



$d(d^{-1})$ -CONTINUOUS MULTI-VALUED MAPPINGS AND SOME FIXED POINT RESULTS

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Abstract. This paper establishes fixed point results for multivalued mappings in ordered bicomplete quasi-metric spaces, thereby generalizing several existing results. To illustrate the significance of these findings, an application to homotopy theory is presented, accompanied by multiple examples.

Keywords. Bicompleteness; Fixed point; Multi-valued mapping; Partial order; Quasi-metric.

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1. INTRODUCTION

Fixed point theory became widely recognized due to the renowned Banach fixed point theorem, originally proposed by Banach [3]. Because of its importance and wide range of applications, this principle has been the focus of much investigation. Several generalized forms have been developed through changes in the metric assumptions or by altering the type of contraction imposed. Nieto et al. [15] developed an extension of the Banach contraction theorem formulated for partially ordered sets, used it to derive existence and uniqueness results for periodic boundary value problems. Harandi and Emami [2] gave a further extension of Nieto et al. results using a control function. We refer to [4, 7, 8, 9, 19] for additional studies on partially ordered sets. The quasi-metric setting is obtained by relaxing the symmetry axiom of the metric, leading to a more flexible structure. Also, every quasi-metric generates a partial order which is called specialization order. Quasi-metric spaces may be regarded as generalized ordered structures, in computer science, they are essential tools, most notably in complexity analysis. Further information can be found in [8, 11, 16, 18, 22, 23, 24]. Schellekens [21] proved the existence of fixed point of contraction mapping on quasi-metric spaces.

Nadler [14] extended the Banach fixed point theorem to the multi-valued mappings, which played a key role in attracting attention to this research direction. Subsequently, Hitzler and

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Seda [12] developed the idea of continuous multi-valued mappings for quasi-metric spaces. Recently, Valero et al. [5] obtained fixed point theorems in the framework of partially ordered bicomplete quasi-metric spaces. Beg et al. [6] further generalized Valero et al. [5] results by using notion of $d(d^{-1})$ -continuity of multi-valued mappings.

This paper is organised as follows: Section 2 introduces the basic definitions and preliminary results required throughout the work. Section 3 extends the results of [6], inspired by the ideas in [2], and establishes a number of new outcomes. The final section illustrates applications of these results within the framework of homotopy theory.

2. PRELIMINARIES

As introduced in [13], a quasi-metric on a nonempty set S is a mapping $d : S \times S \rightarrow \mathbb{R}^+$ that fulfills the following axioms for all ρ, ς, ν in S :

- (i) $d(\rho, \varsigma) = d(\varsigma, \rho) = 0 \Leftrightarrow \rho = \varsigma$,
- (ii) $d(\rho, \nu) \leq d(\rho, \varsigma) + d(\varsigma, \nu)$.

The function d determines a T_0 topology $\tau(d)$ on S , whose basis consists of open d -balls

$$B_d(\rho, r) = \{\varsigma \in S : d(\rho, \varsigma) < r\}, \quad \rho \in S, r > 0.$$

Closed d -balls are defined similarly by

$$B_d[\rho, r] = \{\varsigma \in S : d(\rho, \varsigma) \leq r\}, \quad \rho \in S, r > 0.$$

(S, d) is T_1 in case $d(\rho, \varsigma) = 0 \Leftrightarrow \rho = \varsigma$. Given a quasi-metric d on S , we may construct its symmetrization $d^s : S \times S \rightarrow \mathbb{R}^+$ by $d^s(\rho, \varsigma) = \max\{d(\rho, \varsigma), d^{-1}(\rho, \varsigma)\}$, where d^{-1} is given by $d^{-1}(\rho, \varsigma) = d(\varsigma, \rho)$. For a subset $A \subseteq S$, we use the symbols $cl_d(A)$, $cl_{d^{-1}}(A)$, and $cl_{d^s}(A)$ to indicate the closure of a set A taken in the topologies induced by d, d^{-1} , and d^s , respectively. Moreover, if $A \subseteq S$, the characterization of closure points can be expressed as follows:

$$\rho \in cl_d A \Leftrightarrow d(\rho, A) = \inf_{a \in A} d(\rho, a) = 0$$

and

$$\rho \in cl_{d^{-1}} A \Leftrightarrow d(A, \rho) = \inf_{a \in A} d(a, \rho) = 0.$$

