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EXTENSION OF HARDY-ROGERS FIXED POINT THEOREM

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Dedicated to Professor Stojan Radenović on the occasion of his 75th birthday

Abstract. In this paper, we give an extension of the Hardy-Rogers fixed pont theorem. In particular, we extend the domain for contraction constants. We also investigate our result in the context of compact metric spaces.
Keywords. Complete metric space; Hardy-Rogers type mapping; Fixed point.
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1. INTRODUCTION

Fixed point theory in last two decades has gained great momentum. Areas of research vary from the generalization of metric spaces to the relaxation of the triangle inequality or by eliminating some of the axioms. The existence of a fixed point of contraction with additional conditions on the functions involved is also under the spotlight. For various fixed-point topics in metric spaces, we refer to [3, 8, 10, 12–20]. Recently, Gornicki [4–6] proved various extensions of Kannan's fixed point theorem. Inspired by [5], Gungor and Altun [7] examined the Chatterjea contraction.

In this paper, we aim to obtain some new results by relaxing contraction constants in the Hardy-Rogers fixed point theorem. In particular, we extend the domain for contraction constants. We also investigate our result in the context of compact metric spaces.

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2. Preliminaries

In this section, we list some known definitions and assertions in terms of metric spaces. Banach contraction principle is considered the central result in the metric fixed point theory. It not only can be used to guarantee the existence of various nonlinear differential / integral equations but finds wide applications in numerous engineering fields, such as, image recovery and signal processing. The Banach contraction principle reads as follows.

Theorem 2.1. *[1] Let (X,d) be complete metric space and $T : X \mapsto X$ contraction ie. there exists $\lambda \in [0,1)$ such that $d(Tx,Ty) \leq \lambda d(x,y)$ for all $x,y \in X$. Then*
(1) *T has unique fixed point $w \in X$;*
(2) *Further, for all $x_0 \in X$, iterative sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$ converges to a fixed point w of T.*

Remark 2.2. Condition $d(Tx,Ty) < d(x,y)$ for all $x \neq y$ is not sufficient to ensure existence of a fixed point. As we can see function $T : [1,+\infty) \to [1,+\infty)$, $T(x) = x + \frac{1}{x}$ is fixed point free but fulfills the stated condition. However, in compact metric spaces, this condition indeed implies the existence and uniqueness of fixed points.

Recently, numerous authors developed various generalizations of Banach fixed-point theorem, but the open question remained whether there are contractive conditions for which the operator T is not continuous. Kanan gave a positive answer in 1968 as following.

Theorem 2.3. *[11] Let (X,d) complete metric space, and let $T : X \to X$ mapping such that there exists $\alpha \in \left(0,\frac{1}{2}\right)$ and holds $d(Tx,Ty) \leq \alpha[d(Tx,x)+d(Ty,y)]$ for all $x,y \in X$. Then T has unique fixed point in X.*

Example 2.4. Let $X = \mathbb{R}$, and let d be the usual metric. Define

$$T(x) = \begin{cases} 0, & x \in (-\infty,2], \\ \frac{1}{2}, & x \in (2,+\infty). \end{cases}$$

Then T is not continuous on $\mathbb{R}$, but it is Kannan's contraction with parameter $\alpha = \frac{1}{5}$.

In 1971, Reich [17] obtained a generalization of Banach's and Kannan's fixed point theorem. It reads as follows.

Theorem 2.5. *[17] Let (X,d) be a complete metric space, and let $T : X \to X$ be a mapping with $d(Tx,Ty) \leq ad(x,Tx)+bd(y,Ty)+c(x,y)$ for all $x,y \in X$, where a, b and c are nonnegative and satisfy $a+b+c < 1$. Then T has unique fixed point in X.*

In 1972, Chatterjea [2] proved the following theorem.

Theorem 2.6. *[2] Let (X,d) complete metric space and $T : X \to X$ mapping such that there exists $\alpha \in \left(0,\frac{1}{2}\right)$ and holds $d(Tx,Ty) \leq \alpha[d(x,Ty)+d(y,Tx)]$ for all $x,y \in X$. Then T has unique fixed point in X.*

In 1973, Hardy and Rogers [9] introduced a generalization of Reich's fixed point theorem. In addition, they obtained a generalization of the three previously stated results of Banach, Kannan and Chatterjea.

Theorem 2.7. *[9] Let (X,d) be complete metric space and $T : X \to X$ self mapping. If there exist nonnegative numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and α_5 such that $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 < 1$ and holds*

$$d(Tx,Ty) \le \alpha_1 d(x,y)+\alpha_2 d(x,Tx)+\alpha_3 d(y,Ty)+\alpha_4 d(x,Ty)+\alpha_5 d(y,Tx),$$

then T has a unique fixed point in X.

