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ON WELL-POSEDNESS OF MIXED RANDOM VARIATIONAL INEQUALITIES ON RANDOM SETS

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Dedicated to the memory of Prof. H. Attouch

Abstract. This paper aims to study a class of mixed variational inequalities on random sets, with particular emphasis on issues pertaining to well-posedness. A central point is that the paper does not follow the standard L^p framework (with $1 \leq p < \infty$), which has been used in previous studies and typically specializes to the case $p = 2$ when presenting well-posedness results. Instead, the analysis is carried out entirely in the finer L^∞ Bochner–Lebesgue space. The main contribution of the paper is a novel well-posedness result for the considered mixed random variational inequality, derived from a recent abstract stability theorem. So as another novelty of the paper, perturbations are treated not only with respect to the right-hand side given by linear random forms, but also with respect to the Carathéodory operator and the Carathéodory function, using the concept of Mosco convergence.

Keywords. Carathéodory operator; Coercivity; Stability; Mosco convergence; Nonsmooth boundary condition; Random elliptic boundary value problem; Unilateral boundary condition.

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1. Introduction

While in deterministic regime there is an abundant literature in the field of variational inequalities (or variational inequalities of the first kind, following the terminology of Glowinski [8]) with a large body of applications in economics, finance and game theory, mathematical physics, transportation, and operations research, variational inequalities of the second kind and mixed variational inequalities are less studied; let us mention e.g. [5, 7, 13, 18, 19] for existence theory, iterative solution methods, and applications in mathematical physics, respectively. On contrary, contributions on random mixed variational inequalities (VIs for short) are rather rare. In this note we concentrate on linear mixed VIs with random data extending and supplementing results of [11, 14, 16]. Let us refer to the monograph [15] for a comprehensive treatment of linear and nonlinear variational inequalities of first kind and also of quasivariational inequalities with random data contributing to the fastly growing research field of uncertainty quantification.

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We underline that here we do not follow the general L^p (with $1 \leq p < \infty$) approach of [11, 14, 16] that in the presentation of the well-posedness result is specialized to the case $p = 2$, but throughout work in the finer L^∞ Bochner-Lebesgue space instead. A further important novelty of this paper is the treatment of a random constraint set thus extending some results of [11, 14].

In this contribution we investigate the following class of random VIs in its pathwise formulation: Find for every $\omega \in \Omega$, an element $\hat{y} = \hat{y}_\omega \in K(\omega) \subset H$ depending on $\omega \in \Omega$ such that for all $z \in K(\omega)$,

$$\langle B(\omega,\hat{y}), z - \hat{y}\rangle + \varphi(\omega,z) - \varphi(\omega,\hat{y}) \geq \lambda(\omega)(z - \hat{y})\,, \tag{1.1}$$

Here $(\Omega, \mathscr{A}, \mu)$ is a complete σ-finite measure space and $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ is a real separable Hilbert space identified with the dual space H^*. In contrary to [14], the closed, convex nonvoid constraint set K may depend on $\omega \in \Omega$ as made precise below. Next we have a Carathéodory operator $B : \Omega \times H \to H$, i.e. for each fixed $x \in H$, $B(\cdot, x)$ is measurable with respect to $\mathscr{A}$ and to the Borel algebra $\mathscr{B}(H)$, and for every $\omega \in \Omega$, $B(\omega,\cdot)$ is continuous, and further linear. Thus $B(\omega,\cdot)$ gives rise to

$$\beta(\omega,y,z) := \langle B(\omega)y, z\rangle$$

such that $\beta(\omega,\cdot,\cdot)$ is a continuous bilinear form, and the operator B can also be understood as measurable map from Ω to $\mathscr{L}(H,H)$, the space of linear continuous maps from H to H. Moreover, $\varphi : \Omega \times H \to \mathbb{R}$ is a Carathéodory function such that for every $\omega \in \Omega$, $\varphi(\omega,\cdot)$ is a convex continuous function. Finally $\lambda : \Omega \times H \to \mathbb{R}$ is a Carathéodory function such that for every $\omega \in \Omega$, $\lambda(\omega,\cdot)$ is linear and continuous. Clearly λ can also be seen as measurable map from Ω to H via $\lambda(\omega,z) = \langle\lambda(\omega), z\rangle$.