Choose any point $\rho \in S$. A sequence (ρ_n) is said to d -converge to ρ whenever $\lim_{n \rightarrow \infty} d(\rho, \rho_n) = 0$. Analogously, convergence relative to the topology $\tau_{d^{-1}}$ means $\lim_{n \rightarrow \infty} d(\rho_n, \rho) = 0$. It follows immediately that a sequence (ρ_n) converges to ρ in τ_{d^s} precisely when it does so in both τ_d and $\tau_{d^{-1}}$.

As stated in [13], (S, d) is described as bicomplete exactly when the symmetrized metric space (S, d^s) is complete.

The symbol $\mathcal{P}(S)$ will stand for the class consisting of those subsets of S which are not empty. We write $\mathcal{C}_d(S)$ for the class of nonempty sets that are closed under the topology τ_d , while $\mathcal{C}_{d^{-1}}(S)$ refers to the class of sets closed in the topology $\tau_{d^{-1}}$.

Following [16], for any two sets $U, V \in \mathcal{C}_d(S)$ one can define:

$$\begin{aligned} H_d^-(U, V) &= \sup_{u \in U} d(u, V), & H_d^+(U, V) &= \sup_{v \in V} d(U, v) \\ H_d(U, V) &= \max\{H_d^-(U, V), H_d^+(U, V)\}. \end{aligned}$$

It is observed that H_d^-, H_d^+ and H_d provide extended quasi-pseudometrics on $C_d(S)$.

The notion of continuity, introduced in [6], is given below:

Definition 2.1 (Beg and Eroglu [6]). Consider a partially ordered quasi-metric space $(S, d, \preceq)$. A multi-valued mapping $T : S \rightarrow \mathcal{P}(S)$ is referred to be d (d^{-1})-continuous at ρ whenever the following holds: for any increasing sequence (ρ_n) with $\rho_n \in T(\rho_{n-1})$ and (ρ_n) converges to ρ in symmetrized space (S, d^s) , one can find an element $\varsigma \in T(\rho)$ together with a subsequence (ρ_{n_k}) of (ρ_n) such that (ρ_{n_k}) converges to ς in (S, d) ((S, d^{-1})).

In [17], Romaguera and Sanchis proposed the notion of upper-semicontinuity for multi-valued mappings:

Definition 2.2 (Romaguera and Sanchis [17]). Assume (S, d) is a quasi-metric space and $T : S \rightarrow \mathcal{P}(S)$ a multi-valued operator.

- $\tau(d)$ -upper-semicontinuity: The mapping T is regarded as $\tau(d)$ -upper-semicontinuous whenever the following condition is fulfilled: for every $\rho \in S$ and each set O open in $\tau(d)$ with $T(\rho) \subseteq O$, one can choose a $\tau(d)$ -open neighborhood W of ρ so that $T(W) \subseteq O$,
- $\tau(d^{-1})$ -lower-semicontinuity: Analogously, T is regarded as $\tau(d^{-1})$ -lower-semicontinuous if the next requirement holds: given any $\rho \in S$ and any $\tau(d^{-1})$ -open set O with $T(\rho) \subseteq O$, one can find a neighbourhood $W \in \tau(d^{-1})$ of ρ with $T(W) \subseteq O$.

In [6], the authors compared their notions of d and d^{-1} -continuity with the continuity concept introduced earlier by Romaguera and Sanchis [17].

Proposition 2.3 (Beg and Eroglu, [6]). Suppose (S, d) denotes a quasi-metric space together with a partial order $\preceq$ on S , a multi-valued operator $T : S \rightarrow \mathcal{P}(S)$ that is $\tau(d)$ -upper semicontinuous must also satisfy d -continuity.

