Only agent i self knows $\mathscr{B}_i(\cdot,\cdot)$ - and even he, maybe just partly. In principle and theory, these criteria - typically held hidden - generate a single-period, aggregate, benefit criterion

$$(x, q_{\mathscr{I}}) \in \mathbb{X} \times \mathbb{Q} \mapsto \mathscr{B}_{\mathscr{I}}(x, q_{\mathscr{I}}) := \sup_{(q_i)} \left\{ \sum_{i \in \mathscr{I}} \mathscr{B}_i(x, q_i) \;\middle|\; \sum_{i \in \mathscr{I}} q_i = q_{\mathscr{I}} \right\}, \tag{1.4}$$

denominated in money. Whether benefit $\mathscr{B}_{\mathscr{I}}(x, q_{\mathscr{I}})$ be realized - and problem (1.4) indeed be solved - or not, together $\mathscr{B}_{\mathscr{I}}$ and G define the value function

$$x, x_+ \in X \mapsto v(x, x_{+1}) := \mathscr{B}_{\mathscr{I}}(x, G(x) - x_{+1}), \tag{1.5}$$

which enters (1.2).

Although theoretical, as concept or construct, (1.4) places *efficient use and sharing of short-term quotas* at center stage. *Section 2* spells out this feature. Thereafter the paper is planned as

[3] For duality theory of such models, see [21].

[4] Naturally, the viability domain is part of a "box", meaning $x \in X \subset \mathbb{X}_+ \implies \underline{x} \le x \le \bar{x}$. Below $\underline{x} \ge 0$ the underlying biological system isn't fully viable; above $\bar{x}$ it generates no growth.

follows:
Section 3 shows that short-term *auction*, just prior to the season, permits agents to share $q_{\mathscr{I}}$, each complying with own incentives.
Section 4 models how subsequent, short-term trade may allow them to solve problem (1.4).[5]
Section 5 mandates a principal to set aggregate quotas across seasons.
Section 6 briefly concludes, pointing to closely related fields.

The paper mainly addresses mathematical optimizers and economists. Novelties are diverse - but most on communication or decomposition. Technical ones relate to how *convexity* in each agent's preferences simplifies his expression of economic interest. That property also facilitates synthesis of "market curves". Other, more institutional novelties, come by letting *three* closely coupled "agencies" play complementary roles in solving problem (1.2). Those three are: auctions, markets and principal planning.

2. Short-term Efficiency

This and the next two sections consider a single period of time $t \in \{0,1,...\}$ and corresponding pairs (x_t, q_t). Thus, until Section 5, the frame is that of *one* typical period. To save space and ink, subscript t is tacitly understood and won't be mentioned, whence $(x,q) = (x_t, q_t)$. For benchmark and reference, consider a short-term, best solution (q_i) in (1.4) with $q = q_{\mathscr{I}}$.

Let $\mathbb{Q}^*$ denote the the *dual space* of linear functions $q^* : \mathbb{Q} \to \mathbb{R}$. Recall that a function $\mathscr{B} : \mathbb{Q} \to \mathbb{R} \cup \{-\infty\}$ has a *supgradient*[6] $q^* \in \mathbb{Q}^*$ at $q \in \mathbb{Q}$, written

$$q^* \in \partial\mathscr{B}(q) \text{ iff } \mathscr{B}^*(q^*) := \sup\{\mathscr{B}(\hat{q}) - q^*\hat{q} \mid \hat{q} \in \mathbb{Q}\} = \mathscr{B}(q) - q^*q \text{ with finite value.} \quad (2.1)$$

Proposition 2.1 (on efficiency and equal margins [5], Prop. 4.1). *Since x is fixed here, writing simply $\mathscr{B}_i(\cdot) := \mathscr{B}_i(x,\cdot)$, for any best allocation (q_i) in* (1.4),

$$\partial\mathscr{B}_{\mathscr{I}}(q_{\mathscr{I}}) \subseteq \cap_{i\in\mathscr{I}}\partial\mathscr{B}_i(q_i).$$

Conversely, provided $q_{\mathscr{I}} = \sum_{i\in\mathscr{I}} q_i$, it also holds the turned-around inclusion:

$$\partial\mathscr{B}_{\mathscr{I}}(q_{\mathscr{I}}) \supseteq \cap_{i\in\mathscr{I}}\partial\mathscr{B}_i(q_i).$$

If moreover, $\cap_{i\in\mathscr{I}}\partial\mathscr{B}_i(q_i)$ is non-empty, then (q_i) solves (1.4). □

Proposition 2.1 is, of course, vacuous unless $\partial\mathscr{B}_{\mathscr{I}}(q_{\mathscr{I}}) \neq \emptyset$. On that note, provided each $\mathscr{B}_i(\cdot)$ be concave, so is $\mathscr{B}_{\mathscr{I}}(\cdot)$ as well - and, if the latter is bounded below near $q_{\mathscr{I}}$, then $\partial\mathscr{B}_{\mathscr{I}}(q_{\mathscr{I}})$ comes compact and non-empty. In short, sup-convolution (1.4) *preserves* desirable properties. No less important, it may make some *emerge* - as illustrated next:

Example (*on regularizing convolution*). Here, let $\mathbb{Q} = \mathbb{R}$ and $\mathscr{I} = \{1,2\}$ with $dom\mathscr{B}_i := \mathscr{B}_i^{-1}(\mathbb{R}) = \mathbb{R}_+$. Posit

$$\begin{cases} \mathscr{B}_1(q_1) = q_1 \text{ for } q_1 < 1, & \text{and 0 elsewhere on } dom\mathscr{B}_1, \\ \mathscr{B}_2(q_2) = q_2 \text{ for } q_2 > 2^{-1}, & \text{and 0 elsewhere on } dom\mathscr{B}_2. \end{cases}$$

$\mathscr{B}_i$ is neither closed nor concave. But $\mathscr{B}_{\mathscr{I}}$ is either: $\mathscr{B}_{\mathscr{I}}(q) = q$ on $dom\mathscr{B}_{\mathscr{I}} = [0,+\infty)$. ◊

[5] That problem is one which nobody states or easily takes as his own or designated task.
[6] The *subdifferential* $\underline{\partial}$ - operating on some cost criterion $\mathscr{C}$ - is given by $\underline{\partial}\mathscr{C} = -\partial(-\mathscr{C})$ [17].

