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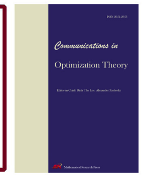

ON THE PERRON-FROBENIUS ROW AND COLUMN EIGENSPACE OF INFINITE DIMENSIONAL STOCHASTIC MATRICES: SOME EXAMPLES

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Dedicated to the memory of Uri G. Rothblum

Abstract. This paper shows, via examples, that the dimension of the row eigenspace for an irreducible infinite dimensional stochastic matrix P corresponding to the Perron-Frobenius eigenvalue 1 can be zero, one, d, or infinite. Infinite dimensionality can arise even when the associated Markov chain is positive recurrent. This is to be contrasted with the finite dimensional setting in which the Perron-Frobenius row and column eigenspace for stochastic matrices is always one dimensional in the presence of irreducibility.

Keywords. Eigenspace; Stochastic matrix; Stationary distribution.

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1. Introduction

A recurring research interest of Uri Rothblum concerned the structure of non-negative finite matrices; see, for example, his early and fundamental work on the algebraic eigenspace of such matrices, as discussed in [8]. The starting point for such analysis is the development of the associated theory when the matrix is irreducible. In this special setting, the algebraic eigenspace has dimension 1, and the corresponding row and column eigenvectors are strictly positive (up to a multiplicative constant).

The current paper shows, via examples, that the situation for infinite dimensional non-negative irreducible matrices can be much more complex. In particular, the algebraic eigenspace for such matrices, even when specialized to positive recurrent stochastic matrices, can be one dimensional, d dimensional with d finite, and even infinite dimensional. Furthermore, the dimension of the row and column eigenspaces can differ. The fact that the eigenspace for positive recurrent Markov chains can include solutions that do not correspond to the stationary distribution of the

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associated Markov chain can complicate some numerical algorithms designed to compute such stationary distributions; see, for example, the algorithm proposed by [4], which searches within the class of square-summable solutions taken relative to a given reference measure (which may include solutions other than the stationary distribution, without imposing further restrictions). Clarifying this point is a main motivation for this paper, and it is discussed further in Section 5. Of course, it is known that when the associated Markov chain is recurrent, then restricting consideration to row eigenvectors that are either summable or non-negative yields a unique row eigenvector (up to a multiplicative constant); see Section 2 for further discussion.

Our examples also establish that an irreducible transient Markov chain can have no non-trivial solutions in its row eigenspace. It is also possible that it can have an infinite dimensional family of positive eigenvectors in its row eigenspace. Thus, unlike the recurrent case, restricting the solution to being positive does not lead to a one dimensional row eigenspace. Similar theory is developed for the column eigenspace of irreducible infinite state stochastic matrices. Section 3 provides examples that illustrate the range of possibilities for the row eigenspace, while Section 4 discusses the same examples from the perspective of the column eigenspace.

2. Review of Recurrence Theory for Markov Chains

Let $X = (X_n : n \geq 0)$ be an irreducible Markov chain on discrete state space S with transition matrix $P = \{P(x,y) : x,y \in S\}$. For $x \in S$, let $P_x(\cdot) = P_x(\cdot|X_0 = x)$, $\mathbb{E}_x(\cdot) = \mathbb{E}_x(\cdot|X_0 = x)$, and let $\tau(x) = \inf\{n \geq 1 : X_n = x\}$ be the first time at which X enters x. The chain is said to be *recurrent* if $P_x(\tau(x) < \infty) = 1$ for some $x \in S$ and *transient* otherwise. A recurrent chain is *positive recurrent* if $\mathbb{E}_x \tau_x < \infty$ for some $x \in S$, and *null recurrent* otherwise.

Furthermore, we say that $\nu = (\nu(y) : y \in S)$ is an *invariant distribution* of P if

$$\sum_x |\nu(x)| P(x,y) < \infty \quad \text{for each } y \in S,$$

and

$$\nu(y) = \sum_x \nu(x) P(x,y). \tag{2.1}$$

for each $y \in S$. We choose to encode such distributions as row vectors, so that (2.1) can be written as $\nu = \nu P$. If an invariant distribution is a probability mass function on S, we say that ν is a *stationary distribution* of P (or X).