Proposition 2.4 (Beg and Eroglu, [6]). Suppose (S, d) denotes a quasi-metric space together with a partial order $\preceq$ on S , a multi-valued operator $T : S \rightarrow \mathcal{P}(S)$ that is $\tau(d^{-1})$ -upper semicontinuous must also satisfy d^{-1} -continuity.

3. MAIN RESULTS

Define Ψ as the collection of continuous mappings $\psi : [0, \infty) \rightarrow [0, 1)$ with the property that whenever $\psi(t_n) \rightarrow 1$, it follows that $t_n \rightarrow 0$.

Theorem 3.1. Assume $(S, d, \preceq)$ is a partially ordered, bicomplete T_1 quasi-metric space and let $T : S \rightarrow \mathcal{P}(S)$ be a multi-valued mapping that is d -continuous. Suppose that there exist elements $\rho_0 \in S$ and $\varsigma_0 \in T(\rho_0)$ with $\rho_0 \preceq \varsigma_0$, and that for every pair $\rho, \varsigma \in S$ satisfying $\rho \preceq \varsigma$, and for each $\rho' \in T(\rho)$, there is some $\varsigma' \in T(\varsigma)$ such that $\varsigma \preceq \varsigma'$, in addition, the condition

$$d^s(\rho', \varsigma') \leq \psi(d(\rho, \varsigma))d(\rho, \varsigma)$$

holds for some $\psi \in \Psi$, then T possesses a fixed point.

Proof. Start with an element $\rho_0 \in S$ and choose $\rho_1 \in T(\rho_0)$ so that $\rho_0 \preceq \rho_1$. From the hypothesis, we may select $\rho_2 \in T(\rho_1)$ with $\rho_1 \preceq \rho_2$ and

$$d^s(\rho_1, \rho_2) \leq \psi(d(\rho_0, \rho_1))d(\rho_0, \rho_1) \leq \psi(d(\rho_0, \rho_1))d^s(\rho_0, \rho_1) < d^s(\rho_0, \rho_1).$$

Continuing the process, there is $\rho_3 \in T(\rho_2)$ such that $\rho_2 \preceq \rho_3$ and

$$d^s(\rho_2, \rho_3) \leq \psi(d(\rho_1, \rho_2))d(\rho_1, \rho_2) \leq \psi(d(\rho_1, \rho_2))d^s(\rho_1, \rho_2) < d^s(\rho_1, \rho_2).$$

Iterating this argument produces an increasing sequence (ρ_n) with $\rho_n \in T(\rho_{n-1})$ for each n , satisfying

$$\begin{aligned} d^s(\rho_n, \rho_{n+1}) &\leq \psi(d(\rho_{n-1}, \rho_n))d(\rho_{n-1}, \rho_n) \leq \psi(d(\rho_{n-1}, \rho_n))d^s(\rho_{n-1}, \rho_n) \\ &< d^s(\rho_{n-1}, \rho_n). \end{aligned} \quad (3.1)$$

Thus, the sequence $\{d^s(\rho_n, \rho_{n+1})\}$ is decreasing, bounded below, and hence convergent to some $\kappa \in [0, \infty)$. Suppose that

$$\lim_{n \rightarrow \infty} d^s(\rho_n, \rho_{n+1}) = \kappa$$

and $\kappa > 0$. From the above inequalities and the continuity of the mapping ψ , we obtain that

$$\frac{d^s(\rho_n, \rho_{n+1})}{d^s(\rho_{n-1}, \rho_n)} \leq \psi(d(\rho_{n-1}, \rho_n)) \quad (3.2)$$

and letting n tend to infinity in (3.2), we have

$$1 \leq \lim_{n \rightarrow \infty} \psi(d(\rho_{n-1}, \rho_n)) = \psi(\lim_{n \rightarrow \infty} d(\rho_{n-1}, \rho_n)) < 1.$$

Therefore, a contradiction is obtained, which implies that $\kappa = 0$. Consequently,

$$\lim_{n \rightarrow \infty} d(\rho_n, \rho_{n+1}) = \lim_{n \rightarrow \infty} d(\rho_{n+1}, \rho_n) = 0.$$

