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ON REGULARIZATION OF THE LAGRANGE MULTIPLIER RULE IN CONVEX CONSTRAINED EXTREMUM PROBLEMS AND ON ITS UNIVERSALITY

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Dedicated to Professor Vladimir Tikhomirov on the occasion of his 90th birthday

Abstract. The regularization of the Lagrange multiplier rule (LMR) in nondifferential form in a convex constrained extremum problem (CEP) with an operator equality constraint in a Hilbert space and a finite number of functional inequality constraints is discussed. The set of admissible elements of the problem under consideration also belongs to a Hilbert space, and its constraints contain additively included parameters, which makes it possible to apply the so-called perturbation method to its study. The main purpose of the regularized LMR is the stable generation of generalized minimizing sequences (GMS) that approximate the exact solution of the problem by means of extremals of the regular Lagrange functional. The regularized LMR can be interpreted as an GMS-generating (regularizing) operator that assigns to each set of input data of the CEP the extremal of its regular Lagrange functional corresponding to this set. In this case, the dual variable in the Lagrange functional is generated in accordance with one or another procedure for stabilizing the dual problem. The main attention in the paper is paid to the discussion of: 1) problems associated with the ill-posedness properties of the classical LMR, as well as its applicability for solving CEPs; 2) the procedure of the dual regularization and its connection with the regularization of the LMR; 3) the procedure for obtaining the classical LMR as a limiting version of its regularized analogue; 4) the possibilities of using the regularized LMR to solve current ill-posed problems. The set of properties of the regularized LMR discussed in the paper allows us to speak about the universality of its classical analogue. On the one hand, the classical LMR is the generally recognized core of the entire theory of extremal problems, and on the other hand, it constitutes the fundamental basis for constructing stable algorithms for solving many ill-posed problems, including extremal ones.

Keywords. Constrained extremum; Convexity; Multiplier rule; Perturbation; Duality; Subdifferential; Minimizing sequence; Regularizing algorithm.

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1. Introduction

Currently, it is considered an established opinion according to which the *Lagrange multipliers rule* (LMR) owes its appearance at the end of the 18th century [1, 2] to the needs of solving problems of very different nature related to the practical activities of people [3, 4, 5]. The

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ideas underlying this classical rule, or, as they also say, *Lagrange's principle* (LP), served as the basis for the creation over the past centuries, through the efforts of many mathematicians, of the theory of extremal problems, see books [3, 4, 5, 6, 7, 8, 9] and also the bibliography in them. A huge role in the development and popularization of the Lagrange's classical ideas within the framework of the general theory of extremal problems in the last more than six decades belongs to the outstanding Russian mathematician Professor Vladimir Mikhailovich Tikhomirov [3, 4, 5, 6, 7, 8, 9].

Below in this paper we will talk about the fact that the LMR, which is the core of the entire theory of extremal problems, should also be considered as the most important direct basis for the construction of new algorithms for the practical solving many extremal problems. The most significant role in this construction is given to the extremals of the Lagrange functionals in constrained extremum problems (CEPs). The "mathematical nature" of the CEPs is such that we can consider their entire set as a typical class of ill-posed problems. It is appropriate to recall here that the concepts of well-posed and ill-posed problems are among the most important in modern mathematics. As it is known, the concept of a well-posed mathematical problem was first introduced into scientific use by J. Hadamard in connection with research in the field of equations of mathematical physics [10, 11]. He called the problem of solving the operator equation

$$Az = h,\ z \in Z, \tag{1.1}$$

where $A : Z \to H$ is a continuous operator, $h \in H$ is a right side of the equation, Z, H are metric spaces, well-posed if: 1) this problem is solvable; 2) its solution is unique; 3) this solution continuously in the metric of Z depends on the perturbations of the operator A and the right side h in the corresponding metrics [10, 11]. In turn, the concept of an ill-posed problem was introduced as a negation of the concept of well-posedness on Hadamard [11, 12, 13, 14]. The urgent need for the ability to solve ill-posed problems clearly emerged by the middle of the 20th century under the influence of the corresponding needs in the development of modern technologies and natural sciences [12, 13, 14, 15]. As a consequence, it gave rise to corresponding intensive research into the possibilities of solving ill-posed problems. In particular, in 1963 the first works of A. N. Tikhonov [12, 13, 14] appeared, in which a regularization method was formulated for solving operator equations of the first kind. According to the regularization method, which from a general point of view can be interpreted as the regularized least squares method, the solution to the original equation (1.1) is approximated by the extremals of the corresponding Tikhonov functional [12, 13, 14].

Various examples [16] indicate that all three classical properties from the definition of an ill-posed problem [10, 11, 14] have a direct relation to the CEPs also. In particular, we encounter the properties of ill-posedness when dealing with the LMR [3, 17, 18]. Speaking about the ill-posedness of the multipliers rule, we primarily mean such properties as its possible *impracticability* and *instability*. We speak about the *impracticability* of the multiplier rule when it is known that this classical rule cannot be written in a given CEP. In fact, a situation "close" to this also arises when we do not know how the multipliers rule can be written for a CEP (for example, this happens when we obtain the multipliers rule under some assumptions and do not know how to do this when these assumptions are weakened). A classical example of the impracticability of the LP can be found in [3, item 3.2.4], see another similar example below in Section 4. In addition, various examples of the impracticability of the LP can be found in [17, 18, 19]. In turn, we understand the *instability* of the classical rule in the sense that admissible elements that satisfy all the relations that make up the multipliers rule behave unstable with respect to perturbations of the input data of extremum problems. Examples of the instability of the multipliers rule can be found in [17, 18, 19].

Below we will adhere to the approach according to which the LMR should be treated as the most important basis for a stable, with respect to errors in the input data, approximation of solutions to the CEPs. Such approximation consists in a constructive indication of the elements of minimizing sequences consisting of the extremals of the regular Lagrange functionals of these problems. It is also based on the methods of the ill-posed problems theory [14]. Here and below, when we speak of a minimizing sequence in an extremal problem, we mean the concept of a *generalized minimizing sequence* (GMS). It coincides with the well-known concept of a *generalized optimal plan* in the sense of [20] in mathematical programming, as well as with the concept of a *minimizing approximate solution* in the sense of J. Warga [21, Chapter III] used in optimal control. The procedure that transforms the classical rule into a universal means of stable solving the CEPs, in accordance with which the specified extremals are selected, we call the regularization of the multipliers rule.

More specifically, the paper considers the regularization of the LMR in nondifferential form as applied to the classical canonical [3, item 3.3.1] convex CEP with an operator equality constraint (i.e., with an equality defined by an operator with an infinite-dimensional image) and with a finite number of functional inequality constraints. The constraints of the problem contain parameters additively included in them. This made it possible, based on the perturbation method [3, item 3.3.2], to study the relationship between the regularization of the multipliers method and the subdifferential properties of its value function and, in particular, to show that the limit passage in the regularized LMR leads to its classical analogue. Here, we restrict ourselves to the case where the image space of the operator defining the equality constraint is a Hilbert one. On the one hand, this allows us to obtain and formulate results more "compactly", and on the other hand, we get the opportunity to consider as specific optimization problems those that are associated with modern, sufficiently meaningful applications.

The paper consists of Introduction and six main Sections, the first of which discusses general considerations related to regularization in CEPs and the universality of the classical LMR. Section 3 is devoted to the formulation of the CEP with a strongly convex objective functional, and it also formulates the concepts and statements required further. Section 4 discusses issues related to questions of the applicability of the classical LMR for solving CEPs in both parametric and individual versions. Section 5 is devoted to the formulation and discussion of the dual regularization method and the regularized LMR in nondifferential form as applied to the considered CEP, and it also discusses how to organize the regularization of the classical rule in the case when the convex objective functional may not be strongly convex. Section 6 shows that the result of the limit passage in the regularized LMR is its classical analogue. Finally, in Section 7, the possibilities of applicability of the regularized LMR for solving the classical ill-posed problem of finding a solution to an operator equation of the first kind are discussed, as well as the connection of the obtained results with the Tikhonov's regularization method.

2. On Regularization in CEPs and on Universality of the LMR in Nondifferential Form

In this Section, we briefly discuss issues related to the regularization in CEPs, as well as with the universality of the classical multipliers rule.

2.1. On the regularization of ill-posed CEPs. The development of methods for solving ill-posed problems [14] has opened up the possibility of overcoming the ill-posedness of CEPs by means of their regularization [16].The well-known approach [16, Chapter 9] to regularization of the CEPs is based on the Tikhonov's stabilization (regularization) method [14, 16, 22] for a general optimization problem (without equality and inequality constraints)

$$\varphi(z) \to \min, \;\; z \in D, \tag{2.1}$$

where $\varphi : D \to \mathbb{R}^1$ is a continuous function, $D \subseteq Z$ is a closed set, for example, of a Hilbert space Z. In the case of convexity of φ and D, the problem (2.1) is a convex optimization problem of general form.

We note important specific features of this approach to the stable solving the CEPs. These include, first of all, the following:

1. The need to transform the original CEP with equality and inequality constraints into a general optimization problem (a convex or nonlinear problem of general form, i.e. without equality and inequality constraints) with its own set D of admissible elements (often D is called the set of geometric constraints), which is constructed in one way or another taking into account the specifics of the equality and inequality constraints of the original CEP. In this situation, the solution to the original problem is approximated by means of extremals of the Tikhonov functional of the auxiliary general problem (2.1), and the "structure" of the original CEP is completely destroyed (for details, see [16, Chapter 9, §2, 4–8]).

2. The need to check the feasibility of a number of "difficult to check" conditions in the CEP. Among them, the so-called strong consistency condition for the statement of the constrained optimization problem (see, for example, the problem (1), (2) and the inequality (9) in [16, Chapter 9, §4], as well as [16, Definition 5.16.3]) can be singled out as, apparently, the most difficult from the point of view of its verification. A sufficient condition for its feasibility is the condition of existence of a saddle point of the Lagrange function of the CEP (see also [16, Lemma 5.16.5]).

2.2. On the regularization of the LMR in nondifferential form in the CEPs. The LMR is the basis on which it is possible to construct a substantially different regularization in the CEPs. In this case, when approximating solutions of the optimization problem, the extremals of its Lagrange function play a central role. To overcome the properties of the ill-posedness of the LMR, it was proposed in [17, 23] to regularize it based on the dual approach to regularization [24, 25]. In this case, the main goal is not the "usual" search for optimal elements, but the constructive generation of a GMS in the optimization problem.