We say that $h = (h(x) : x \in S)$ is a *harmonic function* for P (or X) if

$$\sum_y P(x,y)|h(y)| < \infty,$$

for $x \in S$, and

$$\sum_y P(x,y) h(y) = h(x) \tag{2.2}$$

for $x \in S$. If we encode functions as column vectors, then (2.2) can be written as $Ph = h$. We say that h is *superharmonic* for P (or X) if (2.1) holds and $Ph \leq h$.

The following result summarizes existing theory about the invariant distributions (i.e., row eigenvectors) and harmonic functions associated with stochastic matrix P.

Theorem 2.1. *If P is an irreducible stochastic matrix on a countably infinite state space, then:*

(i) X is positive recurrent if and only if there exists a stationary distribution for P (in which case the stationary distribution must be unique);

(ii) X is positive recurrent if and only if there exists a non-trivial summable invariant distribution ν for P (i.e., $\sum_x |\nu(x)| < \infty$), in which case the summable invariant distribution is unique (up to a multiplicative constant);

(iii) X is recurrent if and only if every bounded (i.e., $\sup\{|h(x)| : x \in S\} < \infty$) superharmonic function is a constant function (i.e., $h(x) = c \in \mathbb{R}$ for $x \in S$);

(iv) X is recurrent if and only if every non-negative superharmonic function is a constant function.

Result (i) is well known; see, for example, [3]. As for (ii), note that the invariance of ν and Fubini's theorem (utilizing the summability) imply that

$$\nu(y) = \sum_{x \in S} \nu(x) \frac{1}{n} \sum_{j=0}^{n-1} P^j(x,y) \tag{2.3}$$

for $y \in S$. Suppose, first, that X is positive recurrent. Then, for $y \in S$,

$$\frac{1}{n} \sum_{j=0}^{n-1} P^j(x,y) \to \pi(y)$$

as $n \to \infty$, where $\pi = (\pi(y) : y \in S)$ is the unique stationary distribution of P. Hence, taking limits in (2.3) and applying the Bounded Convergence Theorem, we find that

$$\nu(y) = \sum_{x \in S} \nu(x) \, \pi(y),$$

so ν is equal to the stationary distribution of X (up to the multiplicative constant $\sum_{x \in S} \nu(x)$).

On the other hand, if X is null recurrent or transient, then

$$\frac{1}{n} \sum_{j=0}^{n-1} P^j(x,y) \to 0$$

as $n \to \infty$ for $y \in S$. Again, taking limits in (2.3) and applying the Bounded Convergence Theorem, we find that $\nu(y) = 0$ for $y \in S$, so that ν is trivial. This is a contradiction of the non-triviality of ν.

As for (iii) and (iv), note that the fact that h is superharmonic guarantees that $(h(X_n) : n \geq 0)$ is a supermartingale adapted to the filtration generated by X. Suppose X is recurrent. Then, the supermartingale convergence theorem ensures that there exists a finite-valued rv M_∞ such that $h(X_n) \to M_\infty$ a.s. as $n \to \infty$; see [2]. Since X is recurrent and irreducible, X visits all states $y \in S$ infinitely often a.s. So, $h(\cdot)$ must be constant. On the other hand, if X is transient, we put $h(x) = P_x(T(z) < \infty)$ for some $z \in S$, where $T(z) = \inf\{n \geq 0 : X_n = z\}$. Then, $h(z) = 1$ and for $x \neq z$, $h(x) \in [0,1]$ and $h(x) = P(x,z) + \sum_{y \neq z} P(x,y)h(y) = (Ph)(x)$. Of course, $h(z) = 1 = \sum_y P(z,y) \geq (Ph)(z)$, so $h \geq Ph$. Furthermore, $h(x) < 1$ for some $x \neq z$, since otherwise z would be recurrent. So, h is non-constant.

3. The Row Eigenspace for Irreducible Stochastic Matrices

As noted in the Introduction, every irreducible stochastic matrix $P = (P(x,y) : x,y \in S)$ with $|S| < \infty$ possesses a unique (nontrivial) solution π to $\pi = \pi P$, where $\pi = (\pi(x) : x \in S)$ can be taken to be positive and stochastic. Our first example, from [6], shows that when $|S| = \infty$, $\pi = \pi P$ may fail to have any non-trivial solution.