To verify that (ρ_n) is Cauchy in (S, d^s) , define $\alpha_n = d^s(\rho_{n-1}, \rho_n)$. By the inequality (3.1), we have

$$\alpha_{n+1} \leq \psi(d(\rho_{n-1}, \rho_n))\alpha_n$$

and

$$\frac{\alpha_{n+1}}{\alpha_n} \leq \psi(d(\rho_{n-1}, \rho_n)). \quad (3.3)$$

Taking the limit as $n \rightarrow \infty$ in (3.3), we obtain that

$$\lim_{n \rightarrow \infty} \frac{\alpha_{n+1}}{\alpha_n} \leq \lim_{n \rightarrow \infty} \psi(d(\rho_{n-1}, \rho_n)) = \psi(\lim_{n \rightarrow \infty} d(\rho_{n-1}, \rho_n)) = \psi(0) < 1.$$

Therefore, the series $\sum_{k=1}^{\infty} \alpha_k$ converges, and hence the sequence (ρ_n) is Cauchy in (S, d^s) . Because (S, d) is bicomplete, there exists $\rho^* \in S$ such that (ρ_n) is convergent to ρ^* in (S, d^s) . By assumption, some $\varsigma^* \in T(\rho^*)$ together with a subsequence (ρ_{n_k}) can be found so that (ρ_{n_k}) converges to ς^* in (S, d) . Consequently,

$$d(\varsigma^*, \rho^*) \leq d(\varsigma^*, \rho_{n_k}) + d(\rho_{n_k}, \rho^*). \quad (3.4)$$

Letting $n_k \rightarrow \infty$ in (3.4) yields $d(\varsigma^*, \rho^*) = 0$. Since the space is T_1 , it follows that $\rho^* = \varsigma^* \in T(\rho^*)$, establishing that it is indeed a fixed point. $\square$

Example 3.2. Take $S = [0, \infty)$ together with the quasi-metric d given by

$$d(\rho, \varsigma) = \begin{cases} \varsigma - \rho, & \text{if } \rho \leq \varsigma \\ \rho, & \text{otherwise.} \end{cases}$$

Define a partial order $\preceq$ on S by $\rho \preceq \varsigma \Leftrightarrow \varsigma \leq \rho$ for all $\rho, \varsigma \in S$. Consider the multi-valued mapping $T : S \rightarrow P(S)$ given by $T(\rho) = [0, \frac{\rho}{\rho+2}]$ for all $\rho \in S$. One can easily see that T is

d -continuous multi-valued mapping. Let $\psi(t) = \frac{1}{t+1}$ for all $t \in [0, \infty)$. Now, if $\rho \preceq \varsigma$, then for each $\rho' \in T(\rho)$ there exists $\varsigma' = 0 \in T(\varsigma)$ such that

$$d^s(\rho', 0) = \rho' \leq \frac{\rho}{2+\rho} \leq \frac{d(\rho, \varsigma)}{1+d(\rho, \varsigma)} = \frac{\rho}{1+\rho}.$$

Hence, all the conditions of Theorem 3.1 are satisfied, and therefore T has a fixed point.

In the preceding theorem, setting $\psi(t) = \sigma$ for some $\sigma \in [0, 1)$ yields, as a corollary a result of Beg and Eroglu, [6]:

Corollary 3.3. *Assume that $(S, d, \preceq)$ is a partially ordered bicomplete T_1 quasi-metric space, and let $T : S \rightarrow \mathcal{P}(S)$ be a d -continuous multi-valued mapping. Suppose there exist elements $\rho_0 \in S$ and $\varsigma_0 \in T(\rho_0)$ such that $\rho_0 \preceq \varsigma_0$. Further, assume that whenever $\rho, \varsigma \in S$ satisfy $\rho \preceq \varsigma$, then for each $\rho' \in T(\rho)$ there is some $\varsigma' \in T(\varsigma)$ with $\varsigma \preceq \varsigma'$, in addition a constant $\sigma \in [0, 1)$ exists for which, $d^s(\rho', \varsigma') \leq \sigma d(\rho, \varsigma)$. Then, T admits a fixed point.*