Note that Proposition 2.1 required *no* continuity, convexity, compactness or differentiability of data. It just needed that $\partial\mathscr{B}_{\mathscr{I}}(q_{\mathscr{I}})$ be non-empty.

Concluding this section are some statements on *Pareto optimality*, *core outcome* and *competitive equilibrium*. These solution concepts connect to *welfare theory* [10]. This is summarized next in a manner adapted to single-season quota trade:

Proposition 2.2 (on Pareto optimality, core solution and competitive equilibrium [13]).
• *Any best solution* (q_i) *to* (1.4) *is a* ***Pareto optimal*** *allocation of* $q_{\mathscr{I}}$.
• *If agent* $i \in \mathscr{I}$ *already "owns"* $\underline{q}_i$*, "coalition"* $I \subseteq \mathscr{I}$ *can - in autarky, by going alone - aim at no less joint benefit than*

$$\mathscr{B}_I(\underline{q}_I) := \sup\left\{ \sum_{i\in I} \mathscr{B}_i(q_i) \;\middle|\; \sum_{i\in I} q_i = \sum_{i\in I} \underline{q}_i =: \underline{q}_I \right\}.$$

On this premise, for any overall ***shadow price*** $q^* \in \partial\mathscr{B}_{\mathscr{I}}(\underline{q}_{\mathscr{I}})$, *the pecuniary profile*

$$i \in \mathscr{I} \mapsto \kappa_i(q^*) := \mathscr{B}_i^*(q^*) + q^*\underline{q}_i,$$

constitutes a ***core outcome*** [13] *in that*

$$\sum_{i\in I} \kappa_i(q^*) \geq \mathscr{B}_I(\underline{q}_I) \;\; \textit{for each} \;\; I \subset \mathscr{I} \;\; \textit{with equality for} \;\; I = \mathscr{I}.$$

• *Still suppose agent* $i \in \mathscr{I}$ *"owns"* $\underline{q}_i$ *with* $\sum_{i\in\mathscr{I}} \underline{q}_i = \underline{q}_{\mathscr{I}}$. *Then, any shadow price* $q^* \in \partial\mathscr{B}_{\mathscr{I}}(\underline{q}_{\mathscr{I}})$, *alongside any optimal allocation* (q_i) *of* $\underline{q}_{\mathscr{I}}$, *constitutes a* ***competitive equilibrium.*** *Indeed, markets clear, and each agent i takes home maximal total profit*

$$\mathscr{B}_i(q_i) + q^*(\underline{q}_i - q_i) = \mathscr{B}_i^*(q^*) + q^*\underline{q}_i. \;\; \square$$

Competitive equilibrium is a *steady state* - one which neither drives nor motivates any further trade. As solution concept, it's central in much microeconomic theory [10]. It mentions though, no mechanisms which might lead agents thereto [5].[7] In the sequel, such a "terminus" models the sharing that goes along with any *end-of-season state*. The next section allows agents, by way of auctions, to take first steps towards reaching the allocation in such a state.

3. Auction of Short-term Quotas

Suppose each resource-user - at the very beginning of any season $t \in \{0,1,...\}$ - enters, or faces, a well organized auction of some specified total quota vector $q \in \mathbb{Q}_+$.

For simplified arguments, any of these agents bids - say, anonymously and independently - for exclusive user-right to at most *one* resource at a time [6].[8] So temporarily, member $i \in \mathscr{I}$ behaves as though $\mathbb{Q} = \mathbb{X} = \mathbb{R}$, and $\mathscr{B}_i : \mathbb{R} \to \mathbb{R}\cup\{-\infty\}$.

Further, but just for simpler notation, consider first *one* typical agent - whose identity *i* won't be mentioned. When the auction opens, he holds no right: $q = 0$. Later, during the auction, already endowed with quantity $q \geq 0$ of the single resource, he wonders whether buying more, say up to limit $\Delta q > 0$, be worth his while. To that end, he may submit some simple *bid* - taking the form of an *order*:

[7] Moreover, many central texts make no mention of money.
[8] Alternatively, at any stage of the auction, user-rights to just *one* resource come up for bids.

Definition 3.1 (on bid as order). *If* $[q,q+\Delta q] \subset dom\mathscr{B} := \mathscr{B}^{-1}(\mathbb{R})$*, a price-quantity pair* $(q^*,\Delta q) \in \mathbb{Q}_+^* \times \mathbb{Q}_+$ *is a* ***bid order*** *if its submitter openly commits to pay unit price* q^* *- accepting, of course, whatever better price* $\hat{q}^* \leq q^*$ *- for any non-negative quantity* $\Delta\hat{q} \leq \Delta q$*.*

Proposition 3.2 (on safe and foresighted bid [6]). *Suppose benefit* $\mathscr{B} : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ *be absolutely continuous* [4] *on the interval* $[q,q+\Delta q] \subseteq dom\mathscr{B}$*. Then, any bid order* $(q^*,\Delta q)$*, with price* $q^* \leq \mathscr{B}'$ *almost everywhere on* $[q,q+\Delta q]$ *is* **safe**: *That is, whatever quantity* $\Delta\hat{q} \in [0,\Delta q]$ *gives him some net improvement*

$$\Delta\mathscr{B} - q^*\Delta\hat{q} := \mathscr{B}(q+\Delta\hat{q}) - \mathscr{B}(q) - q^*\Delta\hat{q} \geq 0. \tag{3.1}$$

If moreover, $\mathscr{B}$ *be concave on the interval* $[q,q+\Delta q]$*, then* $q^* = \min \partial\mathscr{B}(q+\Delta q)$ *is* **foresighted** *or prospective in that improvement* (3.1) *is maximal at* $\Delta\hat{q} = \Delta q$*.* □