Example 1. Suppose $S = \{0,1,2,\cdots\}$ with $P(x,0) = 1-p(x)$ $(0 < p(x) < 1)$ and $P(x,x+1) = p(x)$ for $x \geq 0$. Then, the chain is irreducible and $\pi(x+1) = p(x)\pi(x)$ for $x \geq 0$, so that

$$\pi(x) = \pi(0)\prod_{j=0}^{x-1} p(j).$$

Also,

$$\pi(0) = \sum_{y=0}^{\infty}(1-p(y))\pi(y) = (1-p(0))\pi(0) + \sum_{y=1}^{\infty}\pi(0)\prod_{j=0}^{y-1} p(j)(1-p(y)). \tag{3.1}$$

If $X = (X_n : n \geq 0)$ is the Markov chain having transition matrix P, we let $P_x(\cdot) = P(\cdot|X_0 = x)$. The first return time to state 0 is then defined by $\tau(0) = \inf\{n \geq 1 : X_n = 0\}$. Observe that $P_0(\tau(0) = 1) = 1-p(0)$ and

$$P_0(\tau(0) = n+1) = P_0(X_{n+1} = 0, X_i = i, 1 \leq i \leq n) = \prod_{i=0}^{n-1} p(i)(1-p(n)).$$

for $n \geq 1$. It follows that if $\pi(0) \neq 0$ in (3.1), then upon dividing through by $\pi(0)$, we must conclude that

$$\begin{aligned}
1 &= 1-p(0) + \sum_{y=1}^{\infty}\prod_{j=0}^{y-1} p(j)(1-p(y)) \\
&= P_0(\tau(0) = 1) + \sum_{y=1}^{\infty} P_0(\tau(0) = y+1) \\
&= P_0(\tau(0) < \infty),
\end{aligned}$$

so that a non-trivial solution to (3.1) (and hence $\pi = \pi P$) exists only if X is a recurrent Markov chain. But it is easy to choose $p(j)$ for all j to make X transient. In particular, if $p(j) = \exp(-2^{-j-1})$, then

$$\begin{aligned}
P_0(\tau(0) > n) &= P_0(X_i = i, 1 \leq i \leq n) \\
&= \prod_{j=0}^{n-1}\exp(-2^{-j-1}) \\
&= \exp\left(-\sum_{j=0}^{n-1} 2^{-j-1}\right) \\
&= \exp\left(-1+2^{-n}\right) \to \exp(-1) > 0
\end{aligned}$$

as $n \to \infty$, so that X is transient.

Example 2. Consider an irreducible birth-death chain on $S = \{0,1,\cdots\}$, so that $P(x,x+1) = p(x)$ for $x \geq 0$, $P(x,x-1) = 1-p(x)$ for $x \geq 1$, and $P(0,0) = 1-p(0)$, where $0 < p(x) < 1$ for $x \geq 0$. Then, $\pi = \pi P$ implies that

$$\pi(x) = \pi(0)\prod_{j=0}^{x-1}\frac{p(j)}{1-p(j+1)}$$

for $x \geq 1$.

In this example, $\pi = \pi P$ always has a unique (up to a multiplicative constant) solution, which can be taken to be positive. Of course, if $p(x) = p > 1/2$ for $x \geq 0$, X is transient. So, Examples 1 and 2 establish that a transient, irreducible Markov chain may have no non-trivial solution to $\pi = \pi P$ or a positive solution to $\pi = \pi P$. In particular, existence of a positive solution to $\pi = \pi P$ does not guarantee recurrence. (On the other hand, it is well known that for any irreducible recurrent Markov chain X, $\pi = \pi P$ always has a positive solution which is unique up to a multiplicative constant; see [5]. Furthermore, these positive solutions have probabilistically meaningful interpretations. For example, when the Markov chain is positive recurrent and the eigenvector is normalized to be a probability, each component of the eigenvector can be interpreted as the long-run fraction of time spent in that state by the associated Markov chain.

In [6], the irreducible birth-death chain X on $S = \mathbb{Z}$ with $P(x,x+1) = p = 1 - P(x,x-1)$ for $x \in \mathbb{Z}$ with $0 < p < 1$ is studied. It is shown there that when $p \neq \frac{1}{2}$, the general solution of $\pi = \pi P$ is given by

$$\pi(x) = a + b\left(\frac{p}{q}\right)^x$$

for $a, b \in \mathbb{R}$. In this setting, X is transient and when $a, b > 0$, $\pi = (\pi(x) : x \in S)$ is a positive solution to $\pi = \pi P$. So, even in the presence of irreducibility, $\pi = \pi P$ can have a two-dimensional set of positive solutions.