When T is taken to be single-valued, the next statement follows as a corollary:

Corollary 3.4. *Assume that $(S, d, \preceq)$ is a partially ordered bicomplete T_1 quasi-metric space, and let $T : S \rightarrow S$ be a d -continuous mapping. Suppose there exist elements $\rho_0 \in S$ and $\varsigma_0 = T(\rho_0)$ with $\rho_0 \preceq \varsigma_0$. Further, assume that whenever $\rho, \varsigma \in S$ satisfy $\rho \preceq \varsigma$ and $\varsigma \preceq T(\varsigma)$, the inequality*

$$d^s(T(\rho), T(\varsigma)) \leq \psi(d(\rho, \varsigma))d(\rho, \varsigma)$$

holds for $\psi \in \Psi$. Then, T admits a fixed point.

Theorem 3.5. *Assume $(S, d, \preceq)$ is a partially ordered, bicomplete T_1 quasi-metric space and let $T : S \rightarrow \mathcal{P}(S)$ be a multi-valued mapping that is d^{-1} -continuous. Suppose that there exist elements $\rho_0 \in S$ and $\varsigma_0 \in T(\rho_0)$ with $\rho_0 \preceq \varsigma_0$, and that for every pair $\rho, \varsigma \in S$ satisfying $\rho \preceq \varsigma$, and for each $\rho' \in T(\rho)$, there is some $\varsigma' \in T(\varsigma)$ such that $\varsigma \preceq \varsigma'$, in addition, the condition*

$$d^s(\rho', \varsigma') \leq \psi(d(\rho, \varsigma))d(\rho, \varsigma)$$

holds for some $\psi \in \Psi$, then T possesses a fixed point.

Proof. By an argument analogous to that of Theorem 3.1, the sequence (ρ_n) is Cauchy in (S, d^s) and hence, convergent to ρ^* with respect to d^s . Since T is d^{-1} -continuous, there exists a point $\varsigma^* \in T(\rho^*)$ and subsequence (ρ_{n_k}) such that (ρ_{n_k}) converges to ς^* in (S, d^{-1}) . Thus,

$$d(\rho^*, \varsigma^*) \leq d(\rho^*, \rho_{n_k}) + d(\rho_{n_k}, \varsigma^*).$$

Letting $n_k \rightarrow \infty$ in the above inequality, it follows that $d(\rho^*, \varsigma^*) = 0$. Since S is T_1 , $\rho^* = \varsigma^* \in T(\rho^*)$, confirming that it is indeed a fixed point. $\square$

Using Proposition 2.3 and 2.4, we arrive at the following results:

Corollary 3.6. *Assume $(S, d, \preceq)$ is a partially ordered, bicomplete T_1 quasi-metric space and let $T : S \rightarrow \mathcal{P}(S)$ be a multi-valued mapping that is $\tau(d)$ -upper semicontinuous. Suppose that there exist elements $\rho_0 \in S$ and $\varsigma_0 \in T(\rho_0)$ with $\rho_0 \preceq \varsigma_0$, and that for every pair $\rho, \varsigma \in S$*

satisfying $\rho \preceq \varsigma$, and for each $\rho' \in T(\rho)$, there is some $\varsigma' \in T(\varsigma)$ such that $\varsigma \preceq \varsigma'$, in addition, the condition

$$d^s(\rho', \varsigma') \leq \psi(d(\rho, \varsigma))d(\rho, \varsigma)$$

holds for some $\psi \in \Psi$, then T possesses a fixed point.