Before we move on to our main result, we present a class of functions which play an important role in the fixed-point theory of nonlinear operator.

Definition 2.8. Let (X,d) be a metric space and T be a self mapping on X. We say that T is the Hardy-Rogers type maping if there exist α_1, α_2,α_3, α_4, and $\alpha_5 \in [0,+\infty)$ such that

$$d(Tx,Ty) \le \alpha_1 d(x,y)+\alpha_2 d(x,Tx)+\alpha_3 d(y,Ty)+\alpha_4 d(x,Ty)+\alpha_5 d(y,Tx),\ x,y \in X.$$

Analogously, T is strict Hardy-Rogers type maping if there exist α_1, α_2,α_3, α_4, and $\alpha_5 \in [0,+\infty)$ such that

$$d(Tx,Ty) < \alpha_1 d(x,y)+\alpha_2 d(x,Tx)+\alpha_3 d(y,Ty)+\alpha_4 d(x,Ty)+\alpha_5 d(y,Tx) \text{ for } x \ne y.$$

Remark 2.9. It is easy to see that for $\alpha_1 < 1$ and $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$, we have the classical Banach contraction principle. For $\alpha_1 = \alpha_4 = \alpha_5 = 0$, and $\alpha_2+\alpha_3 < 1$ we see the Kannan contraction; for $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_4+\alpha_5 < 1$, we obtain the Chatterjea contraction.

3. Main Results

Now, we are in a position to state our main results.

Theorem 3.1. *Let (X,d) be a complete metric space, and let $T : X \to X$ be a Hardy-Rogers type mapping with $\alpha_3+\alpha_4 < 1$. Let a and b be two constants such that $0 \le a < 1$ and $b > 0$. If, for arbitrary $x \in X$, there exists $r \in X$ such that*

$$d(r,Tr) \le a \cdot d(x,Tx) \text{ and } d(r,x) \le b \cdot d(x,Tx), \tag{3.1}$$

then T has at least one fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. By (3.1), one sees that there exists x_1 in X such that

$$d(x_1,Tx_1) \le ad(x_0,Tx_0) \text{ and } d(x_1,x_0) \le bd(x_0,Tx_0).$$

Similarly, there exists x_2 such that

$$d(x_2,Tx_2) \le ad(x_1,Tx_1) \text{ and } d(x_2,x_1) \le bd(x_1,Tx_1).$$

Accordingly, we can construct a sequence $\{x_n\} \subset X$ that satisfies

$$d(x_{n+1},Tx_{n+1}) \le ad(x_n,Tx_n) \text{ and } d(x_{n+1},x_n) \le bd(x_n,Tx_n),$$

for all $n \in \mathbb{N}$. From $d(x_{n+1},x_n) \le bd(x_n,Tx_n) \le a^n bd(x_0,Tx_0)$, we easily see that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, one sees that there exists $w \in X$ such that

$\lim_{n\to+\infty} x_n = w$. Further,

$$\begin{aligned}
d(w,Tw) \leq & d(w,x_n)+d(x_n,Tw)\\
\leq & d(w,x_n)+d(x_n,Tx_n)+d(Tx_n,Tw)\\
\leq & d(w,x_n)+d(x_n,Tx_n)+\alpha_1 d(x_n,w)+\alpha_2 d(x_n,Tx_n)+\alpha_3 d(w,Tw)\\
& +\alpha_4 d(x_n,Tw)+\alpha_5 d(w,Tx_n)\\
\leq & (1+\alpha_1)d(w,x_n)+(1+\alpha_2)d(x_n,Tx_n)+\alpha_3 d(w,Tw)\\
& +\alpha_4\left[d(x_n,w)+d(w,Tw)\right]+\alpha_5\left[d(w,x_n)+d(x_n,Tx_n)\right]\\
\leq & (1+\alpha_1+\alpha_4+\alpha_5)d(w,x_n)+(1+\alpha_2+\alpha_5)d(x_n,Tx_n)+(\alpha_3+\alpha_4)d(w,Tw),
\end{aligned}$$

which further implies

$$\begin{aligned}
d(w,Tw) \leq & \frac{1+\alpha_1+\alpha_4+\alpha_5}{1-\alpha_3-\alpha_4}d(w,x_n)+\frac{1+\alpha_2+\alpha_5}{1-\alpha_3-\alpha_4}d(x_n,Tx_n)\\
\leq & \frac{1+\alpha_1+\alpha_4+\alpha_5}{1-\alpha_3-\alpha_4}d(w,x_n)+\frac{1+\alpha_2+\alpha_5}{1-\alpha_3-\alpha_4}a^n d(x_0,Tx_0)\to 0
\end{aligned}$$

as $n\to+\infty$. Therefore, $Tw=w$.