On the other hand, when $p = \frac{1}{2}$, the general solution to $\pi = \pi P$ is $\pi(x) = ax + b$ for $x \in \mathbb{Z}$. For $p = \frac{1}{2}$, X is recurrent. When π is required to be non-negative and non-trivial, this forces $b = 0$, so that the space of non-negative solutions to $\pi = \pi P$ is one dimensional, as noted earlier for recurrent chains.

Our final example extends this Derman example to establish that the row eigenspace can be infinite dimensional, even in the presence of irreducibility and positive recurrence.

Example 3. Consider the Markov chain X defined on state space $S = \{0\} \cup \{(i, j) : i, j \geq 1\}$ with transition probabilities defined by

$$P(x,y) = \begin{cases} p, & x = (i,j), y = (i+1,j), i,j \geq 1 \\ q, & x = (i,j), y = (i-1,j), i \geq 2, j \geq 1 \\ q, & x = (1,j), y = 0, j \geq 1 \\ r_j, & x = 0, y = (1,j), j \geq 1 \\ 1 - \sum_{j=1}^{\infty} r_j, & x = y = 0 \\ 0, & \text{else} \end{cases}$$

where $0 < p < 1$, $r_j > 0$ for $j \geq 1$, and $\sum_{j=1}^{\infty} r_j < 1$. This Markov chain is then irreducible and aperiodic. Note that $\pi = \pi P$ implies that

$$\pi(i, j) = p\pi(i-1, j) + q\pi(i+1, j) \tag{3.2}$$

for $i \geq 2$ and $j \geq 1$. The equation (3.2) has the general solution

$$\pi(i, j) = a_j + b_j (p/q)^i \tag{3.3}$$

for $i, j \geq 1$ and $p \neq q$. In addition to (3.2), $\pi = \pi P$ requires that

$$\pi(1, j) = r_j \pi(0) + q\pi(2, j) \tag{3.4}$$

for $j \geq 1$, and

$$\pi(0) = \left(1 - \sum_{j=1}^{\infty} r_j\right) \pi(0) + q \sum_{j=1}^{\infty} \pi(1, j). \tag{3.5}$$

Plugging (3.3) into (3.4) and by recalling that $p + q = 1$, we find that

$$\begin{aligned}\pi(1, j) - \pi(2, j) =& (1-q)a_j + \left((p/q) - q(p/q)^2\right) b_j \\ =& p\left(a_j + b_j\right) \\ =& r_j \pi(0)\end{aligned} \tag{3.6}$$

for $j \geq 1$. Also, (3.3), (3.5), and (3.6) imply that

$$\begin{aligned}\sum_{j=1}^{\infty} r_j \pi(0) &= q \sum_{j=1}^{\infty} \left(a_j + b_j(p/q)\right) \\ &= \sum_{j=1}^{\infty} \left(qa_j + pb_j\right) \\ &= (q-p) \sum_{j=1}^{\infty} a_j + p \sum_{j=1}^{\infty} (a_j + b_j) \\ &= (q-p) \sum_{j=1}^{\infty} a_j + \sum_{j=1}^{\infty} r_j \pi(0),\end{aligned}$$

from which we conclude that

$$\sum_{j=1}^{\infty} a_j = 0. \tag{3.7}$$

(In manipulating these sums, we are assuming that they are absolutely convergent.)

When $p < q$, (3.3) implies that $\pi(i, j)$ tends to a_j as $i \to \infty$, so that $a_j \geq 0$ for $j \geq 1$ is required for non-negativity of π. On the other hand, (3.7) then forces $a_j = 0$ for $j \geq 1$, so that (3.6) yields $b_j = r_j \pi(0)/p$. It follows that X has a unique stationary distribution π given by

$$\pi(i, j) = \frac{r_j \pi(0)}{p} \left(\frac{p}{q}\right)^i \tag{3.8}$$

for $i, j \geq 1$, where $\pi(0)$ is chosen to make the probabilities sum to one. However, for absolutely summable $(a_j : j \geq 1)$ satisfying (3.7), we note that $b_j = (r_j/p) - a_j$, so that