The central role in the indicated approach, as well as in the regularization of the "constrained extremum problem itself" [16, Chapter 9], belongs to the formal procedure of Tikhonov stabilization (regularization) for the general optimization problem [14, 22]. However, in our case the latter is the dual problem with respect to the original CEP. Formally, it is a concave optimization problem of general type (this is the problem (2.1), where φ is a concave functional, and the min operation is replaced by sup). Its concave objective functional has the form of a typical marginal functional, which allows one to obtain a useful representation for its superdifferential in terms of the original CEP based on methods of nonsmooth analysis. The combined use of these two circumstances generates a procedure for constructing a GMS in the original CEP that is stable to errors in the input data. The elements of this sequence are extremals of the regular Lagrange functional of the original problem, the dual variable of which is generated by the formal Tikhonov regularization indicated above. The listed constructive actions underlie the regularization of the multipliers rule in the CEPs.

Speaking about the formal procedure of Tikhonov stabilization for regularization of the dual problem when justifying the convergence of the entire dual regularization procedure to the solution of the original problem [17, 25], we point out two of its essential features compared to the standard Tikhonov stabilization procedure in [16, Chapter 9], [22] for a convex problem of general form (2.1). Note, first of all, that in both cases we are talking about stabilization of convex problems of the same (up to the sign of the objective functional) general form (without constraints of the equality and inequality types). However, in contrast to the requirements for proving the convergence of the standard Tikhonov stabilization procedure in [16, Chapter 9], [22], firstly, a solution to the dual problem may not exist (the problem may not have a Kuhn–Tucker vector), and secondly, if such a solution does exist, the estimate of the deviation of the perturbed objective function of the dual problem from the exact one required for the standard Tikhonov stabilization [16, Chapter 9], [22]) may be missing. The latter situation is realized in the CEP considered below (see Section 5) in the case of an unbounded set of admissible elements.

The main characteristic properties of the regularization of the LMR discussed in the paper can be formulated as follows. The regularized LMR: 1) is formulated as a existence theorem of GMSs with the above properties in the original problem; 2) generalizes the classical LMR, leads to it in the limit and preserves the "general structure" of the classical analogue; 3) is a condition of ordinary optimality, but expressed in sequential form in terms of regular Lagrange functional; 4) "overcomes" the properties of ill-posedness of its classical analogue and represents an universal regularizing algorithm for solving the CEPs.

2.3. On universality of the LMR. In connection with the above, we must pay attention to the following two important points.

First, despite the above-mentioned ill-posedness of the CEPs in general, in each specific problem, including, possibly, an ill-posed problem, we can, in accordance with "traditions", strive to obtain the usual optimality conditions. As is known, existing methods (see, for example, [3, 4, 5, 6, 7, 8, 9]) for obtaining the LP "assume only an exact specification of the input data" of extremal problems. Therefore, in this "ideal" situation, typical for the theory of extremal problems [3, 4, 5, 6, 7, 8, 9], generally speaking, there is no need to take into account the possible ill-posedness of a specific problem.

Secondly, many of the CEPs, supplied by modern applications [14, 15, 26, 27], are such that the requirement of precise specification of their input data, customary for classical theory, contradicts the physical essence of these problems. In this case, under the conditions of possible ill-posedness of a specific problem and the presence of errors in the specification of its input data, we come to the need to regularize the LMR.

The classical LMR organically unites both directions of optimization theory, corresponding to the two circumstances indicated above. In the first case, it is the usual basis of the entire theory of extremal problems [3, 4, 5, 6, 7, 8, 9]. In the case of ill-posed extremal problems, the classical rule retains this role, but in a regularized form. Thus, we obtain additional evidence in favor of fundamentality and universality of the classical LMR, the "internal potential" of which turned out to be such that, with the appropriate constructive transformation-regularization, it is transformed into a universal means of stable solving ill-posed CEPs.

3. Problem Statement, Necessary Concepts, and Statements

First of all, below the object of our attention will be parametric (i.e., depending on parameters), canonical [3, item 3.3.1] and convex the constrained extremum problem (CEP)

$$(P_{p,r}) \qquad f(z)\to\min,\ Az=h+p,\ g_i(z)\le r_i,\ i=1,\dots,m,\ \ z\in\mathscr{D}\subset Z.$$

Here: $p\in H$, $r=(r_1,\dots,r_m)^*\in\mathbb{R}^m$ are parameters, $f:\mathscr{D}\to\mathbb{R}^1$ is a continuous strongly convex functional with a constant of strong convexity $\kappa>0$, $A:Z\to H$ is a linear bounded operator, $h\in H$ is a given element, $g_i:\mathscr{D}\to\mathbb{R}^1$, $i=1,\dots,m$, are continuous convex functionals, $g(z)\equiv(g_1(z),\dots,g_m(z))^*$, $\mathscr{D}\subseteq Z$ is a convex closed set, Z, H are Hilbert spaces. We will denote the unique solution to the problem $(P_{p,r})$ if it exists, by $z^0_{p,r}$. We will assume that the following condition is satisfied.

Condition $\mathscr{A}$: the functionals $f, g_i, i=1,\dots,m$, are continuous on $\mathscr{D}$, their values are bounded on $\mathscr{D}\cap B_M$ uniformly in $z\in\mathscr{D}\cap B_M$ for every fixed $M>0$, where $B_M\equiv\{z\in Z: \|z\|\le M\}$.

Remark 3.1. All the results obtained in [17] remain valid in the case when the condition of local Lipschitzness of the input data (see condition (1.1) in [17]) is replaced by Condition $\mathscr{A}$ of their continuity and uniform boundedness on bounded subsets of $\mathscr{D}$.

Below, the central role for us will be played by the concept of a *generalized minimizing sequence* (GMS) in the problem $(P_{p,r})$. Recall that by a GMS in the problem $(P_{p,r})$ we mean a sequence $z^i\in\mathscr{D}$, $i=1,2,\dots$ satisfying $f(z^i)\le\beta(p,r)+\delta^i$, $z^i\in\mathscr{D}^{\varepsilon^i}_{p,r}$ for some sequences of nonnegative numbers δ^i, ε^i, $i=1,2,\dots$ converging to zero. Here $\beta(p,r)$ is the generalized value (generalized infimum) of the problem $(P_{p,r})$:

$$\beta(p,r)\equiv\beta_{+0}(p,r)\equiv\lim_{\varepsilon\to+0}\beta_\varepsilon(p,r),\ \beta_\varepsilon(p,r)\equiv\inf_{z\in\mathscr{D}^\varepsilon_{p,r}}f(z),\qquad \beta_\varepsilon(p,r)\equiv+\infty,\text{ if }\mathscr{D}^\varepsilon_{p,r}=\emptyset,$$

$$\mathscr{D}^\varepsilon_{p,r}\equiv\{z\in\mathscr{D}:\|Az-h-p\|\le\varepsilon,\ g_i(z)\le r_i+\varepsilon,\ i=1,\dots,m\},\ \varepsilon\ge0.$$

Obviously, in the general situation $\beta(p,r)\le\beta_0(p,r)\equiv\inf_{z\in\mathscr{D}^0_{p,r}}f(z)$, where the value $\beta_0(p,r)\equiv\{f(z^0_{p,r}),\text{ if } z^0_{p,r}\text{ exists};\ +\infty\text{ otherwise}\}$ is the classical value (classical infimum) of the problem.

Lemma 3.2. *The following statements hold:*

1. *The equality* $\beta_0(p,r)=\beta(p,r)\ \forall(p,r)\in H\times\mathbb{R}^m$ *is true, the functional* $\beta: H\times\mathbb{R}^m\to\mathbb{R}^1\cup\{+\infty\}$ *is lower semicontinuous and convex;*

2. *Let* $\beta(p,r)<+\infty$. *Then for any GMS* z^i, $i=1,2,\dots,$ *the following limit relations are valid in the problem* $(P_{p,r})$ *which is solvable in this case:*

$$f(z^i)\to f(z^0_{p,r})=\beta_0(p,r)=\beta(p,r),\quad z^i\to z^0_{p,r}\ \text{ weakly in } Z,\ i\to\infty.$$

If, in addition to Condition $\mathscr{A}$, the strongly convex functional f is subdifferentiable (in the sense of convex analysis) at points of $\mathscr{D}$, then the above weak convergence is strong: $\|z^i - z^0_{p,r}\| \to 0,\ i \to \infty$.

Proof of the first statement can be found in [28, Lemma 1.2], see also the proof of Corollary 1 in [3, item 3.3.2]. The proof of the second statement is based on standard reasoning for situations of this kind, using the weak compactness of a bounded closed convex set and the weak lower semicontinuity of a continuous convex functional in a Hilbert space.

The following important, in the context of this paper, a statement about the density of subdifferentiability in the sense of convex analysis is also true (see, for example, [29, Theorem 4.3]).

Lemma 3.3. *The subdifferential of a proper convex lower semicontinuous function $h : \mathscr{H} \to \mathbb{R}^1 \cup \{+\infty\}$, where $\mathscr{H}$ is a Hilbert space, is not empty at points of a set dense in $\operatorname{dom} h$.*

4. Parametric and Individual LMRs, Problems of Their Applicability for Solving CEPs

4.1. The parametric and individual LMRs. Obtaining the LP in the problem $(P_{p,r})$ can be approached in two ways. Firstly, we can use the perturbation method [3, item 3.3.2] and relying on the subdifferentiability properties of the value function β to arrive at the so-called parametric (i.e., depending on the parameters p, r) LP [17, 28]. Secondly, we can strive to obtain the LP "personally" in each individual problem $(P_{p,r})$, for example, in the problem $(P_{0,0})$ [3]. Let us introduce the Lagrange functional

$$L_{p,r}(z,\mu_0,\lambda,\mu) \equiv \mu_0 f(z) + \langle \lambda, Az - h - p\rangle + \langle \mu, g(z) - r\rangle,\ z \in \mathscr{D},\ \mu_0 \geq 0,\ \lambda \in H,\ \mu \in \mathbb{R}^m$$

and adopt the notation $\mathbb{R}^m_+ \equiv \{x = (x_1,\dots,x_m)^* \in \mathbb{R}^m : x_i \geq 0, i = 1,\dots,m\}$.