These simple observations invite *two* questions: *First*, should the optimizer indeed buy more input? *Second*, how might he tune the unit price q^* to additional quantity Δq? Answers follow straight-forwardly, granted monotonicity of $\partial\mathscr{B}(\,\cdot\,)$:

Proposition 3.3 (on incremental demand [6]). *Suppose the agent's benefit* $\mathscr{B}(\cdot)$ *be concave and continuous on the interval* $[q,q+\Delta q]$*. Then, if he already holds quantity* q*, a best bid order* $(q^*,\Delta q)$ *should feature* $\Delta q = 0$ *whenever* $q^* \geq \underline{y}^* := \inf \partial\mathscr{B}(q)$*. Otherwise, he may take*

$$\Delta q = \sup\{\Delta\hat{q} > 0 \mid \inf \partial\mathscr{B}(q+\Delta\hat{q}) \leq q^*\}. \ \square$$

There might be several renewable resources, any single one referred to as "species" $s \in S$. Then $\mathbb{X} = \mathbb{R}^S$, with S finite, fixed and non-empty.

Assumption 3.4 (on closure of the auction). *Given seasonal aggregate quota* $q = (q_s) \in \mathbb{Q}_+ := \mathbb{R}_+^S$*, agents' bidding continues, each considering just one species at a time, until each user* $i \in \mathscr{I}$ *finally has acquired some seasonal right* $q_i = (q_{is}) \in \mathbb{Q}_+$ *as his exclusive part of the aggregate quota* $q = \sum_{i\in\mathscr{I}} q_i$*.*

4. Trade of Short-term Quotas

End of auction, each user exits with some short-term rights, already secured as his property. Afterwards he may enter a market where diverse members trade such rights. Suppose $i \in \mathscr{I}$, at some stage, already holds right $q_i \in \mathbb{Q}_+$. He values rights by some *personal price* $q_i^* \in Q_i^* \subseteq \mathbb{Q}^*$. Further, in the same setting, he contemplates an "adjustment" or *deviation* $\Delta q_i \in \Delta Q_i \subseteq \mathbb{Q}$ away from q_i. The two sets Q_i^*, ΔQ_i are somehow adapted, coupled and related to his actual holding q_i. Naturally, ΔQ_i comprises the *no-trade option* $\Delta q_i = 0$.

Might such non-coordinated trades, undertaken on the spot, be worth the agents' while? A theoretical concept of *bid-ask spread* may clarify this. Equip $\mathbb{Q} = \mathbb{R}^S$ with standard inner product $q^*q := q^* \cdot q := \sum_{s\in S} q_s^* q_s$.

Definition 4.1 (bid-ask spread). *Any specified profile* $i \in \mathscr{I} \rightsquigarrow Q_i^* \times \Delta Q_i \subset \mathbb{Q}^* \times \mathbb{Q}$ *of non-empty*

price-quantity sets generates a ***spread***

$$\mathbf{S}_{\mathscr{J}} := \sup_{(\Delta q_i)} \inf_{(q_i^*)} \left\{ \sum_{i\in\mathscr{J}} q_i^* \cdot \Delta q_i \;\middle|\; q_i^* \in Q_i^*,\ \Delta q_i \in \Delta Q_i \ \&\ \sum_{i\in\mathscr{J}} \Delta q_i = 0 \right\}. \tag{4.1}$$

Theorem 4.2 (on spread). (4.1) *is always non-negative. It's minimal, meaning* $\mathbf{S}_{\mathscr{J}} = 0$*, when* $\cap_{i\in\mathscr{J}} Q_i^* \neq \varnothing$.

Conversely, let each price set Q_i^* *be closed convex, and suppose at least one is bounded. Further, let each quantity set be symmetric, meaning* $\Delta Q_i = -\Delta Q_i$*, and contain* $0 \in int(\Delta Q_i)$*. Then,* $\cap_{i\in\mathscr{J}} Q_i^* = \varnothing \Longrightarrow \mathbf{S}_{\mathscr{J}} > 0$.

Proof. Take each $\Delta q_i = 0$ to see that $\mathbf{S}_{\mathscr{J}} \geq 0$. Clearly, any $q^* \in \cap_{i\in\mathscr{J}} Q_i^*$, alongside $\sum_{i\in\mathscr{J}} \Delta q_i = 0$, gives minimal spread $\mathbf{S}_{\mathscr{J}} = 0$.

Conversely, when $\cap_{i\in\mathscr{J}} Q_i^*$ is empty, intersect each price set with one and the same compact Q_i^*. For that, temporarily suppose

$$i \neq j \Longrightarrow Q_i^* \cap Q_j^* \neq \varnothing. \tag{4.2}$$

Now, presuming (4.2), after intersecting each price set with Q_i^*, the product $C := \Pi_{i\in\mathscr{J}} Q_i^*$ becomes compact convex and non-empty. C doesn't intersect the *diagonal*

$$D := \{(q_i^*)_{i\in\mathscr{J}} \mid \text{all } q_i^* \text{ equal}\}.$$

Consequently, C can be strictly separated from D by some non-zero, linear functional $(\Delta q_i)_{i\in\mathscr{J}} \in \mathbb{Q}^{\mathscr{J}}$ in that

$$\inf_{(q_i^*)} \left\{ \sum_{i\in\mathscr{J}} q_i^* \cdot \Delta q_i \mid q_i^* \in Q_i^* \right\} > \sup \left\{ \sum_{i\in\mathscr{J}} q_i^* \cdot \Delta q_i \mid \text{all } q_i^* \text{ equal} \right\}. \tag{4.3}$$

Strict inequality cannot hold here unless $\sum_{i\in\mathscr{J}} \Delta q_i = 0$, hence the right hand side must be nil. If necessary, by positive scaling of $(\Delta q_i)_{i\in\mathscr{J}}$, let each $\Delta q_i \in \Delta Q_i$. Thus, $\mathbf{S}_{\mathscr{J}} > 0$.