$$\pi(i, j) = a_j + \left(\frac{r_j}{p} - a_j\right) \left(\frac{p}{q}\right)^i \tag{3.9}$$

for $i, j \geq 1$, with $\pi(0) = 1$, also solves $\pi = \pi P$. Of course, in view of (3.7) and the fact that $\pi(i, j) \to a_j$ as $i \to \infty$, all the solutions described by (3.9) for which $(a_j : j \geq 1)$ is non-zero fail both to be non-negative and absolutely summable. Regardless, (3.9) describes an infinite dimensional family of solutions to $\pi = \pi P$.

When $p > q$, X is transient. In this case, (3.7) continues to hold, as does (3.9). Furthermore, since $(p/q)^i$ increases in i, the solution given by (3.9) is nonnegative, provided we choose the a_j for all j to satisfy the inequalities $a_j((p/q) - 1) \leq r_j/p$ and $a_j \leq r_j/p$, subject to (3.7). When the choices of $\{a_j : j \geq 1\}$ fail to satisfy one of the inequalities, the solution is of mixed sign.

So, in this transient setting, the family of solutions to $\pi = \pi P$ can be infinite-dimensional. This infinite dimensional family of solutions contains an infinite dimensional subset of nonnegative solutions, and also an infinite dimensional set of mixed sign solutions. Of course, all of these solutions are not summable.

Finally, when $p = q = \frac{1}{2}$, X is null recurrent. In this case,

$$\pi(i,j) = a_j + b_j(i-1), \tag{3.10}$$

$$2\pi(0)r_j = a_j - b_j, \tag{3.11}$$

$$\pi(0)\sum_{k=1}^{\infty} r_k = \frac{1}{2}\sum_{k=1}^{\infty}\pi(1,k) = \frac{1}{2}\sum_{k=1}^{\infty} a_k. \tag{3.12}$$

From (3.11) and (3.12), we see that

$$\sum_{k=1}^{\infty} b_k = 0. \tag{3.13}$$

In view of (3.13), we see that if the b_k for all k are non-zero, then there exist $\{b_k : k \geq 1\}$ of opposite sign, so that (3.10) implies that the solution π is of mixed sign. So, the eigenspace for $\pi = \pi P$ is again infinite dimensional, and the only non-negative solution has $b_k \equiv 0$, so that $\pi(i,j) = 2\pi(0)r_j$ for $i,j \geq 1$.

We note that this example can be easily modified to provide an irreducible stochastic matrix having a row eigenspace of dimension $d-1$, by limiting the state space of our example to $\{0\} \cup \{(i,j) : i \geq 1, 1 \leq j \leq d\}$. (The dimension shrinks by one due to the presence of the constraint (3.7).)

4. The Column Eigenspace for Irreducible Stochastic Matrices

A key difference between the column and row eigenspaces for a stochastic matrix is that the column eigenspace always contains a non-trivial solution, namely the constant vector $e = (e(x) : x \in S)$ for which $e(x) \equiv 1$ for $x \in S$ (since $Pe = e$), while Example 1 shows that the row eigenspace may have no non-trivial solutions.

We now return to the examples of Section 2 to analyze the column eigenspace, when no boundedness or non-negativity restrictions are imposed.

Example 1 (continued). Here, $Ph = h$ translates into the linear system

$$h(0) = (1-p(0))h(0) + p(0)h(1) \tag{4.1}$$

and

$$h(x) = p(x)h(x) + (1-p(x))h(0) \tag{4.2}$$

for $x \geq l$. It is easy to show that (4.1) implies that $h(1) = h(0)$, and that (4.2) successively implies that $h(2), h(3), \cdots$ are also equal to $h(0)$. Hence, the only solution of $Ph = h$ is $h = e$ (up to a multiplicative constant), regardless of whether X is transient or recurrent. So, this example establishes that (iii) and (iv) of Theorem 1 do not hold when superharmonic function is replaced by harmonic function.

Example 2 (continued). Here, the linear system for $Ph = h$ corresponds to

$$h(0) = (1-p(0))h(0) + p(0)h(1)$$

and

$$h(x) = (1-p(x))h(x-1) + p(x)h(x+1)$$

for $x \geq 1$. Again, it is easily seen that the only solution is $h = e$ (up to a multiplicative constant), regardless of whether the birth-death chain is recurrent or transient.