4.1.1. The parametric LMR. Let us formulate the parametric LP in nondifferential form for the problem $(P_{p,r})$. Its proof can be found in [17, Theorem 2.1], [28, Theorem 1.1]. The first part of the parametric LP is, in fact, a formulation of the classical Kuhn–Tucker theorem (see, for example, [3, 16]). However, instead of the concept of a Kuhn–Tucker vector, the equivalent concept of a subdifferential (taken with the opposite sign) of a value function is used here. Thus, the pair of dual variables $(\lambda/\mu_0, \mu/\mu_0)$, which is discussed in both statements of the first part of the theorem, is simultaneously the Kuhn–Tucker vector of the problem $(P_{p,r})$ and a solution of the dual to $(P_{p,r})$ problem. Before formulating the theorem, first of all, we note that below in the paper we use the notation $\partial^\infty h(z)$ for the singular (asymptotic) subdifferential of a convex lower semicontinuous function $h : \mathscr{H} \to \mathbb{R}^1 \cup \{+\infty\}$ at point z, $\mathscr{H}$ be a Hilbert space (see, for example, [30, Chapter 4]), defined by the formula $\partial^\infty h(z) \equiv \{\lambda \in \mathscr{H} : (\lambda, 0) \in N_{\operatorname{epi} h}(z, h(z))\}$, where $N_{\operatorname{epi} h}(z, h(z))$ is the cone of normals (in the sense of convex analysis) to $\operatorname{epi} h$ at point $(z, h(z))$.

Theorem 4.1. [***The parametric LP in nondifferential form***] *Let $(p,r) \in H \times \mathbb{R}^m$ be such point that $\beta(p,r) < +\infty$. Then the following two statements are true.*

1. If $z^0_{p,r} \in \mathscr{D}^0_{p,r} \equiv \{z \in \mathscr{D} : Az - h - p = 0,\ g_i(z) \leq r_i,\ i = 1,\dots,m\}$ is optimal element in the problem $(P_{p,r})$, i.e. $f(z^0_{p,r}) = \beta_0(p,r) = \beta(p,r)$, and $\zeta \in \partial\beta(p,r)$, where $\partial\beta(p,r)$ is a subdifferential (in the sense of convex analysis) of the value function, then for the Lagrange's *multipliers* $\lambda \in H,\ \mu \in \mathbb{R}^m_+$, $(\lambda,\mu) = -\zeta$, *for* $\mu_0 = 1$ *the following relations hold:*

$$L_{p,r}(z^0_{p,r},\mu_0,\lambda,\mu) \le L_{p,r}(z,\mu_0,\lambda,\mu)\ \forall z \in \mathscr{D},\ \ \mu_i(g_i(z^0_{p,r}) - r_i) = 0,\ i = 1,\dots,m. \qquad (4.1)$$

At the same time $-\zeta = (\lambda,\mu)$ *is the Kuhn–Tucker vector of the problem* $(P_{p,r})$.

And, conversely, if $\tilde z \in \mathscr{D}^0_{p,r}$ *is such an element that for some* $\mu_0 > 0$, $\lambda \in H$, $,\mu \in \mathbb{R}^m_+$ *the relations* (4.1) *are satisfied with* $z^0_{p,r}$ *replaced by* $\tilde z$, *then this element is optimal in the problem* $(P_{p,r})$, *that is,* $\tilde z = z^0_{p,r}$, *the pair* $(\lambda/\mu_0, \mu/\mu_0)$ *is the Kuhn–Tucker vector for it and at the same time* $(-\lambda/\mu_0, -\mu/\mu_0) \in \partial\beta(p,r)$.

2. *If* $z^0_{p,r} \in \mathscr{D}^0_{p,r}$ *is the optimal element in problem* $(P_{p,r})$, $(p,r) \in \partial\, dom\beta$ *and* $\zeta \in \partial^\infty\beta(p,r)$, $\zeta \neq 0$, *where* $\partial^\infty\beta(p,r)$ *is the singular subdifferential defined by the formula*

$$\partial^\infty\beta(p,r) \equiv \{(\lambda,\mu) \in H \times \mathbb{R}^m : ((\lambda,\mu),0) \in N_{epi\,\beta}((p,r),\beta(p,r))\},$$

then for the Lagrange multipliers $\lambda \in H,\ \mu \in \mathbb{R}^m_+$, $(\lambda,\mu) = -\zeta$, *the relations* (4.1) *are executed for* $\mu_0 = 0$.

And, conversely, if $\tilde z \in \mathscr{D}^0_{p,r}$ *is an element such that for* $\mu_0 = 0$ *and some* $\lambda \in H,\ \mu \in \mathbb{R}^m_+$, $(\lambda,\mu) \neq 0$, *the relations* (4.1) *are executed with the replacement* $z^0_{p,r}$ *on* $\tilde z$, *then* $(p,r) \in \partial\, dom\,\beta$ *and at the same time* $(-\lambda,-\mu) \in \partial^\infty\beta(p,r)$.

Remark 4.2. The LP formulated in Theorem 4.1 can be considered "obviously useless" for problems $(P_{p,r})$, for which simultaneously we have the equalities $\partial\beta(p,r) = \emptyset$, $\partial^\infty\beta(p,r) = \{0\}$. The latter is quite possible for problems with constraints specified by operators with infinite-dimensional images. From Theorem 4.1 it follows that the usual nondegenerate (regular or nonregular) LP in the problem $(P_{p,r})$ is satisfied only if at least one of the two relations $\partial\beta(p,r) \neq \emptyset$, $\partial^\infty\beta(p,r) \neq \{0\}$ is true. Below, in the problem (4.2) of the example 4.4, the impracticability of the LP will be proven. Taking into account the above, this means simultaneously the emptiness of the subdifferential and the equality to zero of the singular subdifferential of the value function of this problem for $p = 0$, if we formally introduce the parameter $p \in Z$ into the right side of the equation (4.2).

4.1.2. The individual LMR. Classical results related to the impracticability of the individual LP in optimization problems, including nonlinear ones more general than the convex problem $(P_{0,0})$, with an operator equality-constraint and with functional inequality-constraints can be found in [3, §3.2], see also [5].

4.2. On comparison of the parametric and individual LMRs. It is of interest to compare the parametric LP of Theorem 4.1 with the individual analogue in [3, §3.2] in the case when the general nonlinear problem in [3, §3.2] takes on the particular form of a convex programming problem. In general, this comparison problem is quite complex, so let's consider a reasonable simplification of the situation. Let us carry out this comparison for a problem that is classical in the theory of ill-posed problems. Thus consider the problem of finding a normal (minimal on norm) solution to the operator equation

$$(IP) \qquad Az = h,\ z \in \mathscr{D} \subseteq Z,$$

which can be formally written as the equivalent convex programming problem with operator equality constraint

$$(P) \qquad \|z\|^2 \to \inf,\ Az = h,\ z \in \mathscr{D} \subseteq Z.$$

4.2.1. The parametric LMR. The parametric LMR is inextricably linked with the perturbation method (see, for example, [3, item 3.3.2]). It establishes a strict connection between the feasibility of all its relations and the subdifferential properties of the value function of the CEP.

To formulate the parametric LMR, we include the problem (P) into the family of similar problems depending on the parameter $p \in H$ in the equality-constraint

$$(P_p) \qquad \|z\|^2 \to \inf,\ Az = h + p,\ \ z \in \mathscr{D} \subseteq Z.$$

The problem (P) is included in this family when $p = 0$, that is $(P) = (P_0)$. Solutions to the problem (P_p), if they exist, will be denoted by z^0_p.

The problem (P_p) is a special case of the problem $(P_{p,r})$ with $f(\cdot) = \|\cdot\|^2$, $g(\cdot) = 0$, $r = 0$, and the value function $\beta(p,r)$, $(p,r) \in H \times \mathbb{R}^m$ takes the form $\beta(p)$, $p \in H$. The parametric nondegenerate (regular or irregular) LMR in the problem $(P) = (P_0)$ in accordance with Theorem 4.1 can be formally written if and only if at least one of the two relations $\partial\beta(0) \neq \emptyset$, $\partial^\infty\beta(0) \neq \{0\}$ holds. However, unfortunately, checking the feasibility of the required subdifferential properties of the value function is a difficult independent mathematical problem.

4.2.2. The individual LMR. In this case, the main object of study is the usual or, as one might also say, individual (independent of parameters) the CEP (P). The main assumption of the individual LMR [3, 5] is associated with the requirement that the image of the operator defining equality is closed.

As applied to the problem (P) in the case $\mathscr{D} = Z$, the classical LMR for smooth problems with equalities from the book [3, item 3.2.2], see also [5, Corollary 1], under the condition that Z, H are Hilbert spaces, is formulated as follows.

Proposition 4.3. *If the point z^0 is a solution to the problem (P) and $R(A) = \overline{R(A)}$, then there is a nondegenerate set $(\mu_0, \lambda) \in \mathbb{R}^1_+ \times H$ such that $2\mu_0 z^0 + A^*\lambda = 0$. If in this case $R(A) = H$, then in the last equality we can assume $\mu_0 > 0$. Since the problem (P) is convex, the equality $2\mu_0 z^0 + A^*\lambda = 0$ is equivalent to the inequality*

$$L(z^0, \mu_0, \lambda) \leq L^0(z, \mu_0, \lambda)\ \ \forall z \in Z, \quad L(z, \mu_0, \lambda) \equiv \mu_0\|z\|^2 + \langle \lambda, Az - h \rangle.$$

Unfortunately, the main requirement $R(A) = \overline{R(A)}$ of this assertion is quite strict, and, as noted in [3, item 3.2.4], failure to fulfill this condition of closure can lead to the fact that the LMR is not true at all. The condition of closure condition cannot be satisfied, for example, for completely continuous operators A [31, p. 225, Theorem 1], "most often" encountered when considering various substantive ill-posed problems [14, 26, 27].

In this context, it is also of interest to compare two LMRs, namely, the parametric LMR of Theorem 4.1 as applied to the problem (P_p) and Proposition 4.3 formulated for the problem (P). To simplify the situation, we consider the problem (P) under the condition $\mathscr{D} = Z$.