Returning to the auxiliary hypothesis (4.2), suppose it be violated when $i \neq j$. Then, in the price space $\mathbb{Q}^*$, separate the compact subset Q_i^* strictly from Q_j^* by some non-zero $\Delta q_j \in \mathbb{Q}$ and posit $\Delta q_i := -\Delta q_j$. Upon annulling all other deviations, except Δq_i, Δq_j, (4.3) still holds. From here on, repeat above arguments to complete the proof. □

Some remarks on spread follow next; they can be perused or skipped.

• (*Saddle-value*). When all sets $Q_i^*, \Delta Q_i$ are convex, the bilinear criterion inside (4.1) fits saddle-point theory. If each ΔQ_i also be compact, spread obtains as lop-sided, max-inf saddle value; see Theorem 1.86 in [15].

• (*Steepest ascent*). The special instance $Q_i^* = \partial \mathscr{B}_i(q_i)$, $\Delta Q_i = \{\Delta q_i \mid \|\Delta q_i\| \leq 1\}$ reduces spread to *steepest directional ascent*:

$$\mathbf{S}_{\mathscr{J}} = \max \left\{ \sum_{i\in\mathscr{J}} \mathscr{B}_i'(q_i; \Delta q_i) \;\middle|\; \sum_{i\in\mathscr{J}} \Delta q_i = 0 \ \text{ and each } \|\Delta q_i\| \leq 1 \right\}.$$

• (*Competitive equilibrium*). Viewed via Prop. 2.2 with $Q_i^* = \partial \mathscr{B}_i(q_i)$, *competitive equilibrium* amounts to $\mathbf{S}_{\mathscr{J}} = 0$, realized with each $\Delta q_i = 0$.

• (*Improving trade*). If $\mathbf{S}_{\mathscr{J}} > 0$, the price set $Q_i^* = \partial \mathscr{B}_i(q_i)$ may not fit. Indeed, if $\mathscr{B}_i(\cdot)$

comes strictly concave near q_i, any order price $q_i^* \in \partial\mathscr{B}_i(q_i)$, applied to any sufficiently small "quantity" $\Delta q_i \neq 0$, sets agent i strictly back:

$$\mathscr{B}_i(q_i+\Delta q_i) - q_i^*(\Delta q_i) < \mathscr{B}_i(q_i).$$

Such set-back stems from the myopic nature of $\partial\mathscr{B}_i(q_i)$. Instead of valuing Δq_i by any status quo price, selected from $\partial\mathscr{B}_i(q_i)$, it appears wiser to use some $q_i^* \in \partial\mathscr{B}_i(q_i+\Delta q_i)$. That is, the agent had better replace whatever *ex ante price*, adapted to the pre-trade *point of departure* q_i, with another *ex post price,* fitting the after-trade *point of arrival* $q_i+\Delta q_i$. Put differently: prudent pricing should be foresighted or *prospective*. This feature was already brought out in Proposition 3.2 for the instance $\mathbb{Q}=\mathbb{R}$.

• (*On persistence of spread*). Let $\mathbf{S}_{\mathscr{I}}(\mathbf{q})$ denote the spread at $\mathbf{q}=(q_i)\in\mathbf{Q}:=\mathbb{Q}^{\mathscr{I}}$ for $Q_i^*=\partial\mathscr{B}_i(q_i)$, presumed non-empty for each i. Now, replace $\mathbf{S}_{\mathscr{I}}(\mathbf{q})$ with a corresponding near-enough-instance $\mathbf{S}_{\mathscr{I}}(\mathbf{q}+\Delta\mathbf{q})$ to obtain:

Proposition 4.3 (on persistence of strictly positive spread). *Suppose each correspondence $\partial\mathscr{B}_i(\cdot)$, $i\in\mathscr{I}$, has compact, non-empty values, and it's outer semicontinuous near q_i* [17], *with $\mathbf{S}_{\mathscr{I}}(\mathbf{q})>0$. Then, after sufficiently small displacement $\Delta\mathbf{q}=(\Delta q_i)\in\mathbf{Q}$, still $\mathbf{S}_{\mathscr{I}}(\mathbf{q}+\Delta\mathbf{q})>0$.* ♢

Reasonably presume that *trade always complies with incentives.* That is: *nobody accepts a deal which sets him back*. On this premise, how could improvements obtain? For that - and for simple arguments - consider deals which are elementary in *two* ways. *First*, whenever $\mathbf{S}_{\mathscr{I}}>0$, there exists a minimal subset $I\subseteq\mathscr{I}$, not a singleton, for which still $\mathbf{S}_I>0$. *Second*, along some smallest subspace of $\mathbb{Q}$, the corresponding *partial price sets* still do not intersect. In that space, choose $\Delta Q_i=-\Delta Q_i$, with $0\in relint(\Delta Q_i)$, so as to maintain $\mathbf{S}_I>0$.

As upshot, spread applies to few parties - say, just *two* - who trade few resources - say, just *one*. An instance of this utterly reduced complexity is singled out next:

On trade of one resource, driven by bilateral spreads. Let two resource-users i, j have concave criteria $\mathscr{B}_i(\cdot)$, $\mathscr{B}_j(\cdot)$, either finite-valued near their respective holdings q_i, q_j. Suppose their actual spread be strictly positive, meaning $\partial\mathscr{B}_i(q_i)\cap\partial\mathscr{B}_j(q_j)=\emptyset$. Suppose moreover, that temporarily, they just consider some *single-good trade*. So, $\mathbb{Q}$ is one-dimensional here. Further, by outer semicontinuity and Prop. 4.3, with no loss of generality, take the interval $\partial\mathscr{B}_i(q_i+\Delta q_i)\subset\mathbb{R}$ to be strictly above $\partial\mathscr{B}_j(q_j-\Delta q_j)$ for sufficiently small $\Delta q_i,\Delta q_j>0$. Consequently, both intervals being closed, it holds:

$$q_i^* := \min\partial\mathscr{B}_i(q_i+\Delta q_i) > \max\partial\mathscr{B}_j(q_j-\Delta q_j) =: q_j^*.$$