Corollary 3.7. Assume $(S, d, \preceq)$ is a partially ordered, bicomplete T_1 quasi-metric space and let $T : S \rightarrow \mathcal{P}(S)$ be a multi-valued mapping that is $\tau(d^{-1})$ -lower semicontinuous. Suppose that there exist elements $\rho_0 \in S$ and $\varsigma_0 \in T(\rho_0)$ with $\rho_0 \preceq \varsigma_0$, and that for every pair $\rho, \varsigma \in S$ satisfying $\rho \preceq \varsigma$, and for each $\rho' \in T(\rho)$, there is some $\varsigma' \in T(\varsigma)$ such that $\varsigma \preceq \varsigma'$, in addition, the condition

$$d^s(\rho', \varsigma') \leq \psi(d(\rho, \varsigma))d(\rho, \varsigma)$$

holds for some $\psi \in \Psi$, then T admits a fixed point.

If $C_d(S)$ or $C_{d^{-1}}(S)$ is used in place of $P(S)$, the T_1 -assumption in the preceding theorems may be relaxed.

Theorem 3.8. Assume $(S, d, \preceq)$ is a partially ordered, bicomplete quasi-metric space and let $T : S \rightarrow C_d(S)$ be a multi-valued mapping that is d -continuous. Assume further there exist elements $\rho_0 \in S$ and $\varsigma_0 \in T(\rho_0)$ with $\rho_0 \preceq \varsigma_0$, and that for every pair $\rho, \varsigma \in S$ satisfying $\rho \preceq \varsigma$, and for each $\rho' \in T(\rho)$, there is some $\varsigma' \in T(\varsigma)$ such that $\varsigma \preceq \varsigma'$, in addition, the condition

$$d^s(\rho', \varsigma') \leq \psi(d(\rho, \varsigma))d(\rho, \varsigma), \quad (3.5)$$

holds for some $\psi \in \Psi$, then T possesses a fixed point.

Theorem 3.9. Assume $(S, d, \preceq)$ is a partially ordered, bicomplete quasi-metric space and let $T : S \rightarrow C_{d^{-1}}(S)$ be a multi-valued mapping that is d^{-1} -continuous. Suppose that there exist elements $\rho_0 \in S$ and $\varsigma_0 \in T(\rho_0)$ with $\rho_0 \preceq \varsigma_0$, and that for every pair $\rho, \varsigma \in S$ satisfying $\rho \preceq \varsigma$, and for each $\rho' \in T(\rho)$, there is some $\varsigma' \in T(\varsigma)$ such that $\varsigma \preceq \varsigma'$, in addition, the condition

$$d^s(\rho', \varsigma') \leq \psi(d(\rho, \varsigma))d(\rho, \varsigma) \quad (3.6)$$

holds for some $\psi \in \Psi$, then T possesses a fixed point.

4. APPLICATION TO HOMOTOPY

Fixed point theory is a well-established branch of mathematics with broad applications in many diciplines. On the other hand, homotopy theory represents a central topic within algebraic topology. Motivated by earlier works that investigated connections between fixed point results and homotopy theory (see [1, 9, 10, 20]), we now illustrate how our findings can be applied in the homotopy setting.

Theorem 4.1. Let $(S, d, \preceq)$ be a bicomplete partially ordered quasi-metric space. Let Θ be a nonempty τ_d -open subset of comparable elements of S . Suppose that there exists $\psi \in \Psi$ and $T : cl_d\Theta \times [0, 1] \rightarrow C_d(S)$ is a d -continuous multi-valued mapping in the first variable and the conditions below are met:

- (1) $\rho \notin T(\rho, t)$ for every $\rho \in \partial_d\Theta$, $t \in [0, 1]$ and there exists $\rho_0 \in \Theta$ such that $\rho_0 \preceq T(\rho_0, \cdot)$ ($\partial_d\Theta$, denotes the boundary of Θ in (S, τ_d)).
- (2) T satisfies (3.5) in Theorem 3.8 for the first variable.

- (3) There exists $L \geq 0$, such that the inequality below is satisfied for all $t_1, t_2 \in [0, 1]$ and $\rho \in S$, $H_{d^s}(T(\rho, t_1), T(\rho, t_2)) \leq L |t_1 - t_2|$.
- (4) For each $t \in [0, 1]$, $H_{d^s}(T(\rho, t), T(\varsigma, t)) \leq \psi(d(\rho, \varsigma))d(\rho, \varsigma)$, for all comparable elements ρ, ς .
- (5) If for some $\rho \in \Theta$ there exists $\varsigma \in S$ with $d(\rho, \varsigma) \leq r$ for some $r \geq 0$, then $\rho \preceq \varsigma$.
- (6) If $\rho \in T(\rho, t)$, then $T(\rho, t) = \{\rho\}$.