Let T be a Hardy-Rogers type mapping with $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5<1$ on complete metric space (X,d), and let $r=Tx$ for any $x\in X$. Then

$$\begin{aligned}
d(r,Tr) \leq & \alpha_1 d(x,r)+\alpha_2 d(x,Tx)+\alpha_3 d(r,Tr)+\alpha_4 d(x,Tr)+\alpha_5(r,Tx)\\
= & (\alpha_1+\alpha_2+\alpha_4)d(x,Tx)+(\alpha_3+\alpha_4)d(r,Tr)
\end{aligned}$$

Therefore

$$d(r,Tr)\leq \frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3-\alpha_4}d(x,Tx) \tag{3.2}$$

We exchange α_2 with α_3 and α_4 with α_5 in (3.2) to obtain

$$d(r,Tr)\leq \frac{\alpha_1+\alpha_3+\alpha_5}{1-\alpha_2-\alpha_5}d(x,Tx).$$

It follows that

$$a=\min\left\{\frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3-\alpha_4},\frac{\alpha_1+\alpha_3+\alpha_5}{1-\alpha_2-\alpha_5}\right\}$$

satisfies condition (3.1), since

$$0\leq a<1,\quad d(r,Tr)\leq a\cdot d(x,Tx) \text{ and } d(Tx,x)=d(x,Tx)(\text{ for } b=1).$$

Now, for arbitrary $x_0\in X$, we can define a sequence $x_{n+1}=Tx_n$. From Theorem 3.1, one sees that this sequence is convergent, that is, $\lim_{n\to+\infty}=w$ and $Tw=w$. Let u be another fixed point of T. Then

$$\begin{aligned}
0<d(Tu,Tw) \leq & \alpha_1 d(u,w)+\alpha_2 d(u,Tu)+\alpha_3 d(w,Tw)+\alpha_4 d(u,Tw)+\alpha_5 d(w,Tu)\\
= & (\alpha_1+\alpha_4+\alpha_5)d(u,w).
\end{aligned}$$

In view of $\alpha_1+\alpha_4+\alpha_5<1$, we conclude $d(u,w)=0$, a contradiction. Hence, T has a unique fixed point.

Example 3.2. Let d be the usual metric on $\mathbb{R}$. Define a function $T : [0,1] \to [0,1]$ by

$$T(x) = \begin{cases} 0, & 0 \le x < 1, \\ \frac{1}{2}, & x = 1. \end{cases}$$

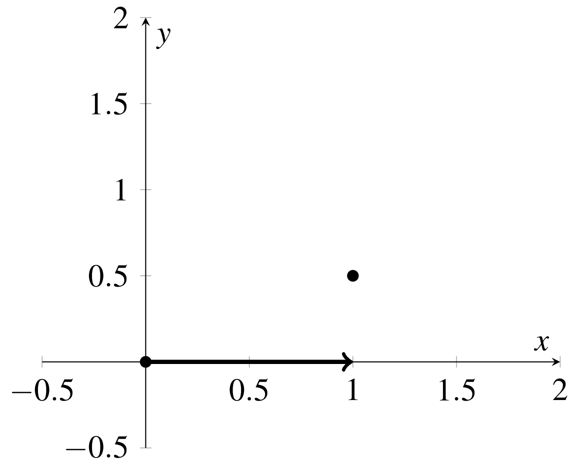

FIGURE 1. Graph of function T

Obviously, T is a discontinuous function, so the Banach contraction principle does not apply. Note that

$$d\left(T\frac{1}{2}, T1\right) = d\left(0, \frac{1}{2}\right) = \frac{1}{2}.$$

Evidently,

$$d\left(T\frac{1}{2}, T1\right) = \frac{1}{2}\left(d\left(\frac{1}{2}, T\frac{1}{2}\right) + d(1, T1)\right) = \frac{1}{2}\left(d\left(\frac{1}{2}, 0\right) + d\left(1, \frac{1}{2}\right)\right) = \frac{1}{2},$$

so the Kanan contraction condition is not fulfilled. In the same manner, one has

$$d\left(T\frac{1}{2}, T1\right) = \frac{1}{2}\left(d\left(\frac{1}{2}, T1\right) + d\left(1, T\frac{1}{2}\right)\right) = \frac{1}{2}\left(d\left(\frac{1}{2}, \frac{1}{2}\right) + d(1, 0)\right) = \frac{1}{2}.$$

Plainly, T does not satisfy Chatterjea's condition. However, T is the Hardy-Rogers contraction with parameters $\alpha_1 = \frac{1}{10}, \alpha_2 = \frac{1}{9}, \alpha_3 = \frac{1}{8}, \alpha_4 = \frac{1}{7}$, and $\alpha_5 = \frac{1}{2}$. Therefore, our obtained result is stronger than Banach's, Kannan's, and Chatterjea's fixed point theorems.