Agent i, upon considering $q_i+\Delta q_i$ as post-trade *prospective point*, *bids* $(q_i^*,\Delta q_i)$. Thereby he *commits* to pay any unit price $\leq q_i^*\in\partial\mathscr{B}_i(q_i+\Delta q_i)$ for whatever non-negative quantity $\leq\Delta q_i$. His counterpart j *asks* $(q_j^*,\Delta q_j)$, accepting thereby to be paid unit price $\geq q_j^*\in\partial\mathscr{B}_j(q_j-\Delta q_j)$ for handing over whatever non-negative quantity $\leq\Delta q_j$. So, a bilateral trade, executed at any unit price $q^*\in[q_j^*,q_i^*]$ for any quantity $\Delta q\in(0,\min\{\Delta q_i,\Delta q_j\}]$, gives no party reason to complain. Provided each $\mathscr{B}_i$, $\mathscr{B}_j$ be strictly concave, either party sees some strict improvement - as brought out by Prop. 3.2.

To elaborate on this argument, let ∂_s denote the *partial supdifferential*, taken with respect to q_s.

Assumption 4.4 (on bilateral, repeated trade of *one* resource at a time). *Any user trades, at any time, at most one resource $s \in S$, not necessarily the same. Consequently, all transactions are bilateral, each realized by acceptable payment for suitable quantity of one species. Trade unfolds, in parallel and quickly, until*

$$\partial_s \mathscr{B}_i(q_i) \cap \partial_s \mathscr{B}_j(q_j) \neq \varnothing \text{ for each species } s \in S \text{ and every agent pair } i, j \in \mathscr{I}.$$

Proposition 4.5 (on *one* price for each resource). *Suppose benefit $\mathscr{B}_i : \mathbb{Q} \to \mathbb{R} \cup \{-\infty\}$, $i \in \mathscr{I}$, be concave and supdifferentiable on its domain $\mathscr{B}_i^{-1}(\mathbb{R})$. Then, under* Assumption 4.4*, trade stops at some feasible allocation (q_i) of the short-term total quota iff there is widespread agreement on every single price, meaning*

$$\cap_{i \in \mathscr{I}} \partial_s \mathscr{B}_i(q_i) \neq \varnothing \text{ for each species } s \in S. \tag{4.4}$$

If moreover, (4.4) *implies that $\cap_{i \in \mathscr{I}} \partial \mathscr{B}_i(q_i) \neq \varnothing$, trade stops only when spread $\mathbf{S}_{\mathscr{I}} = 0$.*

Proof. Select and fix any $s \in S$. When user $i \in \mathscr{I}$ holds quota $q_i = (q_{is}) \in \mathbb{Q}$, his partial supdifferential $\partial_s \mathscr{B}_i(q_i)$ is a closed *interval* on the real line - possibly unbounded. Let $\underline{q}^*_{is}$, $\bar{q}^*_{is} \in \mathbb{R} \cup \{\pm\infty\}$ be the lower and upper limits of that interval. When voluntary trade stops, by Assumption 4.4, every two among these intervals intersect. Hence

$$\underline{q}^*_s := \max_i \underline{q}^*_{is} \leq \min_j \bar{q}^*_{js} =: \bar{q}^*_s.$$

Consequently, any price $q^*_s \in [\underline{q}^*_s, \bar{q}^*_s]$ clears the quota market for species s.[9] □

It's worthwhile to note the flexibility and generality afforded by the supdifferential notation. To illustrate, suppose some member i cannot possibly use species s. So, if forced to, $\mathscr{B}_i(q_i) = -\infty$ whenever $q_{is} \neq 0$. Then $\partial_s \mathscr{B}_i(q_i)\big|_{q_{is}=0} = \mathbb{R}$, whence agent i doesn't bite in (4.4).

Also worth notice is that smoothness - maybe merely partial - of some $\mathscr{B}_i$, makes for more precise selection of clearing prices. In particular, the last claim in Prop. 4.5 merits some addition:

Proposition 4.6 (on *one* price for quota). *Let $q_{\mathscr{I}} = \sum_{i \in \mathscr{I}} q_i$ and suppose intersection* (4.4) *reduces to a singleton. Then $\mathscr{B}_{\mathscr{I}}$, presumed concave, is differentiable at $q_{\mathscr{I}}$.*

Proof. By Proposition 2.1, each partial supdifferential $\partial_s \mathscr{B}_{\mathscr{I}}(q_{\mathscr{I}})$ is a singleton, hence $\mathscr{B}_{\mathscr{I}}(\cdot)$ has a partial derivative at $q_{\mathscr{I}}$, separately in each component $s \in S$. Now, the assertion follows from Theorem 3.8.2 in [12]. □

5. Long Term Efficiency

Henceforth suppose a benevolent, competent *principal* manages the resources over discrete time - with infinite horizon. That task of his entails shifts in this paper's orientation, bordering

[9] This constructive argument repeats insight from *Helly's teorem* on the intersection of closed convex subsets in a Euclidean space.

on many fields:
Economics, considered so far, revolved around seasonal allocation - in the optic of *short-term micro*. Enter here similar but complementary concerns, related to *long-term macro*.
Mathematics, invoked above, centered on *calculus in finite dimensions*. Enter here *functional analysis and infinite dimensions*.
Dynamics and intertemporal allocation, overlooked hitherto, scarcely trouble many and myopic resource-users. By contrast, resource stability and yields are the principal's chief concerns.
Capitalism - much contended and disputed - enters somehow back door. The reason is simple: Natural resources constitute *endowments*. Irrespective of their ownership or use, some party better care for *resource rent*. So, whether by intention or not, the principal acts as though *capitalist* - somehow as ideal, sole and perpetual owner. He has, however, no role or say in consuming, saving or distributing dividends.