First, note that the condition $R(A) = \overline{R(A)}$, $R(A) \neq H$ of Proposition 4.3 implies the presence of a nonzero element in the singular subdifferential $\partial^\infty\beta(0)$. Indeed, in this case $\operatorname{dom}\beta = R(A)$ and for any $x \in R(A)$ the normal cone $N_{R(A)}(x)$ coincides [32, Proposition 4.1.9] with an orthogonal subspace $R(A)^\perp$ containing a nonzero element. This circumstance, in accordance with the first statement of the second part of Theorem 4.1, ensures the feasibility of the nonregular $(\mu_0 = 0)$ LMR (4.1) for $\mathscr{D} = Z$.

Secondly, the condition $R(A) = \overline{R(A)} = H$, as can be seen, ensures that the subdifferential $\partial\beta(0)$ is nonempty. This is due to the fact that the functional β, in accordance with Lemma

3.2, is convex and finite at each point $x \in H$, i.e. $\operatorname{dom}\beta = H$, and therefore $\partial\beta(x) \neq \emptyset$ due to Corollary 2.3 and Theorem 4.2 in [29], which, in turn, due to the validity in given the situation of the first statement of the first part of Theorem 4.1, ensures the feasibility of the regular ($\mu_0 > 0$) LMR (4.1) for $\mathscr{D} = Z$.

Thus, the conditions $R(A) = \overline{R(A)}$, $R(A) \neq H$ and $R(A) = \overline{R(A)} = H$ of Proposition 4.3 (the classical LMR for smooth problems with equalities [3, item 3.2.2]) are also sufficient for the applicability of the corresponding necessary conditions for the extremum of Theorem 4.1. At the same time, it can be argued that there is such an extensive class of problems, for example of the form (P), for which the nondegenerate classical parametric LMR of Theorem 4.1 can be written (that is, in (4.1) set $(\mu_0, \lambda) \neq 0$), but at the same time the LMR of Proposition 4.3 (for smooth problems with equalities [3, item 3.2.2]) cannot be applied. In the case $\mathscr{D} = Z$, such problems include, for example, problems in which $R(A) \neq \overline{R(A)}$, but, at the same time, either $\partial^\infty\beta(0) \neq \{0\}$, or $\partial\beta(0) \neq \varnothing$, or both these relations are executed jointly. So, to summarize what was said above in this Section, already when writing out the formal LMR in both the "individual" and "parametric" versions in the problem of finding a normal solution to an operator equation of the first kind (IP), we encounter difficulties of a fundamental nature.

4.3. On problems of applying the classical LMR for practical solving CEPs. As noted above, the LMR owes its birth to the needs of solving practical problems. However, its direct application for the practical solving many current modern CEPs, arising, in particular, in natural science applications, faces problems of a fundamental nature. Let us highlight two, in our opinion, the most important, such problems associated with the *impracticability* and *instability* of the classical rule.

4.3.1. Impracticability of the LMR. In order to apply the LMR in a specific problem, you must first of all be able to write it down. Here we often encounter the problem of the impracticability of LMR. We speak on the impracticability of the LMR when it is known that this classical rule cannot be written down in a particular CEP. Close in meaning to the situation of the impracticability of the LMR is the one when we do not know how the LMR is written in a specific problem, but, at the same time, we cannot say that the classical rule is not true (for example, this happens when we obtain the multipliers rule for some assumptions and do not know how to do this when weakening these assumptions). Let us give an example of the impracticability associated with the "simplest" infinite-dimensional CEP with an operator (that is, specified by an operator with an infinite-dimensional image) constraint-equality. This is the problem (P) already considered above under the condition $\mathscr{D} = Z$, which under this condition coincides with the classical ill-posed problem of finding a nomal solution to the equation (IP). It is intended to demonstrate that it is precisely in the class of important (including from an applied point of view) problems with operator constraints that there are meaningful problems in which the LP is "impossible to write", and therefore its formal application in such situations is completely "pointless". An example of the impracticability of the classical rule is contained, for example, in the book [3, item 3.2.4], other informative and more complex examples can be found in [18].

Let us ask ourselves a formally natural question: is it possible to apply the constructions of the LMR or, in other words, the LP, to solve this ill-posed problem? The fundamental difficulties that arise along this path begin when trying to formally write down the LP in the CEP (P).

Example 4.4. Let us consider the problem of minimizing a strongly convex functional with the operator equality constraint

$$\|z\|^2 \to \min,\ Az = h,\ z \in Z \tag{4.2}$$

with an injective and self-adjoint operator $A : Z \to Z$, Z is a Hilbert space such that $R(A) \neq Z$ (for example, A can be the Fredholm integral operator with a closed symmetric kernel). In accordance with Theorem 3.1 in [33] in this case the equation $Az = h,\ z \in Z$ is densely solvable. Let us show that in the problem (4.2), for h chosen in a certain way, the LP is not satisfied, which, as is easy to see, results in the unsolvability of the corresponding dual problem. Let $z^0 \in Z$, but $z^0 \notin R(A)$. Then consider the problem (4.2) with $h = Az^0$. In such a problem, the classical LP in differential form [3, item 3.2.2], and, as a consequence, in nondifferential form, is not satisfied. Indeed, if this were not so, then there would be a nondegenerate pair of multipliers $(\mu_0, \lambda) \in \mathbb{R}^1_+ \times Z$ such that $2\mu_0 z^0 + A\lambda = 0$. In this case, for $\mu_0 = 0$ we obtain $\lambda = 0$ due to the injectivity of A, and for $\mu_0 = 1$, accordingly, the contradictory equality $z^0 = -1/2A\lambda$, which proves the impracticability of the classical LP in the problem (4.2) with a chosen h.

As a specific problem of constrained optimization of the form (4.2), we can take, for example, the problem of finding a normal solution to the Fredholm integral equation of the first kind (similar problems are among the classic ones in the theory of ill-posed problems [14])

$$\|z\|^2_{2,(0,1)} \to \min,\ \int_0^1 K(x,s)z(s)ds = h(x),\ 0 \le x \le 1,\ z \in L_2(0,1) \tag{4.3}$$

with symmetric kernel $K(x,s) = \{(1-x)s,\ 0 \le s \leqslant x;\ x(1-s),\ x \leqslant s \leqslant 1\}$ and with $Z = L_2(0,1)$. In this case, in accordance with Picard's theorem (see, for example, [34, p. 148]) the equation (4.3) due to the closedness of the kernel can only be uniquely solvable (see the analysis of example 1 on p. 149 in [34]). Therefore, in accordance with what was said above, we can state that for any $h(\cdot) = \int_0^1 K(\cdot,s)z(s)ds$ with such $z \in L_2(0,1)$, which are not continuous functions (the corresponding equivalence class does not contain a continuous function), the LP is not satisfied in the CEP (4.3).

So, when we consider the problem (4.2) such a situation arises. Firstly, since the relation $R(A) \neq \overline{R(A)}$ is satisfied, i.e. the image of the operator A is not closed, then it is impossible to use classical LP for smooth problems with equalities [3, item 3.2.2]. Secondly, if instead of this individual problem we take its "parametric analogue"

$$\|z\|^2 \to \min,\ Az = h + p,\ z \in Z,$$

then, on the one hand, in this case the classical LP for smooth problems with equalities for the same reason cannot be used for any p for which this problem is solvable. On the other hand, in the set of all such solvability points, those points p everywhere densely lie for which a nondegenerate parametric LP can be written in this problem. These are all those points at which the value function is subdifferentiable (in the sense of convex analysis), see Lemma 3.3. At the same time, we are not aware of any practically verifiable conditions, the fulfillment of which could guarantee the indicated subdifferentiability at an arbitrarily chosen point p.

4.3.2. Instability of the LMR. The instability of solutions to CEPs is well known, see various examples in [16]. In particular, we note that the problem of finding a normal (minimal in norm) solution to an operator equation of the first kind, considered above in Example 4.4, is a classical ill-posed problem. A detailed analysis of its instability under perturbation of the input data can be found, in fact, in any educational book on the theory of ill-posed problems (see, for example, [14, 16]). In fact, a consequence of this instability is the *instability* of the corresponding LMR, which we defined above as the *instability* of the extremals of the Lagrange's functionals with respect to the errors in the input data of CEPs. For various meaningful examples of the LMR instability in the CEPs with functional and operator constraints, see [17, 18, 19].

5. Regularization of the LMR in Nondifferential Form in the Convex CEP

All the circumstances noted in the previous Section explain the emergence of fundamental difficulties in the formal approach to solving CEPs in the way of approximating their exact solutions by extremals of the Lagrange's functionals of perturbed problems. However, the "internal potential" of the classical LMR allows us to organize its natural transformation-regularization, which, firstly, overcomes these difficulties and, secondly, preserves the "usual structure" of the classical result.

The main idea of the regularization of the classical rule for the problem $(P_{p,r})$ discussed below is to make the set of dual variables of the problem dependent on the magnitude of the perturbation of the input data so that, as this numerical value tends to zero, the extremals of the perturbed Lagrange's functionals taken sequentially form a generalized minimizing sequence (GMS) of the problem. Moreover, in the case of the problem $(P_{p,r})$ with a strongly convex objective functional, such GMS inevitably converges to an exact solution of the original (exact) problem. First of all, below we consider the regularization of the LMR in the case of a strongly convex objective functional.