Turning to Derman's birth-death example on $S = \mathbb{Z}$, the equations describing the column eigenspace take the form

$$h(x) = (1-p)h(x-1) + ph(x+1)$$

for $x \in \mathbb{Z}$. When $p \neq q$, the general solution is

$$h(x) = \tilde{a} + \tilde{b}\left(\frac{q}{p}\right)^x$$

for $x \in \mathbb{Z}$, whereas when $p = q = \frac{1}{2}$, the general solution is $h(x) = \tilde{a} + \tilde{b}x$ for $x \in \mathbb{Z}$. In both cases, the eigenspace is two-dimensional. When we restrict to bounded solutions, $\tilde{b}$ must equal zero, so the eigenspace becomes one dimensional. Of course, when X is recurrent (i.e., $p = \frac{1}{2}$), non-negativity forces $\tilde{b} = 0$, so the eigenspace again becomes one dimensional under this restriction.

Example 3 (continued). For this Markov chain, the column eigenspace is described by the equations

$$h(0) = \left(1 - \sum_{k=1}^{\infty} r_k\right) h(0) + \sum_{k=1}^{\infty} r_k h(1,k), \tag{4.3}$$

$$h(1,j) = qh(0) + ph(2,j), \tag{4.4}$$

$$h(i,j) = qh(l-1,j) + ph(i+1,j) \tag{4.5}$$

for $i \geq 2$, $j \geq 1$. When $p \neq q$, the general solution of (4.5) is given by

$$h(i,j) = \tilde{a}_j + \tilde{b}_j \left(\frac{q}{p}\right)^i \tag{4.6}$$

for $i, j \geq 1$. The equation (4.4) then takes the form

$$\tilde{a}_j + \tilde{b}_j = h(0) \tag{4.7}$$

for $j \geq 1$, while (4.3) becomes

$$\sum_{k=1}^{\infty} r_k h(0) = \sum_{k=1}^{\infty} \frac{r_k}{p}\left(p\tilde{a}_k + q\tilde{b}_k\right) \tag{4.8}$$

$$= \sum_{k=1}^{\infty} r_k\left(\tilde{a}_k + \tilde{b}_k\right) + \sum_{k=1}^{\infty} r_k \tilde{b}_k \frac{q-p}{p}. \tag{4.9}$$

In view of (4.7), we conclude that

$$\sum_{k=1}^{\infty} r_k \tilde{b}_k = 0 \tag{4.10}$$

(assuming that all sums involving $\{\tilde{a}_k : k \geq 1\}$ and $\{\tilde{b}_k : k \geq 1\}$ are absolutely convergent). When X is positive recurrent (so that $p < q$), $(q/p)^i \to \infty$ as $i \to \infty$, so that imposing boundedness on h implies that $\tilde{b}_j = 0$ for $j \geq 1$, and hence $\tilde{a}_j = h(0)$ (due to (4.7)). So, the only bounded harmonic function are constants, in alignment with the thereby discussed earlier in this section. Similarly, if h is non-negative, (4.10) implies that $\tilde{b}_k = 0$ for $k \geq 1$ (since otherwise there are at least two

$\tilde{b}_k$ for some k of opposite sign, so that h is of mixed sign), again implying that $h = e$ is the only such harmonic function (up to a multiplicative constant). However, the key point is that

$$h(l,i,j) = h(0) + \tilde{b}_j \left(\left(\frac{q}{p} \right)^i - 1 \right) \tag{4.11}$$

with the $\{\tilde{b}_j : k \geq 1\}$ satisfying (4.10), is an infinite dimensional family of solutions sitting within the column eigenspace of this irreducible positive recurrent Markov chain.

When X is transient (so that $(p > q)$), then $(q/p)^i \to 0$ as $i \to \infty$. In the presence of summability of the $\tilde{a}_j$ and $\tilde{b}_j$ ranging from $j = 1, 2, \cdots$, $(\tilde{b}_j : j \geq 1)$ is a bounded sequence, so that (4.11) is then bounded. So, X then possesses an infinite dimensional family of bounded solutions within its column eigenspace. Furthermore, since $\left(\tilde{b}_j((q/p)^i - 1) : i, j \geq 1\right)$ is bounded, (4.11) established that if we choose $h(0)$ sufficiently positive, then the general solution (4.11) is not only bounded but non-negative. So, this class of transient irreducible chains produces an infinite-dimensional column eigenspace of bounded non-negative solutions.