On these issues, for simplicity and transparency, henceforth consider just *one* resource. So, from here on, $\mathbb{Q} = \mathbb{X} = \mathbb{R}$. Further, for the same reasons, suppose no agent's benefit depends on the actual state of the stock: $\mathscr{B}_i(x, q_i) = \mathscr{B}_i(q_i)$.[10]

At best, the principal ought maximize present-value (1.2), using (1.5) as single-period criterion. To acceptable approximation, he "knows" the growth function $G(\cdot)$ (1.3) - and observes, all along, the evolving resource state. However, because $\mathscr{B} := \mathscr{B}_{\mathscr{J}}$ (1.4) is hard to identify, he must contend with some surrogate version B.

To that end, suppose he records, during the auction, each *accepted bid*. Every such record has the simple price-quantity form $(q^*, \Delta\hat{q}) \in \mathbb{R}^2_+$, spelled out in Section 3.[11] Together these constitute a finite set $\mathscr{D}$ of "dividends". The latter defines the *practical* benefit - that is, an imperfect surrogate

$$q \in \mathbb{Q} \mapsto B(q) := \sup\left\{ \sum q^* \cdot \Delta\hat{q} \;\middle|\; \sum \Delta\hat{q} \leq q,\ (q^*, \Delta\hat{q}) \in \mathscr{D} \right\} \tag{5.1}$$

instead of the *theoretical* counterpart $\mathscr{B}$ (1.4). Note that B comes closed concave. It's increasing and piecewise affine - irrespective of underlying data. Its supdifferential $\partial B(\cdot)$ becomes a decreasing staircase.

Assumption 5.1 (on the overall planner's program). *Given $x_0 \in X$ and $B(\cdot)$ (5.1), the principal aims at present value*

$$V(0) := \sup\left\{ \sum_{t=0}^{\infty} \delta^t B(q_t) \;\middle|\; x_{t+1} + q_t \leq G(x_t) \right\}. \tag{5.2}$$

Since $x_t, q_t \in X$, provided problem (5.2) be feasible, it has a best solution.

Its value should display some stability, preferably of differential sort. To that end, perturb the constraint in (5.2) by some deviation $\Delta q_t \in \mathbb{Q}$ to get $x_{t+1} + q_t \leq G(x_t) + \Delta q_t$. Naturally, the profile $\Delta\mathbf{q} := (\Delta q_t)$ belongs to the linear space $\mathbf{Q} := l^\infty(\mathbb{Q})$ of bounded sequences in $\mathbb{Q}$. Thus, given x_0, there emerges a family of present-value problems

$$(\Delta q_t) = \Delta\mathbf{q} \mapsto V(\Delta\mathbf{q}) := \sup\left\{ \sum_{t=0}^{\infty} \delta^t B(q_t) \;\middle|\; x_{t+1} + q_t \leq G(x_t) + \Delta q_t \right\},$$

[10] Extensions are easy, but create complexity.
[11] And each is voluntary, whence it complies with incentives.

comprising (5.2) as special instance. Because $B(\cdot)$ (5.1) was denominated in money, perturbation profile $\Delta\mathbf{q}$ ought be valued in corresponding terms. Henceforth such valuation takes the form

$$(\Delta q_t) = \Delta\mathbf{q} \mapsto \mathbf{q}^*(\Delta\mathbf{q}) := \sum_{t=0}^{\infty} \delta^t q_t^*(\Delta q_t) \;\text{ with }\; \mathbf{q}^* = (q_t^*). \tag{5.3}$$

Some remarks on (5.3), following next, can be skipped or perused.

• (*On rents collected and shared*). An optimal profile $\mathbf{q} = (q_t)$, solving (5.2), generates a flow $t \to B(q_t)$ of payments for quantities taken out. All along, the principal - qua auctioneer or capitalist - collects these payments, called *resource rents*, from users for their short-term quotas. Thus, throughout, such *rent* splits into $B(q_t)$ for the "owner" and $\mathscr{B}(q_t) - B(q_t)$ shared among the users. The specific sharing, not addressed, depends on underlying data and auction details.

• (*On estimation of resource rent*). Besides setting seasonal quotas, the principal has *two* more tasks: *One,* to collect payments B (5.1) for user-rights; *the second*, to estimate the larger resource rent (1.4) $\mathscr{B}_{\mathscr{J}} := \mathscr{B} \geq B$. He must, however, contend with some intermediate version $\bar{B} \in [B, \mathscr{B}]$. For a reduced gap $\mathscr{B} - \bar{B}$, the set $\mathscr{D}$ in (5.1) could also compromise all *realized deals* during subsequent quota trade.

• (*On pricing of take-out profiles*). For argument, suppose trajectories $\Delta\mathbf{q} = (\Delta q_t)$ be traded on some fictitious market. Naturally, since each non-zero instance $t \to \Delta q_t \geq 0$ is desirable, its price better be non-negative, meaning all $q_t^* \geq 0$. Most naturally, $\mathbf{q}^*(0) = 0$. And $\Delta\mathbf{q} \to 0$ should imply $\mathbf{q}^*(\Delta\mathbf{q}) \to 0$. Finally, for consistency, pricing ought preclude *gaining something for nothing*. That is: no bundling (or unbundling) of any $\Delta\hat{\mathbf{q}}$, $\Delta\mathbf{q}$ into (resp. from) $\Delta\hat{\mathbf{q}} + \Delta\mathbf{q}$ should generate economic gain. In financial jargon: no trade should permit *arbitrage*. Therefore, as operator, $\mathbf{q}^*$ *must be continuous, linear and non-negative*.

• (*On reflexive pricing*). Every perturbation profile $\Delta\mathbf{q} = (\Delta q_t)$ belongs to the space l^∞ of bounded real sequences in $\mathbb{Q}$. So, the norm $\|\Delta\mathbf{q}\| := \sup_t \|\Delta q_t\|$ is finite. As defined by (5.3), operator $\mathbf{q}^*$ belongs to the *dual space* l^1, composed of all sequences (q_t^*) for which $\sum_{t=0}^{\infty} \delta^t \|q_t^*\| < +\infty$. The original space $\mathbf{Q} := l^\infty$ is, however, *not* reflexive: Its continuous dual $\mathbf{Q}^* = l^{\infty *}$ contains l^1 as strict subspace [4].