The objective of the present contribution is to derive a novel well-posedness result for the considered mixed random VI from the recent stability theorem in [14], where we handle perturbations not only with respect to the right hand side given by linear random forms, but also with respect to to the Carathéodory operator B and with respect to the Carathéodory function φ using the concept of Mosco convergence.

As an application we treat a scalar non-smooth random boundary value problem that captures all the difficulties of a Tresca frictional unilateral problems that result in unilateral boundary conditions and a non-smooth convex sublinear functional. Based on our theory, we present a concrete stability result, where we give explicit conditions on the given functions and random variables that are involved in the perturbed Carathéodory operator B, the Carathéodory function φ, and in the perturbed linear random form λ.

The outline of the paper is as follows. The subsequent section collects some preliminaries about Mosco convergence and about stability of VIs in the deterministic range. Section 3 is concerned with solvability in a general complete σ-finite measure space. In particular we show that under specific assumptions on the data the unique solution $\hat{y} : \omega \in \Omega \mapsto \hat{y}(\omega) \in K(\omega)$ lies in an appropriate Lebesgue-Bochner space [22, section 4.2], see Theorem 3.1. In section 4 we focus to a probability space $(\Omega, \mathbb{A}, P)$ and provide a stability result for perturbations in the Carathéodory operator B, the Carathéodory function φ, and in the right hand side λ, see Theorem 4.1. Section 5 applies the above theory to a non-smooth random boundary value problem and presents a concrete stability result, see Theorem 5.2. The paper ends in section 6 with an outlook to some open directions of research in the field of random VIs.

2. PRELIMINARIES

2.1. **The concept of Mosco convergence.** In this article we use the concept of epi-convergence of extended real-valued convex lower semicontinuous proper functions in the sense of Mosco [3, 21] ("Mosco convergence") in the framework of a real Banach space V. Note that such a convex lower semicontinuous function $F : V \to \mathbb{R} \cup \{+\infty\}$ is called proper iff $F \not\equiv \infty$ on V; this means that the effective domain of F in the sense of convex analysis ([23]),

$$\operatorname{dom} F := \{v \in V : F(v) < +\infty\}$$

is nonempty, closed and convex.

DEFINITION 2.1. *Let F_n $(n \in \mathbb{N}), F : V \to \mathbb{R} \cup \{+\infty\}$ be convex lower semicontinuous proper functions. Then F_n are said to be* Mosco convergent *to F, written $F_n \xrightarrow{\mathrm{M}} F$, if and only if the subsequent two hypotheses hold:*

(M1) *If $v_n \in V$ $(n \in \mathbb{N})$ weakly converges to v for $n \to \infty$, then*

$$F(v) \le \liminf_{n\to\infty} F_n(v_n)\,.$$

(M2) *For any $v \in V$ with $F(v) < \infty$ there exist $v_n \in V$ $(n \in \mathbb{N})$ strongly converging to v for $n \to \infty$ such that*

$$F(v) = \lim_{n\to\infty} F_n(v_n)\,.$$

The Mosco convergence of the sum of two Mosco converging convex lower semicontinuous proper functions is not obvious. We can refer to [4, Theorem 7] for such a result under an extra condition related to the Brézis-Crandall-Pazy condition.

Instead we shall use the following simpler result on Mosco convergence of the sum of two convex lower semicontinuous proper functions.