5.1. Exact and perturbed CEPs, GMS-generating (regularizing) operator. Consider next the set F of all possible sets of input data $\mathsf{f} \equiv \{f, A, h, g\}$, $g(z) \equiv (g_1(z), \dots, g_m(z))^*$, each of which consists of a continuous strongly convex on $\mathscr{D}$ functional f with independent of elements of F strong convexity constant $\kappa > 0$, a linear bounded operator A, an element h and functionals g_i, $i = 1, \dots, m$, convex on $\mathscr{D}$, satisfy the condition $\mathscr{A}$. Moreover, the values of f, A, g are bounded on $\mathscr{D} \cap B_M$ uniformly on $\mathsf{f} \in \mathsf{F}$. Let us define sets of unperturbed (f^0) and perturbed (f^δ) input data, respectively: $\mathsf{f}^0 \equiv \{f^0, A^0, h^0, g^0\}$ and $\mathsf{f}^\delta \equiv \{f^\delta, A^\delta, h^\delta, g^\delta\}$, $\delta \in (0, \delta_0]$, $\delta_0 > 0$ is a some number. We will assume that

$$|f^\delta(z) - f^0(z)| \le C\delta(1 + \|z\|^2),\ \|A^\delta z - A^0 z\| \le C\delta(1 + \|z\|)\ \ \forall z \in \mathscr{D}, \tag{5.1}$$
$$\|h^\delta - h^0\| \le C\delta,\ |g^\delta(z) - g^0(z)| \le C\delta(1 + \|z\|^2)\ \ \forall z \in \mathscr{D},$$

where $C > 0$ does not depend on δ, $g^\delta \equiv (g_1^\delta, \dots, g_m^\delta)^*$.

The problem $(P_{p,r})$, the set $\mathscr{D}^\varepsilon_{p,r}$, corresponding to input data f^δ, $\delta \ge 0$, we will denote by $(P^\delta_{p,r})$, $\mathscr{D}^{\delta,\varepsilon}_{p,r}$, $\mathscr{D}^{0,\varepsilon}_{p,r} \equiv \mathscr{D}^\varepsilon_{p,r}$. The problem $(P^\delta_{p,r})$ for $\delta > 0$ will be called perturbed; for $\delta = 0$, accordingly, we will talk about the exact problem. For the generalized value of the exact problem $(P^0_{p,r})$, we will use the notation $\beta(p,r)$ introduced above.

Let us introduce the definition of regularizing algorithm [35] in the convex programming problem $(P^0_{p,r})$.

Definition 5.1. An operator $R_{p,r}(\cdot,\cdot,\cdot,\cdot,\delta)$ depending on $\delta \in (0, \delta_0)$, which assigns to each quadruple $(f^\delta, A^\delta, h^\delta, g^\delta)$ satisfying the estimates (5.1), element $R_{p,r}(f^\delta, A^\delta, h^\delta, g^\delta, \delta) \equiv z^\delta \in \mathscr{D}$ such that $f^0(z^\delta) \to \beta(p,r)$, $\|A^0 z^\delta - h^0 - p\| \to 0$, $\min_{x \in \mathbb{R}^m_-} |g^0(z^\delta) - x - r| \to 0$, $\delta \to 0$, is called regularizing in the problem $(P^0_{p,r})$.

Remark 5.2. Definitions of regularizing algorithms for mathematical programming problems with a finite number of functional constraints such as equality and inequality can be found, for example, in [16, Chapter 9]. These definitions are given in the case of problems of the first type (that is, problems in which only the lower bound is sought, see [16, Chapter 9, p. 802]) and of the second type (that is, problems in which we are looking for both the lower bound and the optimal element, see [16, Chapter 9, p. 837]). From a formal point of view, Definition 5.1 occupies an intermediate position between the two above definitions [16, Chapter 9]. Unlike the definition of [16, Chapter 9, p. 802] in Definition 5.1 we are talking not only about approaching the lower bound of the problem, but also, in parallel, about fulfilling its constraints "in the limit" with the simultaneous representation of "converging" as $\delta \to 0$ both "on function" and "on constraints" of elements $R_{p,r}(f^\delta, A^\delta, h^\delta, g^\delta, \delta) \equiv z^\delta \in \mathscr{D}$. At the same time, in contrast to the definition of [16, Chapter 9, p. 837] in Definition 5.1 there is no talk about any convergence (strong, weak) as $\delta \to 0$ of the elements of the family $R_{p,r}(f^\delta, A^\delta, h^\delta, g^\delta, \delta) \equiv z^\delta \in \mathscr{D}$ to any specific element, for example, to the exact solution of the problem $(P^0_{p,r})$ if the latter exists. Such convergence (strong, weak) is already a consequence of the fact that the elements $R_{p,r}(f^\delta, A^\delta, h^\delta, g^\delta, \delta) \equiv z^\delta \in \mathscr{D}$ for $\delta \to 0$ converge simultaneously both "on function" and "on constraints", as well as additional properties of the input data of the problem.

Since the main goal of this paper is to construct an GMS in the problem $(P^0_{p,r})$, and the family $\{z^\delta \in \mathscr{D} : \delta \in (0, \delta_0]\}$ from Definition 5.1 is not a sequence, then in addition to the definition of the regularizing operator in the problem $(P^0_{p,r})$ introduced above, we introduce its "trace", namely, the definition of GMS-generating (regularizing) operator [35] in the problem $(P^0_{p,r})$.

Definition 5.3. Let $\delta^k \in (0, \delta_0)$, $k = 1, 2, \dots$, be a sequence of positive numbers converging to zero. An operator $R_{p,r}(\cdot,\cdot,\cdot,\cdot,\delta^k)$ depending on δ^k, $k = 1, 2, \dots$, assigning to each set of input data $(f^{\delta^k}, A^{\delta^k}, h^{\delta^k}, g^{\delta^k})$ satisfying the estimates (5.1) for $\delta = \delta^k$, element $z^{\delta^k} \in \mathscr{D}$, is called GMS-generating in the problem $(P^0_{p,r})$ if the sequence z^{δ^k}, $k = 1, 2, \dots$, is a GMS in this problem.

5.2. Regularization of the LMR in the convex CEP. From Lemma 3.2 and Definition 5.3 it follows that the presence in the problem $(P^0_{p,r})$ of a GMS-generating operator in the sense of this definition provides, under the condition of subdifferentiability of f^0, the possibility of arbitrarily exact approximation of the solution $z^0_{p,r}$ to the problem $(P^0_{p,r})$ as its disturbance tends to zero. For this reason, any necessary or sufficient conditions for the existence of a GMS in the problem $(P^0_{p,r})$ are simultaneously, respectively, necessary and sufficient conditions for the existence of a solution in this problem and the possibility of its arbitrarily accurate approximation as the problem perturbation tends to zero. In the regularized LMR formulated below, we are talking about just such necessary and sufficient conditions for the existence of a GMS in the problem $(P^0_{p,r})$. To formulate it, we first introduce the regular Lagrange's functional

$$L^\delta_{p,r}(z, \lambda, \mu) \equiv f^\delta(z) + \langle \lambda, A^\delta z - h^\delta - p \rangle + \langle \mu, g^\delta(z) - r \rangle, \ z \in \mathscr{D}, \lambda \in H, \ \mu \in \mathbb{R}^m,$$

the dual functional

$$V^\delta_{p,r}(\lambda, \mu) \equiv \inf_{z \in \mathscr{D}} L^\delta_{p,r}(z, \lambda, \mu), \ \lambda \in H, \ \mu \in \mathbb{R}^m,$$

as well as the dual problem

$$V^\delta_{p,r}(\lambda, \mu) \to \sup, \ \lambda \in H, \ \mu \in \mathbb{R}^m_+.$$

Due to the strong convexity and continuity in z of the Lagrange functional for any pair $(\lambda,\mu)\in H\times\mathbb{R}^m_+$ the value of $V^\delta_{p,r}(\lambda,\mu)$ is achieved on a single element $z^\delta[\lambda,\mu]\equiv\text{argmin}\{L^\delta_{p,r}(z,\lambda,\mu),\ z\in\mathscr{D}\}$, and this element does not depend on h^δ.

5.2.1. Dual regularization in the CEP with a strongly convex objective functional. The central role in the formulation of the regularized LMR is played by the convergence theorem of the dual regularization procedure in the problem $(P^0_{p,r})$. Let the problem $(P^0_{p,r})$ be solvable, but the solvability of the problem dual to $(P^0_{p,r})$ is not assumed. Let us denote by $(\lambda^{\delta,\alpha}_{p,r},\mu^{\delta,\alpha}_{p,r})$ unique in $H\times\mathbb{R}^m_+$ a point giving a maximum on this set for $\alpha>0$ to the strongly concave functional

$$R^{\delta,\alpha}_{p,r}(\lambda,\mu)\equiv V^\delta_{p,r}(\lambda,\mu)-\alpha\|\lambda\|^2-\alpha|\mu|^2,\ (\lambda,\mu)\in H\times\mathbb{R}^m_+.$$

Let the matching condition be satisfied

$$\frac{\delta}{\alpha(\delta)}\to 0,\ \alpha(\delta)\to 0,\ \delta\to 0. \tag{5.2}$$

Theorem 5.4. [***Regularizing dual algorithm***] *Let the problem $(P^0_{p,r})$ be solvable and $z^0_{p,r}$ be its solution. Regardless of whether the problem dual to $(P^0_{p,r})$ is solvable or not, if the matching condition (5.2) is satisfied, the following two statements hold.*

1. Convergence in the main variable. *The following relations are satisfied*

$$\alpha(\delta)\|(\lambda^{\delta,\alpha(\delta)}_{p,r},\mu^{\delta,\alpha(\delta)}_{p,r})\|^2\to 0,\ f^0(z^\delta[\lambda^{\delta,\alpha(\delta)}_{p,r},\mu^{\delta,\alpha(\delta)}_{p,r}]) \tag{5.3}$$

$$\le f^0(z^0_{p,r})+\theta_1(\delta),\ \theta_1(\delta)\to 0,\ \delta\to 0,$$

$$\|A^0z^\delta[\lambda^{\delta,\alpha(\delta)}_{p,r},\mu^{\delta,\alpha(\delta)}_{p,r}]-h^0-p\|\le\theta_2(\delta)\to 0,\ \delta\to 0, \tag{5.4}$$

$$g^0_i(z^\delta[\lambda^{\delta,\alpha(\delta)}_{p,r},\mu^{\delta,\alpha(\delta)}_{p,r}])\le r_i+\theta_3(\delta),\ \ \theta_3(\delta)\to 0,\ \delta\to 0, \tag{5.5}$$

$$\langle(\lambda^{\delta,\alpha(\delta)}_{p,r},\mu^{\delta,\alpha(\delta)}_{p,r}),(A^\delta z^\delta[\lambda^{\delta,\alpha(\delta)}_{p,r},\mu^{\delta,\alpha(\delta)}_{p,r}]-h^\delta,g^\delta(z^\delta[\lambda^{\delta,\alpha(\delta)}_{p,r},\mu^{\delta,\alpha(\delta)}_{p,r}]))\rangle\to 0,\ \delta\to 0,$$

where $\theta_j(\cdot)$, $j=1,2,3$, some nonnegative functions of a positive argument.