Any member $\sigma \in l^{\infty *} \diagdown l^1$, called a *Banach limit* [18, 20], satisfies the following property: For any time t and decomposition $\Delta q = (\Delta q_0, .., \Delta q_t, 0, 0, ..) + (0, .., 0, \Delta q_{t+1}, \Delta q_{t+2}, ..)$, the price $\sigma(\Delta\mathbf{q})$ is unaffected by the initial part $(\Delta_0, .., \Delta_t, 0, 0, ..)$. Such singular valuation is *asymptotic*, settled beyond any finite horizon. Plainly, this makes little economic sense.

• (*On shadow price profiles*). View $\mathbf{q}^* \in l^1_+$ as *shadow price profile* iff

$$\sup\{V(\Delta\mathbf{q}) - \mathbf{q}^*(\Delta\mathbf{q}) \mid \Delta\mathbf{q} \in l^\infty\} = V(\mathbf{0}). \tag{5.4}$$

This happens iff $\mathbf{q}^*$ is a supgradient of $V(\cdot)$ at 0; that is, iff $\mathbf{q}^* \in \partial V(\mathbf{0})$.

For coherence and consistency, despite having to work with V (5.2) instead of $\mathscr{V}$ (1.2), it's desirable that the principal's *implicit price* $\mathbf{q}^* = (q_t^*)$ of resource profiles accords the *explicit price* q_t^* which clears the quota market at time t; see Prop. 4.5&6. As argued below, chances appear good for such coincidence. $\Diamond$

Perturbation $\Delta\mathbf{q} = (\Delta q_t)$ brings up the standard *Lagrangian*

$$L(\mathbf{x}, \mathbf{q}, \mathbf{q}^*) := \sup_{(\Delta q_t)} \left\{ \sum_{t=0}^{\infty} \delta^t [B(q_t) - q_t^* \cdot \Delta q_t] \mid x_{t+1} + q_t \leq G(x_t) + \Delta q_t \right\} \tag{5.5}$$

by choosing the cheapest $\Delta q_t \geq x_{t+1} + q_t - G(x_t)$ at price $q_t^* \geq 0$. Hence, with *net growth* $g(x) := G(x) - x$,

$$L(\mathbf{x}, \mathbf{q}, \mathbf{q}^*) = \sum_{t=0}^{\infty} \delta^t [B(q_t) + q_t^* \{g(x_t) - q_t - (x_{t+1} - x_t)\}] \text{ if } q^* \in l_+^1, \ +\infty \text{ otherwise.}$$

Proper use of the Lagrangian requires some *constraint qualification*. Among these, granted concave functions, as here, the *Slater condition* amounts to strict feasibility. It would suffice that the resources be *productive* in that, from the initial x_0, some state $x \in X$ can be reached, after finite time, at which $G(x) > x$.

As usual, $V(0) = \sup_{\mathbf{x},\mathbf{q}} \inf_{\mathbf{q}^*} L(\mathbf{x}, \mathbf{q}, \mathbf{q}^*)$ and $\inf_{\mathbf{q}^*} \sup_{\mathbf{x},\mathbf{q}} L(\mathbf{x}, \mathbf{q}, \mathbf{q}^*) \geq V(0)$. Chief interest revolves around saddle points of L. It obtains forthwith:

Proposition 5.2 (on max-min solutions). *Provided single-period payoff B and net growth g both be closed concave functions, Lagrangian $L(\mathbf{x}, \mathbf{q}, \mathbf{q}^*)$* (5.5) *becomes concave in $(\mathbf{x}, \mathbf{q})$ and linear in $\mathbf{q}^*$. Further, if there is a shadow price $\mathbf{q}^* \in \partial V(0)$, it holds*

$$V(0) = \max_{\mathbf{x},\mathbf{q}} \inf_{\mathbf{q}^*} L(\mathbf{x}, \mathbf{q}, \mathbf{q}^*) = \inf_{\mathbf{q}^*} \max_{\mathbf{x},\mathbf{q}} L(\mathbf{x}, \mathbf{q}, \mathbf{q}^*).$$

If moreover, the set $\mathbf{Q}^*$ of suitable prices $\mathbf{q}^*$ comes compact - say, in the weak topology generated by l^∞ - then L has a max-min saddle point $(\mathbf{x}, \mathbf{q}, \mathbf{q}^*)$.[12] *Any such $\mathbf{q}^*$ is a Lagrange multiplier, meaning it gives no duality gap because $V(0) = \sup_{(\mathbf{x},\mathbf{q})} L(\mathbf{x}, \mathbf{q}, \mathbf{q}^*)$.* □

To bridge with optimal control, consider the single-period *Hamiltonian*

$$H(x_t, q_t, q_t^*) := B(x_t, q_t) + q_t^*[g(x_t) - q_t], \ -\infty \text{ if } (x_t, q_t) \notin X \times Q, \text{ and } +\infty \text{ if } q_t^* \not\geq 0.$$

H records aggregate payment for resource use plus shadow value of savings. Use H to rewrite Lagrangian (5.5) as

$$L(\mathbf{x}, \mathbf{q}, \mathbf{q}^*) = \sum_{t \in T} \delta^t \{H(x_t, q_t, q_t^*) - q_t^*(x_{t+1} - x_t)\}. \tag{5.6}$$

Now, optimality conditions come straightforwardly in the form of a discrete-time, Prontryagin maximum principle [9]:

Theorem 5.3 (on optimal system control). *Given $x_0 \in X$, suppose $0 \leq t \mapsto (x_t, q_t) \in X \times Q$ solves* (5.2)*, and that there exists some shadow price $\mathbf{q}^* = (q_t^*) \in \partial V(0)$. Then, it holds for each t,*

$$\begin{cases} \textit{primal dynamic:} & x_{t+1} - x_t & = & g(x_t) - q_t, \\ \textit{dual dynamic:} & q_t^* - \delta^{-1} q_{t-1}^* & \in & -\frac{\partial}{\partial x} H(x_t, q_t, q_t^*), \ t > 0, \\ \textit{maximum condition:} & q_t & \in & \arg\max H(x_t, \cdot, q_t^*), \textit{ and} \\ \textit{price equal margins:} & q_t^* & \in & \cap_{i \in \mathscr{I}} \frac{\partial}{\partial q_{it}} B_i(x_t, q_{it}). \end{cases} \tag{5.7}$$