LEMMA 2.2. *Let $F_{i,n}$ $(n \in \mathbb{N}), F_i : V \to \mathbb{R} \cup \{+\infty\}$ $(i = 1,2)$ be convex lower semicontinuous proper functions. Suppose that for $n \to \infty$, $F_{1,n} \xrightarrow{\mathrm{M}} F_1$; $(F_{2,n}; F_2)$ satisfy* (M1) *and there holds*

(C) *If $F_2(w) < \infty$ and $w_n \in V$ $(n \in \mathbb{N})$ strongly converges to w for $n \to \infty$, then*

$$F_2(w) = \lim_{n\to\infty} F_{2,n}(w_n)\,.$$

Then $F_n := F_{1,n} + F_{2,n} \xrightarrow{\mathrm{M}} F := F_1 + F_2$.

Proof. To show (M1) for $(F_n; F)$ let $v_n \rightharpoonup v$ (weak convergence) in V. Then by (M1) for $(F_{1,n}; F_1)$ and $(F_{2,n}; F_2)$,

$$\begin{aligned} F(v) &= F_1(v) + F_2(v) \\ &\le \liminf_{n\to\infty} F_{1,n}(v_n) + \liminf_{n\to\infty} F_{2,n}(v_n) \\ &\le \liminf_{n\to\infty} [F_{1,n}(v_n) + F_{2,n}(v_n)] \\ &= \liminf_{n\to\infty} F_n(v_n)\,. \end{aligned}$$

To show (M2) for $(F_n; F)$ let $v \in V$ with $F(v) < \infty$. Then $F_i(v) < \infty; i = 1,2$. Since $(F_{1,n}; F_1)$ satisfies (M2), there exist $v_n \in V$ such that $v_n \to v$ (strong convergence) and $F_1(v) = \lim_{n\to\infty} F_{1,n}(v_n)$. By (C), also $F_2(v) = \lim_{n\to\infty} F_{2,n}(v_n)$ and the conclusion follows. ☐

Needless to say, for linear functionals λ, λ_n $(n \in \mathbb{N})$ in the dual V^*, Mosco convergence of $\lambda_n \xrightarrow{\mathrm{M}} \lambda$ follows from strong convergence $\lambda_n \to \lambda$ for $n \to \infty$.

COROLLARY 2.3. *Let F_n $(n \in \mathbb{N}), F : V \to \mathbb{R} \cup \{+\infty\}$ be convex lower semicontinuous proper functions. Suppose that for $n \to \infty$, $F_n \xrightarrow{\mathrm{M}} F$. Moreover let λ, λ_n $(n \in \mathbb{N})$ linear functionals in the dual V^* such that $\lambda_n \to \lambda$ for $n \to \infty$. Then $\tilde{F}_n := F_n + \lambda_n \xrightarrow{\mathrm{M}} \tilde{F} := F + \lambda$.*

Proof. Apply Lemma 2.2 to

$$F_{1,n} := F_n, F_{2,n} := \lambda_n; F_1 := F, F_2 := \lambda \,.$$

□

2.2. **Stability of deterministic extended real-valued variational inequalities.** In this subsection we recall from [14] a general stability result for the following class of (deterministic) linear extended real-valued VIs in a real reflexive Banach space V with norm $\|\cdot\|$ and dual V^*: Find an element $\hat{v}$ such that

$$\langle L\hat{v}, v - \hat{v} \rangle_{V^* \times V} + F(v) - F(\hat{v}) \geq 0 \qquad \forall v \in V \,. \tag{2.1}$$

where $L : V \mapsto V^*$ is linear and continuous, and further $F : V \to \mathbb{R} \cup \{+\infty\}$ is a convex lower semicontinuous function that is supposed to be proper (i.e. $F \not\equiv \infty$ on V).