If the strongly convex functional f^0 is also subdifferentiable (in the sense of convex analysis) at points of $\mathscr{D}$, then we have also the limit relation

$$\|z^\delta[\lambda^{\delta,\alpha(\delta)}_{p,r},\mu^{\delta,\alpha(\delta)}_{p,r}]-z^0_{p,r}\|\to 0,\ \delta\to 0. \tag{5.6}$$

In other words, regardless of whether the dual problem is solvable or not, the algorithm given by the equality $R_{p,r}(f^\delta,A^\delta,h^\delta,g^\delta,\delta)=z^\delta[\lambda^{\delta,\alpha(\delta)}_{p,r},\mu^{\delta,\alpha(\delta)}_{p,r}]$, is regularizing in the sense of Definition 5.1, and in the case of subdifferentiability of f^0 at points of $\mathscr{D}$ there is also strong convergence (5.6). If there is no such subdifferentiability, then we can only talk about weak convergence $z^\delta[\lambda^{\delta,\alpha(\delta)}_{p,r},\mu^{\delta,\alpha(\delta)}_{p,r}]$ to $z^0_{p,r}$ for $\delta\to 0$.

2. Convergence in the dual variable. *If the problem dual to $(P^0_{p,r})$ is solvable, the limit relation holds*

$$(\lambda^{\delta,\alpha(\delta)}_{p,r},\mu^{\delta,\alpha(\delta)}_{p,r})\to(\lambda^0_{p,r},\mu^0_{p,r}),\ \ \delta\to 0, \tag{5.7}$$

where $(\lambda^0_{p,r},\mu^0_{p,r})\in H\times\mathbb{R}^m_+$ is the minimal in norm element among all elements that deliver the maximum to the functional $V^0_{p,r}$ on $H\times\mathbb{R}^m_+$.

For the proof of this Theorem, see [17, Theorem 3.1], [36, Sect. 2].

Remark 5.5. In the formulation of Theorem 3.1 in [17] the terms obtained in its proof were "lost": the quantity $\psi(\delta)$ (see inequalities (3.23), (3.26) in [17]) was "lost" in the expression for $\psi_1(\delta)$ in the formula corresponding to (5.3); the value $C\delta(2+L)$ was "lost" on the right side of the inequality corresponding to (5.4); the value $C\delta(1+L^2)$ was "lost" on the right side of the inequality corresponding to (5.5). Taking into account the notation of the first statement of Theorem 5.4 and Theorem 3.1 in [17] we can write the equalities $\theta_1(\delta) = \psi_1(\delta) = \psi(\delta) + C\delta(1+L^2)$, $\theta_2(\delta) = \phi(\delta, \alpha(\delta)) + C\delta(1+L)$, $\theta_3(\delta) = \phi(\delta, \alpha(\delta)) + C\delta(1+L^2)$. All these "lost" quantities are also restored in Theorem 1 given in [35], which is an analogue of Theorem 3.1 in [17], as well as the first statement of the theorem formulated above.

Remark 5.6. In the dual regularization method, the problem

$$V^{\delta}_{p,r}(\lambda,\mu) - \alpha(\delta)\|(\lambda,\mu) - (\widetilde{\lambda},\widetilde{\mu})\|^2 \to \sup, \ (\lambda,\mu) \in H \times \mathbb{R}^m_+,$$

where $(\widetilde{\lambda},\widetilde{\mu}) \in H \times \mathbb{R}^m_+$ is an arbitrary fixed element, can be taken as a regularized perturbed dual problem. Then, in accordance with the classical theory of Tikhonov stabilization (see, for example, [16, Chapter 9, §4, Theorem 2]), in this case as a limit point $(\lambda^0_{p,r}, \mu^0_{p,r})$ of the limit relation (5.7) is the element that provides the minimum value to the functional $\|(\lambda,\mu) - (\widetilde{\lambda},\widetilde{\mu})\|^2$, $(\lambda,\mu) \in H \times \mathbb{R}^m_+$ among all solutions of the problem dual to $(P^0_{p,r})$. In this case, all the statements made above related to the dual regularization procedure remain valid.

Remark 5.7. Let us finally again point out the essential features of the regularization procedure for the dual problem [17, 25] when proving the convergence of the entire dual regularization procedure to the solution of the original problem in comparison with the standard Tikhonov stabilization procedure (see [16, Chapter 9], [22]). First of all, note that, in both cases, we are talking about the stabilization of identical (up to the sign of the objective functional) convex problems of general form (without equality and inequality constraints). First, a solution to the dual problem may not exist (the problem may not have a Kuhn–Tucker vector). Second, if such a solution exists, there may be no necessary (for the standard Tikhonov stabilization, see [16, Chapter 9], [22]) estimate for the deviation of the perturbed objective function of the dual problem from the exact one. The latter situation is realized in the problem dual to the problem $(P^0_{p,r})$ in the case of an unbounded set $\mathcal{D}$.

5.2.2. Regularized LMR in convex CEP with a strongly convex objective functional. Now, after formulating the convergence theorem of the dual regularization method, we can formulate and discuss the regularized LMR for the problem $(P^0_{p,r})$. It will have the form of necessary and sufficient conditions for the existence of a bounded GMS in this problem. These conditions can also be simultaneously treated as necessary and sufficient conditions for ordinary optimality, but expressed in sequential form.

Theorem 5.8. [***The regularized LMR***] *Let $\delta^k \in \mathbb{R}^1$, $k = 1,2,\ldots$, is an arbitrary sequence of nonnegative numbers converging to zero. For the existence of a bounded GMS in the problem $(P^0_{p,r})$ it is necessary and sufficient for the existence of such a sequence $(\lambda^k, \mu^k) \in H \times \mathbb{R}^m_+$, $k = 1,2,\ldots$, for which the relations hold*

$$\delta^k\|(\lambda^k,\mu^k)\|^2 \to 0, \ z^{\delta^k}[\lambda^k,\mu^k] \in \mathcal{D}^{\delta^k,\varepsilon^k}_{p,r}, \ \varepsilon^k \to 0, \tag{5.8}$$

$$\langle(\lambda^k,\mu^k), A^{\delta^k} z^{\delta^k}[\lambda^k,\mu^k] - h^{\delta^k} - p, g^{\delta^k}(z^{\delta^k}[\lambda^k,\mu^k]) - r\rangle \to 0, \ k \to \infty$$

and the sequence $z^{\delta^k}[\lambda^k,\mu^k]$, $k=1,2,\dots$ is bounded. This sequence $z^{\delta^k}[\lambda^k,\mu^k]$, $k=1,2,\dots$, is the desired GMS in the problem $(P^0_{p,r})$. In the case of subdifferentiability of f^0 on $\mathcal{D}$, regardless of whether the problem dual to $(P^0_{p,r})$ is solvable or not, $z^{\delta^k}[\lambda^k,\mu^k]\to z^0_{p,r}$, $k\to\infty$. In other words, the operator $R_{p,r}(\cdot,\cdot,\cdot,\cdot,\delta^k)$, which is given by the equality $R_{p,r}(f^{\delta^k},A^{\delta^k},h^{\delta^k},g^{\delta^k},\delta^k)=z^{\delta^k}[\lambda^k,\mu^k]$, is a GMS-generating (regularizing) in the problem $(P^0_{p,r})$ in the sense of Definition 5.3 and generates a GMS converging to $z^0_{p,r}$ for $k\to\infty$ in the case of subdifferentiability of f^0 on $\mathcal{D}$. In addition, we have the limit relation

$$V^0_{p,r}(\lambda^k,\mu^k)\to\sup_{(\lambda,\mu)\in H\times\mathbb{R}^m_+}V^0_{p,r}(\lambda,\mu)=f^0(z^0_{p,r}),\ k\to\infty.$$

If the problem $(P^0_{p,r})$ is solvable, as a sequence $(\lambda^k,\mu^k)\in H\times\mathbb{R}^m_+$, $k=1,2,\dots$, we can take the sequence $(\lambda^{\delta^k,\alpha(\delta^k)}_{p,r},\mu^{\delta^k,\alpha(\delta^k)}_{p,r})$, $k=1,2,\dots$, $\delta^k/\alpha(\delta^k)\to 0$, $k\to\infty$, generated by the dual regularization algorithm of the first statement of Theorem 5.4 (Theorem 3.1 in [17]). In this case $(\lambda^{\delta,\alpha(\delta)}_{p,r},\mu^{\delta,\alpha(\delta)}_{p,r})\equiv\operatorname{argmax}\{V^\delta_{p,r}(\lambda,\mu)-\alpha(\delta)\|(\lambda,\mu)\|^2,\ (\lambda,\mu)\in H\times\mathbb{R}^m_+\}$ and the matching condition (5.2) is satisfied. Besides, in the case of the existence of a bounded GMS and the solvability of the problem dual to $(P^0_{p,r})$, the limit relation $(\lambda^k,\mu^k)\to(\lambda^0_{p,r},\mu^0_{p,r})$, $k\to\infty$ is true, where $(\lambda^0_{p,r},\mu^0_{p,r})\in H\times\mathbb{R}^m_+$ is a normal (minimal in norm) solution to the dual problem.

For the proof of this theorem, see [17, Theorems 4.1, 4.2] and [36, Theorem 1].

Remark 5.9. In accordance with Remark 5.6 instead of the functional

$$R^{\delta,\alpha}_{p,r}(\lambda,\mu)\equiv V^\delta_{p,r}(\lambda,\mu)-\alpha\|\lambda\|^2-\alpha|\mu|^2,\ (\lambda,\mu)\in H\times\mathbb{R}^m_+$$

in the dual regularization algorithm one can use the functional

$$R^{\delta,\alpha}_{p,r}(\lambda,\mu)\equiv V^\delta_{p,r}(\lambda,\mu)-\alpha\|\lambda-\tilde\lambda\|^2-\alpha|\mu-\tilde\mu|^2,\ (\lambda,\mu)\in H\times\mathbb{R}^m_+,$$

where $(\tilde\lambda,\tilde\mu)\in H\times\mathbb{R}^m_+$ is an arbitrary fixed point. It can be noted that in this case, all statements of Theorem 5.8, except the last two, will remain in force. At the same time, these last two statements must be reformulated as follows.