Proof. By assumption, the series in (5.6) is absolutely summable. Hence differentiation can be done term by term. On that basis, by $\frac{\partial}{\partial q_t^*} L(\mathbf{x}, \mathbf{q}, \mathbf{q}^*) = 0$, the primal system equation holds for all t. Further, note that L features x_t in *two* terms only, singled out next: $L(\mathbf{x}, \mathbf{q}, \mathbf{q}^*) = \cdots +$

$$\delta^t \{H(x_t, q_t, q_t^*) - q_t^*(x_{t+1} - x_t)\} + \delta^{t-1} \left\{H(x_{t-1}, q_{t-1}, q_{t-1}^*) - q_{t-1}^*(x_t - x_{t-1})\right\} + \cdots.$$

[12] Invoke Theorem 1.86 in [15].

From this and $\frac{\partial}{\partial x_t} L(\mathbf{x},\mathbf{q},\mathbf{q}^*) = 0$, the adjoint equation on dual dynamics follows. Finally, because $L(\mathbf{x},\mathbf{q},\mathbf{q}^*)$ is additively separable with respect to $\mathbf{q}$, and should be maximized in each q_t, the maximum condition is immediate. Finally, coincidence, at each time, of user cost and shadow price derives from Proposition 3.1. □

Chief concerns revolve around steady states and stability [16] of system (5.7). Arguments invoke the *reduced Hamiltonian* $\mathscr{H}(x,q^*) := \max_q H(x,q,q^*)$ and its generalized differential, defined by $(x^*,q^*) \in \partial\mathscr{H}(x,q^*) := \partial_x\mathscr{H}(x,q^*) \times \partial_{q^*}\mathscr{H}(x,q) \Longleftrightarrow$

$$\begin{cases} \mathscr{H}(\hat{x},q^*) \leq \mathscr{H}(x,q^*) + x^*(\hat{x}-x) & \text{for all } \hat{x} \in \mathbb{X}, \text{ by concavity in } x, \\ \mathscr{H}(x,\hat{q}^*) \geq \mathscr{H}(x,q^*) + q^*(\hat{q}^*-q^*) & \text{for all } \hat{q}^* \in \mathbb{Q}^*, \text{ by convexity in } q^*. \end{cases}$$

In these terms, system (5.7) moves by the dynamics

$$x_{t+1} - x_t \in \partial_{q^*}\mathscr{H}(x_t,q_t^*) \quad \& \quad q_t^* - \delta^{-1}q_{t-1}^* \in -\partial_x\mathscr{H}(x_t,q_t^*). \tag{5.8}$$

Hamiltonian dynamics (5.8) resemble classical instances of primal-dual programming. Broadly, convergence to a *golden-rule, long-run equilibrium* requires strong concavity of payoff $\mathscr{B}(x,q)$ or growth $g(x)$ in x [16].[13] Further, δ should be large enough - as stressed by [22]. With $\delta = 1/(1+\rho)$, $\rho > 0$, a steady state (x,q,q^*) solves the "equation" system:

$$\begin{cases} g(x) & = & q & \text{(stationary stock levels),} \\ \rho q^* & \in & \partial_x H(x,q,q^*) & \text{(Hotelling's rule), and} \\ q^* & \in & \partial_q B(x,q) & \text{(optimal take-out).} \end{cases}$$

6. Concluding remarks

Several sciences, besides *biology* and *ecology*, consider management of renewable commons. Among those, *economic theory* advocates monetized markets to facilitate individual choice and decentralized decisions. As complement, *political science* brings out that suitable institutions are needed - in fact, indispensable [14]. Less evident, but important, is the role of *optimization theory*. To wit, underlying *convexity* - alias *concave criteria and dynamics* - can make for *decreasing margins,* at *two* levels:
First - in the small, at the lower level - via individual preferences and technologies so that users could share short-term quotas as a sequence of competitive equilibria;
second - in the large, at the upper level - via stock growth so that the principal could face an aggregate, long-term saddle problem of concave-convex sort.

Attentions and perspectives split accordingly. One one side, individual users may well be myopic, contending with repeated, money-based trade of seasonal quotas. On the other side, the principal can concentrate on aggregate take-outs, savings and system dynamics. Thus, decentralized trade may unfold *in season* subject to centralized choice of aggregates *across seasons*. So, a sequence of perfectly competitive, seasonal quota markets can comply with - and be subordinated to - principal planning of time profiles.

Arguments for this broad outline of industrial organization presumed that:
∘ User rights be clearly defined, perfectly divisible, marketable and transferable.
∘ Qualified users operate, directly or indirectly, with monetary criteria [10].

[13] Invoking uncertainty as to the far distant future, Weitzman [22] argued that the largest posssible discount factor should be used.

◦ For practical purposes, there are no single-period externalities and transaction costs.
◦ Aggregate quotas obtain via dynamic programming or optimal control [9].
◦ Discounting is moderate [16, 22].
◦ Compliance is controlled, oversight is perfect, and deviations are suitably penalized.

For links to chief market theory, recall that the concept of *competitive equilibrium* presumes well defined, fully respected *property rights*. Absent such rights, many commons will easily be mismanaged. Towards mitigation, this paper proposes that a principal acts *as though* sole owner. At any stage, he - and maybe he alone - makes competent decisions as to reductions or savings in renewable stocks.

Important, but not discussed here, are issues on irreversible investments in long-lived, highly specialized gear [3]. Also overlooked was uncertainty as to stock dynamics [19]. And frankly, convexity assumptions, while crucial here above, may sometimes be hard to justify.

Given these loose ends - and the pressing needs for improvement - this paper should be read and seen as an invitation to consider diverse aspects of common-property resource management.

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