Finally, if $p = q = \frac{1}{2}$, then X is a null recurrent irreducible Markov chain. The general solution to (4.5) then takes the form

$$h(i,j) = \tilde{a}_j + \tilde{b}_j(i-1) \tag{4.12}$$

for $i, j \geq 1$, while (4.4) then becomes

$$\tilde{a}_j - \tilde{b}_j = h(0) \tag{4.13}$$

for $j \geq 1$, and (4.3) reduces to

$$\sum_{k=1}^{\infty} r_k h(0) = \sum_{k=1}^{\infty} r_k \tilde{a}_k = \sum_{k=1}^{\infty} r_k \left(\tilde{a}_k - \tilde{b}_k \right) + \sum_{k=1}^{\infty} r_k \tilde{b}_k. \tag{4.14}$$

In view of (4.13), (4.14) implies the validity of (4.10). So, the general solution to $Ph = h$ is given by

$$h(i,j) = h(0) + \tilde{b}_j i$$

for $i, j \geq 1$, subject to the constraint (4.10) on $\{\tilde{b}_j : j \geq 1\}$. Hence, the column eigenspace is again infinite dimensional. A similar argument to that used in the positive recurrent case establishes that when the solution is required to be bounded or non-negative, then $h = e$ is the only such solution of $Ph = h$ (up to a multiplicative constant).

Our final example shows that the dimension of the row and column eigenspace can differ, even when X is irreducible.

Example 4. Suppose that

$$P = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots & p_r & 0 & 0 & 0 & \cdots \\ p_0 & p_1 & p_2 & \cdots & p_r & 0 & 0 & 0 & \cdots \\ 0 & p_0 & p_1 & \cdots & p_{r-1} & p_r & 0 & 0 & \cdots \\ 0 & 0 & p_0 & p_1 \cdots & p_{r-2} & p_{r-1} & p_r & 0 & \cdots \\ \vdots & \vdots & \vdots & & \ddots & & & \vdots & \vdots \end{pmatrix}$$

where $(p_i : 0 \leq i \leq r)$ is a probability mass function, with all the p_i's $(0 \leq i \leq r)$ being positive. Then, $\pi = \pi P$ implies that

$$\pi(0) = \pi(0)p_0 + \pi(1)p_0,$$

so that $\pi(1)$ can be solved in terms of $\pi(0)$. Also,

$$\pi(i) = p_0\pi(i+1) + p_1\pi(i) + \cdots + p_i\pi(0),$$

for $1 \leq i \leq r$, and

$$\pi(i) = p_0\pi(i+1) + p_1\pi(i) + \cdots + p_r\pi(i-r)$$

for $i > r$, so that it follows that $\pi(i+1)$ can always be expressed in terms of $\pi(i), \cdots, \pi(0)$ for $i \leq r$ and $\pi(i), \cdots, \pi(i-r)$ for $i \geq r$. Consequently, the solution space for $\pi = \pi P$ is one dimensional.

On the other hand, the linear system $Ph = h$ translates into

$$h(0) = p_0 h(0) + p_1 h(1) + \cdots + p_r h(r) \tag{4.15}$$

and

$$h(i) = p_0 h(i-1) + p_1 h(i) + \cdots + p_r h(i+r-1) \tag{4.16}$$

for $i \geq 1$. Let $h(0), \cdots, h(r)$ be any solution of (33) and (34) with $i = 1$. With two equations and $r+1$ unknowns $h(0), \cdots, h(r)$, the solution space is r-dimensional. Equation (34) with $i \geq 2$ then permits one to solve for $h(i+r-1)$ in terms of $h(i-1), \cdots, h(i+r-2)$, thereby verifying that the solution space for $Ph = h$ is r dimensional for this example. So, if $r \geq 2$, the column eigenspace for this transition matrix has a different dimension than the row eigenspace. Note that this conclusion holds regardless of whether the chain is recurrent or transient.