Whenever $T(\cdot, 0)$ admits a fixed point in Θ , so does $T(\cdot, 1)$.

Proof. Assume $T(\cdot, 0)$ possesses a fixed point in Θ . Let us define the set

$$I = \{t \in [0, 1] : \rho \in T(\rho, t) \text{ for some } \rho \in \Theta\}.$$

Since $0 \in I$, it can be easily seen that I is nonempty. Establishing that I is both open and closed allows us to use the connectedness of $[0, 1]$ to conclude $I = [0, 1]$; this yields the existence of a fixed point of $T(\cdot, 1)$. Now, let's first show that I is closed in $[0, 1]$. For this aim, let $\{t_n\}$ be a sequence in I with $t_n \rightarrow t^* \in [0, 1]$ as $n \rightarrow \infty$. It must be proven that $t^* \in I$. Since $t_n \in I$ for all $n \in \{1, 2, 3, \dots\}$, there exist $\rho_n \in \Theta$ with $\rho_n \in T(\rho_n, t_n)$.

Now, we show that (ρ_n) is a d^s -Cauchy sequence. Assume on the contrary that

$$\limsup_{m, n \rightarrow \infty} d^s(\rho_m, \rho_n) > 0.$$

Then, we have

$$\begin{aligned} d^s(\rho_n, \rho_m) &= H_{d^s}(T(\rho_n, t_n), T(\rho_m, t_m)) \\ &\leq H_{d^s}(T(\rho_n, t_n), T(\rho_n, t_m)) + H_{d^s}(T(\rho_n, t_m), T(\rho_m, t_m)) \\ &\leq L |t_n - t_m| + \psi(d(\rho_n, \rho_m))d(\rho_n, \rho_m) \\ &\leq L |t_n - t_m| + \psi(d(\rho_n, \rho_m))d^s(\rho_n, \rho_m). \end{aligned}$$

That is,

$$d^s(\rho_n, \rho_m) \leq \left(\frac{L}{1 - \psi(d(\rho_n, \rho_m))} \right) |t_n - t_m|.$$

Since every convergent sequence in $[0, 1]$ is a Cauchy sequence, $\{t_n\}$ is a Cauchy sequence in $[0, 1]$. Passing to limit $m, n \rightarrow \infty$ in the above inequality, we deduce $\limsup_{m, n \rightarrow \infty} d^s(\rho_m, \rho_n) = 0$ and this contradicts with our assumption. Therefore, (ρ_n) must be Cauchy sequence in (S, d^s) . Owing to the bicompleteness of S , one can find $\rho^* \in cl_{d^s}\Theta$ (and hence $\rho^* \in cl_d\Theta$) satisfying $\lim_{n \rightarrow \infty} d^s(\rho_n, \rho^*) = 0$. Furthermore, we have

$$\begin{aligned} d^s(\rho_n, T(\rho^*, t^*)) &= H_{d^s}(T(\rho_n, t_n), T(\rho^*, t^*)) \\ &\leq H_{d^s}(T(\rho_n, t_n), T(\rho_n, t^*)) + H_{d^s}(T(\rho_n, t^*), T(\rho^*, t^*)) \\ &\leq L |t_n - t^*| + \psi(d(\rho_n, \rho^*))d(\rho_n, \rho^*) \\ &\leq L |t_n - t^*| + \psi(d(\rho_n, \rho^*))d^s(\rho_n, \rho^*). \end{aligned}$$

Letting $n \rightarrow \infty$ in the following, we deduce

$$\lim_{n \rightarrow \infty} d^s(\rho_n, T(\rho^*, t^*)) = d^s(\rho^*, T(\rho^*, t^*)) = 0.$$

Thus, $\rho^* \in T(\rho^*, t^*)$. Thus, $t^* \in I$ and I is closed in $[0, 1]$. Now we prove that I is open in $[0, 1]$.