If the problem $(P^0_{p,r})$ is solvable, as a sequence $(\lambda^k,\mu^k)\in H\times\mathbb{R}^m_+$, $k=1,2,\dots$, we can take the sequence $(\lambda^{\delta^k,\alpha(\delta^k)}_{p,r},\mu^{\delta^k,\alpha(\delta^k)}_{p,r})$, $k=1,2,\dots$, $\delta^k/\alpha(\delta^k)\to 0$, $k\to\infty$, generated by the dual regularization algorithm of Theorem 3.1 in [17] taking into account Remarks 5.6, 5.9. In this case $(\lambda^{\delta,\alpha(\delta)}_{p,r},\mu^{\delta,\alpha(\delta)}_{p,r})\equiv\operatorname{argmax}\{V^\delta_{p,r}(\lambda,\mu)-\alpha(\delta)\|(\lambda,\mu)-(\widetilde\lambda,\widetilde\mu)\|^2,\ (\lambda,\mu)\in H\times\mathbb{R}^m_+\}$, $\delta/\alpha(\delta)\to 0$, $\alpha(\delta)\to 0$, $\delta\to 0$. Then, in the case of the existence of a bounded GMS and the solvability of the problem dual to $(P^0_{p,r})$, the limit relation holds $(\lambda^k,\mu^k)\to(\lambda^0_{p,r},\mu^0_{p,r})$, $k\to\infty$, in which $(\lambda^0_{p,r},\mu^0_{p,r})$ is an element that minimizes the functional $\|(\lambda,\mu)-(\tilde\lambda,\tilde\mu)\|^2$, $(\lambda,\mu)\in H\times\mathbb{R}^m_+$ on the set of all solutions to the dual problem.

So, in conclusion of this Remark, it can be argued that due to the arbitrariness in the choice of $(\widetilde\lambda,\widetilde\mu)\in H\times\mathbb{R}^m_+$, in the case of the existence of a bounded GMS and the solvability of the dual to $(P^0_{p,r})$ problem, without loss of generality, within the framework of Theorem 5.8, we can assume that the limit relation $(\lambda^k,\mu^k)\to(\lambda^0_{p,r},\mu^0_{p,r})$, $k\to\infty$, holds as $k\to\infty$, in which any pre-selected and fixed solution to the dual problem can be taken (for example, normal, i.e., minimal in norm in the case of $(\tilde\lambda,\tilde\mu)=0$) as $(\lambda^0_{p,r},\mu^0_{p,r})\in H\times\mathbb{R}^m_+$. Moreover, due to the coincidence of the set of all solutions of the dual to $(P^0_{p,r})$ problem or, in other words, the set of all its Kuhn–Tucker vectors, with its subdifferential $\partial\beta(p,r)$ taken with the opposite sign (i.e. with $-\partial\beta(p,r)$, see, for example, [28]), as $(\lambda^0_{p,r},\mu^0_{p,r}) \in H\times\mathbb{R}^m_+$ any pre-selected and fixed element from $-\partial\beta(p,r)$ can be taken.

5.2.3. Regularization of the LMR in nondifferential form in the case of a not strongly convex objective functional. To conclude this Section, we will briefly discuss how the LMR can be regularized in nondifferential form in the case of a not strongly convex objective functional. The regularized LMR generates the GMS in the problem $(P^0_{p,r})$ from the extremals of the Lagrange functionals $L^{\delta^k}_{p,r}(\cdot,\lambda^k,\mu^k)$, $k=1,2,\ldots$. Due to the strong convexity on $\mathscr{D}$ for each $k=1,2,\ldots$ of these functionals, the problem of their minimizing is a classical well-posed problem (in terms of the theory of ill-posed optimization problems [16]). Firstly, it has a unique solution and, secondly, each minimizing sequence in this minimization problem strongly converges to this solution. In the absence of strong convexity of the objective functional, we cannot guarantee the correctness of the problem of minimizing Lagrange's functionals $L^{\delta^k}_{p,r}(\cdot,\lambda^k,\mu^k)$, $k=1,2,\ldots$, and for this reason, formally following the dual regularization procedure in this case gives rise to its pronounced “nonconstructiveness”, as well as a similar “nonconstructiveness” of the corresponding regularized LMR. Regularization of the LMR in problems of convex programming and optimal control with not strongly convex objective functionals was considered by the author in the papers [37, 38]. In continuation and development of the papers [37, 38], in [35, 39] another approach to the regularization of the LMR in problems of convex programming and optimal control with not strongly convex objective functionals was proposed. In [35, 39], in parallel with the regularization of the dual problem at the same time, regularization of the corresponding Lagrange functional is carried out. In this case, the problem $(P^\delta_{p,r})$ is formally replaced by a family of the CEPs depending on the regularization parameter $\varepsilon>0$

$$(P^{\varepsilon,\delta}_{p,r}) \qquad f^\delta(z)+\varepsilon\|z\|^2\to\min,\ A^\delta z=h^\delta+p,\ g^\delta_i(z)\le r_i,\ i=1,\ldots,m,\ \ z\in\mathscr{D}\subseteq Z.$$

In other words, the problem $(P^\delta_{p,r})$ is replaced by a problem with a Tikhonov-regularized convex functional f^δ. Therefore, from a formal point of view, for each fixed $\varepsilon>0$ we are in a situation in which the same approach can be applied to the family $(P^{\varepsilon,\delta}_{p,r})$ as was applied to the family $(P^\delta_{p,r})$ with a strongly convex objective functional f^δ, as a result of which we obtained Theorems 5.4, 5.8. The papers [35, 39] show how fulfillment simultaneously with the matching condition (5.2) and a similar condition of consistent simultaneous tending to zero of the parameters ε and δ leads to the corresponding variants of the dual regularization method and to the regularized LMR in relation to the problem $(P^0_{p,r})$, the objective functional of which may not be strongly convex. In this case, the corresponding GMSs in the problem $(P^0_{p,r})$ generated by these variants are formed from the extremals of the Lagrange functionals of the problems $(P^{\varepsilon,\delta}_{p,r})$ that are strongly convex in z for $\varepsilon>0$ each of which in this case is the only solution to the corresponding well-posed minimization problem.

6. The Classical LMR as a Limiting Version of the Regularized Analogue

This Section is devoted to obtaining the classical LMR in the nondifferential form based on passage to the limit in the relations of the regularized LMR. We will pass to the limit in the relations of Theorem 5.8, while considering, to simplify the presentation, the functional f^0 to be subdifferentiable (in the sense of convex analysis) at points of the set $\mathscr{D}$.

Let us first consider the case when $\beta(p,r) < +\infty$ and the subdifferential $\partial\beta(p,r)$ is not empty (see Lemma 3.3), i.e. when the problem dual to $(P^0_{p,r})$ is solvable. In this situation, we use Theorem 5.8 taking into account Remark 5.9 and fix a sequence $(\lambda^k,\mu^k) \in H\times\mathbb{R}^m_+$, $k = 1,2,\dots$ referred in this remark. It converges as $k\to\infty$ to the element $(\lambda^0_{p,r},\mu^0_{p,r}) \in (-\partial\beta(p,r))$. Then, due to the simultaneous convergence $z^{\delta^k}[\lambda^k,\mu^k] \to z^0_{p,r},\ k\to\infty$, the passage to the limit in the third condition (limit relation) in (5.8) gives

$$\langle(\lambda^0_{p,r},\mu^0_{p,r}),(A^0z^0_{p,r}-h^0-p,g^0(z^0_{p,r})-r)\rangle = \langle\mu^0_{p,r},g^0(z^0_{p,r})-r\rangle = 0,$$

whence we clearly derive the usual relations of complementary slackness

$$\mu^0_{p,r,i}(g^0_i(z^0_{p,r})-r_i) = 0,\ \ \mu^0_{p,r,i}\ge 0,\ \ i = 1\dots,m.$$

Moreover, $(\lambda^0_{p,r},\mu^0_{p,r})$ is an element that minimizes the functional $\|(\lambda,\mu)-(\tilde\lambda,\tilde\mu)\|^2$, $(\lambda,\mu)\in H\times\mathbb{R}^m_+$ on the set of all solutions to the problem dual to $(P^0_{p,r})$ for some $(\widetilde{\lambda},\widetilde{\mu})\in H\times\mathbb{R}^m_+$. At the same time, since the element $z^{\delta^k}[\lambda^k,\mu^k]$ minimizes the functional $L^{\delta^k}_{p,r}(z,\lambda^k,\mu^k)$, $z\in\mathscr{D}$, then, obviously, the limit element has the same property, i.e.

$$L^0_{p,r}(z^0_{p,r},\lambda^0_{p,r},\mu^0_{p,r}) \le L^0_{p,r}(z,\lambda^0_{p,r},\mu^0_{p,r})\ \forall z\in\mathscr{D}. \tag{6.1}$$

Thus, in the situation under consideration, we have obtained all the relations of the classical LP in the nondifferential form of Theorem 4.1. Naturally, if the input data of the problem $(P^0_{p,r})$, i.e. the functionals f^0, g^0_i, $i = 1,\dots,m$ are Frechét differentiable at points of the set $\mathscr{D}$, then from these relations all relations of the classical LP can be obtained in differential form.

Let us further assume that $\beta(p,r) < +\infty$, $\partial\beta(p,r) = \emptyset$, and the singular (asymptotic) subdifferential $\partial^\infty\beta(p,r)$ contains a nonzero element. In this case, we use the well-known representation for the singular subdifferential of a convex lower semicontinuous functional (see, for example, [30, p. 82])

$$\partial^\infty\beta(p,r) = w - \limsup_{(p',r')\xrightarrow{\beta}(p,r),t\downarrow 0} t\partial\beta(p',r')$$

$$\equiv \{w-\lim_{k\to\infty} t_k\zeta_k : t_k\downarrow 0, \zeta_k\in\partial\beta(p^k,r^k),(p^k,r^k)\xrightarrow{\beta}(p,r)\},$$

where the symbol $(p',r')\xrightarrow{\beta}(p,r)$ means that $((p',r'),\beta(p',r'))\to((p,r),\beta(p,r))$, symbol $t\downarrow 0$ means convergence to zero from the right, and the symbol $w-\lim\limits_{k\to\infty} t_k\zeta_k$ means the weak convergence of the elements $t_k\zeta_k$ as $k\to\infty$.