In next theorem, we investigate fixed points of strict Hardy-Rogers type continuous mappings on compact metric spaces.

Theorem 3.3. *Let (X,d) be a compact metric space, and let $T : X \to X$ continuous mapping. If T is a strict Hardy-Rogers type mapping with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \le 1$, then T has unique fixed point w and for each $x \in X$ sequence $\{T^n x\}$ converges to w.*

Proof. Define a function $g : X \to [0,+\infty)$ by $g(x) = d(x,Tx)$. Note that X is compact. Then, there exists $w \in X$ such that $f(w) = \inf\{g(x) : x \in X\}$. If $w \neq Tw$, then

$$\begin{aligned} d(Tw,T^2w) <&\alpha_1 d(w,Tw) + \alpha_2 d(w,Tw) + \alpha_3 d(Tw,T^2w) + \alpha_4 d(w,T^2w) + \alpha_5 d(Tw,Tw) \\ <&(\alpha_1+\alpha_2+\alpha_4)d(w,Tw) + (\alpha_3+\alpha_4)d(Tw,T^2w). \end{aligned}$$

Thus

$$d(Tw,T^2w) < \frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3-\alpha_4} d(w,Tw) \tag{3.3}$$

By symmetry, we may exchange α_2 with α_3 and α_4 with α_5 in (3.3) to obtain

$$d(Tw,T^2w) < \frac{\alpha_1+\alpha_3+\alpha_5}{1-\alpha_2-\alpha_5} d(w,Tw).$$

In view of

$$\min\left\{\frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3-\alpha_4}, \frac{\alpha_1+\alpha_3+\alpha_5}{1-\alpha_2-\alpha_5}\right\} \leq 1,$$

we have $g(Tw) = d(Tw,T^2w) < d(w,Tw) = g(w)$, a contradiction. Hence, w is a unique fixed point.

For second part of proof, we define a sequence $x_n = T^n x$. The case for $x = w$ is trivial. Let $x \neq w$. Then,

$$\begin{aligned} d(T^{n+1}x,T^n x) <&\alpha_1 d(T^n x,T^{n-1}x) + \alpha_2 d(T^n x,T^{n+1}x) + \alpha_3 d(T^{n-1}x,T^n x) \\ &+\alpha_4 d(T^n x,T^n x) + \alpha_5 d(T^{n-1}x,T^{n+1}x) \\ &<(\alpha_1+\alpha_3+\alpha_5)d(T^n x,T^{n-1}x) + (\alpha_2+\alpha_5)d(T^n x,T^{n+1}x), \end{aligned}$$

which implies

$$d(T^{n+1}x,T^n x) < kd(T^n x,T^{n-1}x) < d(T^n x,T^{n-1}x) < \ldots < d(Tx,x),$$

where

$$k = \min\left\{\frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3-\alpha_4}, \frac{\alpha_1+\alpha_3+\alpha_5}{1-\alpha_2-\alpha_5}\right\}.$$

Therefore, sequence $z_n = d(T^{n+1}x,T^n x)$ is nonegative and decreasing and thus convergent. Let $\lim_{n\to+\infty} z_n = z > 0$. In view of the compactness of X, one sees that sequence $\{T^n x\}$ contains a convergent subsequence $\{T^{n_i}x\}$ such that $T^{n_i}x \to q \in X$ as $i \to +\infty$. Hence

$$0 < z = \lim_{i\to+\infty} d(T^{n_i+1}x,T^{n_i}x) = d(Tq,q),$$

that is, $q \neq w$. Moreover, by (3.3), one has

$$z = \lim_{i\to+\infty} (T^{n_i+2}x,T^{n_i+1}x) = d(T^2q,Tq) < d(Tq,q) = z,$$

which is a contradiction. Thus $z = 0$ and $\lim_{n\to+\infty} d(T^{n+1}x,w) \leq z = 0$. Finally, we have that sequence $T^{n+1}x$ converges to w. □

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