Let $t_0 \in I$ and $\rho_0 \in \Theta$ with $\rho_0 \in T(\rho_0, t_0)$. Since Θ is open in (S, τ_d) , there exists $r_0 > 0$ such that $B_d(\rho_0, r_0) \subseteq \Theta$. Choose an $\varepsilon > 0$ with $\varepsilon < \frac{(1-\psi(r_0))r_0}{L}$. Take $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and for all $\rho \in B_d[\rho_0, r_0] = \{\rho \in S : d(\rho_0, \rho) \leq r_0\}$ we will prove that $T(\rho, t) \subseteq B_d[\rho_0, r_0]$. Suppose that $\varsigma \in T(\rho, t)$. Then, we have

$$\begin{aligned} d(\rho_0, \varsigma) &= d(T(\rho_0, t_0), \varsigma) \\ &\leq d^s(T(\rho_0, t_0), \varsigma) \\ &\leq H_{d^s}(T(\rho_0, t_0), T(\rho, t)) \\ &\leq H_{d^s}(T(\rho_0, t_0), T(\rho, t_0)) + H_{d^s}(T(\rho, t_0), T(\rho, t)) \\ &\leq \psi(d(\rho_0, \rho))d(\rho_0, x) + L |t_0 - t| \leq \psi(r_0)r_0 + L\varepsilon = r_0. \end{aligned}$$

Thus, $\varsigma \in B_d[\rho_0, r_0]$ and for each $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, $T(\cdot, t) : B_d[\rho_0, r_0] \rightarrow C_d(B_d[\rho_0, r_0])$. Since all the hypothesis of Theorem 3.8 hold, $T(\cdot, t)$ has a fixed point in $cl_d\Theta$ and from the condition (1), the fixed point lies in Θ . Therefore, for any $t_0 \in I$, there exists $\varepsilon > 0$ and $(t_0 - \varepsilon, t_0 + \varepsilon) \subseteq I$. Thus, I is open in $[0, 1]$. Consequently, the claim follows. $\square$

Similarly we can prove the following result.

Theorem 4.2. *Let $(S, d, \preceq)$ be a bicomplete partially ordered quasi-metric space. Let Θ be a nonempty $\tau_{d^{-1}}$ -open subset of comparable elements of S . Suppose that there exists $\psi \in \Psi$ and $T : cl_{d^{-1}}\Theta \times [0, 1] \rightarrow C_{d^{-1}}(S)$ is a d^{-1} -continuous multi-valued mapping in the first variable and the conditions below are satisfied:*

- (1) $\rho \notin T(\rho, t)$ for every $\rho \in \partial_{d^{-1}}\Theta$, $t \in [0, 1]$ and there exists $\rho_0 \in \Theta$ such that $\rho_0 \preceq T(\rho_0, \cdot)$ ($\partial_{d^{-1}}\Theta$, denotes the boundary of Θ in $(S, \tau_{d^{-1}})$).
- (2) T satisfies (3.6) in Theorem 3.9 for the first variable.
- (3) There exists $L \geq 0$, such that the inequality below is satisfied for all $t_1, t_2 \in [0, 1]$ and $\rho \in S$, $H_{d^s}(T(\rho, t_1), T(\rho, t_2)) \leq L |t_1 - t_2|$.
- (4) For each $t \in [0, 1]$, $H_{d^s}(T(\rho, t), T(\varsigma, t)) \leq \psi(d(\rho, \varsigma))d(\rho, \varsigma)$, for all comparable elements ρ, ς .
- (5) If for some $\rho \in \Theta$ there exists $\varsigma \in S$ with $d(\rho, \varsigma) \leq r$ for some $r \geq 0$, then $\rho \preceq \varsigma$.
- (6) If $\rho \in T(\rho, t)$, then $T(\rho, t) = \{\rho\}$.

Whenever $T(\cdot, 0)$ admits a fixed point in Θ , so does $T(\cdot, 1)$.

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