Let us rewrite the inequality (6.1) in the form

$$L^0_{p,r}(z^0_{p,r},1,\lambda^0_{p,r},\mu^0_{p,r}) \le L^0_{p,r}(z,1,\lambda^0_{p,r},\mu^0_{p,r})\ \forall z\in\mathscr{D}, \tag{6.2}$$

where for $\nu\ge 0$ the notation $L^0_{p,r}(z,\nu,\lambda,\mu)\equiv \nu f^0(z)+\langle(\lambda,\mu),(A^0z-h^0-p,g^0(z)-r)\rangle$ is accepted. Let's multiply the inequality (6.2) by $s>0$

$$L^0_{p,r}(z^0_{p,r},s,s\lambda^0_{p,r},s\mu^0_{p,r}) \le L^0_{p,r}(z,s,s\lambda^0_{p,r},s\mu^0_{p,r}). \tag{6.3}$$

We will proceed as follows. For any weak limit point of the form

$$(\tilde{\lambda}_{p,r}, \tilde{\mu}_{p,r}) = w - \lim_{k\to\infty,\,(p^k,r^k)\xrightarrow{\beta}(p,r),\,s_k\downarrow 0} s_k(\lambda^k_{p^k,r^k}, \mu^k_{p^k,r^k})$$

with $(\lambda^k_{p^k,r^k}, \mu^k_{p^k,r^k}) \in -\partial\beta(p^k,r^k)$ we can write as a result of the obvious passage to the limit in (6.3) as $k\to\infty$, after substituting (p^k,r^k), $(\lambda^k_{p^k,r^k}, \mu^k_{p^k,r^k})$ and s_k into this inequality instead of (p,r), $(\lambda^0_{p,r}, \mu^0_{p,r})$ and s respectively

$$L^0_{p,r}(z^0_{p,r}, 0, \tilde{\lambda}_{p,r}, \tilde{\mu}_{p,r}) \le L^0_{p,r}(z, 0, \tilde{\lambda}_{p,r}, \tilde{\mu}_{p,r}). \tag{6.4}$$

During this passage to the limit, it was taken into account that $z^0_{p^k,r^k} \to z^0_{p,r}$, $k\to\infty$, which is a consequence of weak convergence of $z^0_{p^k,r^k}$ to $z^0_{p,r}$, numerical convergence of $f^0(z^0_{p^k,r^k})$ to $f^0(z^0_{p,r})$ for $k\to\infty$, subdifferentiability at points $\mathscr{D}$ and strong convexity of f^0.

In addition, during this passage to the limit, we took into account the fact that, by virtue of Remark 5.9, we can take any element from $(-\partial\beta(p^k,r^k))$ as the element $(\lambda^k_{p^k,r^k}, \mu^k_{p^k,r^k})$. At the same time, due to the conditions of complementary slackness $\mu_{p^k,r^k,i}(g^0_i(z^0_{p^k,r^k}) - r^k_i) = 0,\ i = 1,2,\ldots,m$, as a result of passage to the limit as $k\to\infty$ and the limit relations $z^0_{p^k,r^k} \to z^0_{p,r},\ k\to\infty$ we obtain $\tilde{\mu}_{p,r,i}(g^0_i(z^0_{p,r}) - r_i) = 0,\ i = 1,2,\ldots,m$, which together with (6.4) means the satisfiability of an irregular nondegenerate LP. At the same time, we approximated the solution $z^0_{p,r}$ of the problem $(P^0_{p,r})$ by points $z^0_{p^k,r^k}$ that provide the minimum value to the Lagrange's functionals $L^0_{p^k,r^k}(z, \lambda^k_{p^k,r^k}, \mu^k_{p^k,r^k})$, $z\in\mathscr{D}$.

And finally, if the last of the three possible situations here is realized, when $\beta(p,r) < +\infty$, $\partial\beta(p,r) = \emptyset$ and at the same time $\partial^\infty\beta(p,r) = \{0\}$, then this means that the classical LP of Theorem 4.1 in the problem $(P^0_{p,r})$ with thus chosen (p,r) is not executed.

7. Regularization of the LMR and the Classical Ill-posed Problem of Solving an Operator Equation of the First Kind

As noted above in Introduction and Section 2, CEPs as a whole constitute a typical class of ill-posed problems. In this Section we will consider what the regularization of the LMR gives as applied to the classical ill-posed problem (IP) of finding a normal solution to an operator equation of the first kind on a pair of Hilbert spaces. This problem was discussed above in Section 4. More precisely, below we will discuss the application of the above regularization in the case of two the CEPs , equivalent to the original classical ill-posed problem (IP).

So, let's introduce the "exact" and "approximate" problems (IP)

$$(IP^\delta) \qquad A^\delta z = h^\delta,\ z\in\mathscr{D}\subseteq Z,$$

with given linear bounded operator $A^\delta : Z\to H$ and element $h^\delta\in H$ such that $\|A^\delta - A^0\|\le\delta$, $\|h^\delta - h^0\|\le\delta$, $A\equiv A^0$, $h = h^0$. Here $\delta\in(0,\delta_0],\ \delta_0 > 0$, is a numerical parameter characterizing the degree of deviation of the input data of the approximate problem (IP^δ) from the input data of the exact problem $(IP) = (IP^0)$. Let us finally assume that the problem $(IP) = (IP^0)$ has an exact normal solution $z^0\in\mathscr{D}$.

From a formal point of view, the normal solution to the problem (IP) can be sought by solving two the CEPs. One of them is the problem of nonconvex minimization with one functional equality constraint $\|z\|^2 \to \min,\ \|Az-h\|^2 = 0,\ z \in \mathscr{D}$ or an equivalent quadratic convex optimization problem with one functional inequality-constraint

$$(CE1) \qquad \|z\|^2 \to \min,\ \|Az-h\|^2 \le 0,\ z \in \mathscr{D}.$$

The other problem is a minimization one with an operator (i.e., specified by an operator with an infinite-dimensional image in the case of infinite-dimensional H) equality-constraint

$$(CE2) \qquad \|z\|^2 \to \min,\ Az = h,\ z \in \mathscr{D}.$$

All these three constrained optimization problems are equivalent from the point of view that their solutions (the only ones) either do not exist at the same time, or exist and coincide with each other.

Next we introduce the corresponding families of problems depending on $\delta \ge 0$

$$(CE1^\delta) \qquad \|z\|^2 \to \min,\ \|A^\delta z - h^\delta\|^2 \le 0,\ z \in \mathscr{D},$$

$$(CE2^\delta) \qquad \|z\|^2 \to \min,\ A^\delta z = h^\delta,\ z \in \mathscr{D}.$$

The problem $(CE1^\delta)$ coinsides with the problem $(P^\delta) = (P^\delta_{0,0})$ of Section 5 for $f^\delta(\cdot) = \|\cdot\|^2$, $A^\delta = 0$, $h^\delta = 0$, $g^\delta(z) = g^\delta_1(z) = \|A^\delta z - h^\delta\|^2$, $m = 1$. In turn, the problem $(CE2^\delta)$ coinsides with the problem $(P^\delta) = (P^\delta_{0,0})$ of Section 5 for $f^\delta(\cdot) = \|\cdot\|^2$, $g^\delta_i(z) = 0$, $i = 1,\dots,m$.

In each of the two problems $(CE1^0)$ and $(CE2^0)$, Theorems 5.4, 5.8 can be applied and corresponding variants of the LMRs can be obtained, which are simultaneously corresponding regularizing algorithms in the sense of Definitions 5.1, 5.3. The GMSs constructed in accordance with them, consisting of extremals of the corresponding Lagrange functionals, strongly converge to the solution z^0 of the original problem (IP^0). In accordance with the above, it is natural to treat these regularized LMRs in the same way as the usual optimality conditions, but expressed in a (regularized) sequential form. Let us recall, finally, that the formal writing of the classical LP for the problem $(CE2^0)$, as shown in Section 4, is fraught with difficulties of a fundamental nature. It can also be argued that we will not receive any “significant information” about the solution z^0 of the problem $(CE1^0)$ when trying to formally write down the classical LMR for it. This is due to the peculiarity of this problem with “one-dimensional inequality”, which consists in the impossibility of establishing any “practically verifiable” properties of its regularity. And finally, we note that both problems have the “same” property of instability of the solution z^0 as the “generating” problem (IP^0). The possibilities of solving the problem (IP) based on the regularization of the LMR for the problem $(CE2)$ were discussed in detail in [40].

In conclusion, let us point out the important relationship between the problem $(CE1)$ and the Tikhonov regularization method, which, in turn, indicates the inextricable connection of this method, classical for the theory of ill-posed problems, with Lagrange multipliers and with duality. This relationship was the subject of a fairly detailed discussion in the paper [41]. In this paper, in particular, it was shown that the apparatus associated with the theory of duality and the LMR, consistently applied to the problem $(CE1)$, allows us to look at the classical regularization method from a slightly different angle, compared to the traditional approach [14, 26, 27]. In more precise terms, the Tikhonov regularization method itself and its classical version as the generalized residual principle [26, 27] are nothing more than different versions of solving a problem that is dual to the “simplest” quadratic optimization problem $(CE1)$. In this case, just as this happens within the framework of the dual regularization method of Theorem 5.4, the approximation of the exact solution of the problem (IP) or, what is the same thing, the exact solution of the exact problem $(CE1)$, occurs through extremals of the regular Lagrange's functional for the problem $(CE1)$, but in contrast to Theorem 5.4 without regularization of the problem dual to $(CE1)$ itself. Or, to put it somewhat differently, the paper [41] shows that Tikhonov's regularization method can be naturally interpreted as a method of stable approximation of the exact solution by extremals of the Lagrange functional for the problem $(CE1)$ with the simultaneous construction of a maximizing sequence of Lagrange multipliers, while the Lagrange multiplier is the reciprocal of the regularization parameter in the Tikhonov method. Thus, in [41] the convergence theorem of the Tikhonov regularization method is given the form of a statement in the form of duality with respect to the problem $(CE1)$.

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