5. An Algorithmic Consequence

Suppose that we wish to numerically compute the stationary distribution π of an irreducible transition matrix $P = (P(x,y) : x, y \in S)$. If $|S| < \infty$, then π is the unique solution of

$$\pi = \pi P$$

subject to

$$\sum_x \pi(x) = 1.$$

When $|S| = \infty$ and X is an irreducible positive recurrent Markov chain such that each column of P contains a finite number of non-zero entries (as is common in most stochastic models), then the following algorithm is a natural generalization of the above finite state space variant.

Let $S_n = \{x_1, \ldots, x_n\} \subset S$ be a finite approximation to S, and suppose we solve

$$\pi_n(y) = \sum_{x \in S} \pi_n(x) P(x,y) \tag{5.1}$$

for $y \in S_n$, subject to

$$\sum_{x \in \mathscr{S}_n} \pi_n(x) = 1, \tag{5.2}$$

where $\mathscr{S}_n = \{x \in S : P(x,y) > 0 \text{ for some } y \in S_n\}$. However, in part because of the uniqueness issues raised earlier in this paper, one can not expect this algorithm to be convergent as $n \to \infty$ (in the sense that $\pi_n(y) \to \pi(y)$ as $n \to \infty$ for each $y \in S$).

Example 5. Suppose that $S=\{0,1,2,\dots\}$ and that P takes the form

$$P=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This transition matrix arises in the setting of the $M/G/1$ queue and is known to have a unique stationary distribution; see [1]. If $S_n=\{0,1,\dots,n-1\}$, then (5.1) corresponds to

$$\pi_n(x)=\frac{1}{2}\pi_n(x-1)+\frac{1}{2}\pi_n(x+2) \tag{5.3}$$

for $x\geq 1$, whereas

$$\pi_n(0)=\frac{1}{2}\pi_n(0)+\frac{1}{2}\pi_n(2), \tag{5.4}$$

subject to (5.2). The general solution of (5.3) is

$$\pi_n(x)=\alpha+\beta z_1^x+\gamma z_2^x,$$

for $\alpha,\beta,\gamma\in\mathbb{R}$,where $z_1=(-1+\sqrt{5})/2$ and $z_2=-(1+\sqrt{5})/2$. Then, (5.4) forces $\alpha=0$. Note that the unique stationary distribution is

$$\pi(x)=(1-z_1)z_1^x,$$

while the general solution of (5.3), (5.4), and (5.1) is

$$\pi_n(x)=\left(1-\frac{\gamma(1-z_2^{n+2})}{1-z_2}\right)\frac{(1-z_1)}{(1-z_1^{n+2})}z_1^x+\gamma z_2^x$$

for $\gamma\in\mathbb{R}$. If, for example, $\gamma=1$, $\pi_n(x)\not\to\pi(x)$ as $n\to\infty$.

However, our earlier discussion in this paper makes clear the key role played by non-negativity in generating unique solutions to the linear system $\pi=\pi P$. In view of this, suppose we add on the non-negativity constraints

$$\pi_n(x)\geq 0 \tag{5.5}$$

for $x\in\mathscr{S}_n$ to (5.1) and (5.2). In the presence of (5.5), we see that we must then set $\gamma=0$, so that

$$\pi_n(x)=\frac{(1-z_1)z_1^x}{(1-z_1^{n+2})},$$

and consequently $\pi_n(x)$ now converges to $\pi(x)$ as $n\to\infty$ in Example 5.

In fact, if the modified algorithm (5.1), (5.2), and (5.5) is convergent, so that there exists $\pi_\infty=(\pi_\infty(x):x\in S)$ such that $\pi_n(x)\to\pi_\infty(x)$ as $n\to\infty$ for each $x\in S$, then π_∞ must equal the stationary distribution π (since π_∞ clearly then is a non-negative solution to $\pi=\pi P$), so that π_n then is guaranteed to converge to the stationary distribution. Convergence to a limit π_∞ is guaranteed in the presence of tightness (see, for example, [9]), for which a sufficient condition is to add on a constraint of the form $\sum_x\pi_n(x)f(x)\leq c$, to (5.1), (5.2), and (5.5), where f has sub-level sets of finite cardinality. Typically, the choice of c would be determined on the basis of a known inequality

$$\sum_x \pi_n(x)f(x) \le c' \tag{5.6}$$

for $c' < c$; see [7] for a discussion of how inequalities of the form (5.6) can be generated.

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