Here we assume that the linear operator L is coercive, that is, there exists some $c_L > 0$ such that

$$\langle Lv, v \rangle_{V^* \times V} \geq c_L \|v\|^2 \qquad \forall v \in V \,. \tag{2.2}$$

Clearly coercivity in the sense of (2.2) implies uniqueness of the solution $\hat{u}$ of (2.1). Note by the separation theorem it can be shown that F is conically minorized, that is, it enjoys the estimate

$$F(v) \geq -c_F(1 + \|v\|), v \in V$$

with some $c_F > 0$. Hence (2.2) implies the asymptotic coercivity condition in [13], too. Thus the existence result [13, Theorem 5.2] applies to the bifunction $\varphi_L(u,v) := \langle Lu, v - u \rangle_{V^* \times V}$ to conclude the following

THEOREM 2.4. *Suppose (2.2). Then the VI (2.1) is uniquely solvable.*

Next we deal with the stability of the solution $\hat{v}$, the solution of (2.1) with respect to the extendend real-valued function F. So we are led to introduce the solution map $\mathbb{S}$ by $\mathbb{S}(F) := \hat{v}$, the solution of (2.1).

In view of our applications it is not hard to require that the functions F_n are uniformly conically minorized, that is, there holds the estimate

$$F_n(v) \geq -d_0\,(1 + \|v\|), \; \forall n \in \mathbb{N}, v \in V \tag{2.3}$$

with some $d_0 \geq 0$.

THEOREM 2.5. [14, Theorem A.2] *Suppose that the linear continuous operator L is coercive in the sense of (2.2) with constant $c_L > 0$. Let $F, F_n : V \to \mathbb{R} \cup \{+\infty\}$ $(n \in \mathbb{N})$ be convex lower semicontinuous proper functions that are uniformly conically minorized; let $F_n \xrightarrow{\mathrm{M}} F$. Then strong convergence $\mathbb{S}(F_n) \to \mathbb{S}(F)$ holds.*

To obtain a stability result applicable to the mixed random VI (1.1) we set

$$F(v) := f(v) - \lambda(v) + \delta(v|K)$$

for a given convex lower semicontinuous function $f : V \to \mathbb{R}$ and a given continuous linear form λ, while using the indicator function $\delta(\cdot|K)$ for a given closed convex set $K \subset V$ in the sense of convex analysis ([23]). Thus the VI (2.1) becomes: Find an element $\hat{v} \in K$ such that

$$\langle L\hat{v}, v - \hat{v}\rangle_{V^* \times V} + f(v) - f(\hat{v}) \geq \lambda(v - \hat{v}) \qquad \forall v \in K. \tag{2.4}$$

Likewise as above, unique solvability of the VI (2.4) leads to consider the solution map $\tilde{\Sigma}$ by $\tilde{\Sigma}(f, \lambda) := \hat{v}$, the solution of (2.4). This leads to the following consequence.

COROLLARY 2.6. *Suppose that the linear continuous operator L is coercive in the sense of (2.2) with constant $c_L > 0$. Let $\lambda, \lambda_n \in V^*$ be continuous linear forms, and let $f, f_n : V \to \mathbb{R}$ $(n \in \mathbb{N})$ be convex lower semicontinuous functions that are uniformly conically minorized. Suppose $\lambda_n \to \lambda$ in E^* and $f_n \xrightarrow{\text{M}} f$. Then strong convergence $\tilde{\Sigma}(f_n.\lambda_n K_n) \to \tilde{\Sigma}(f, \lambda, K)$ holds.*

Proof. Apply Corollary 2.3 and then Theorem 2.5 to arrive at the conclusion. □

3. SOLVABILITY OF MIXED RANDOM VARIATIONAL INEQUALITIES IN L^∞ BOCHNER-LEBESGUE SPACE

From now on we assume that the set-valued function $K : \Omega \ni \omega \mapsto K(\omega)$ is *measurable* (a "random" set), this means, for any open set $O \subset H$, its inverse image

$$K^{-1}(O) := \{\omega \in \Omega : K(\omega) \cap O \neq \emptyset\}$$

is measurable.

We also assume that B alias β is *coercive*, that is, for any $\omega \in \Omega$ there is some constant $c_0(\omega) > 0$ such that

$$c_0(\omega)\, \|y\|^2 \leq \beta(\omega, y, y) = \langle B(\omega, y), y\rangle, \quad \forall y \in H. \tag{3.1}$$

Next we fix $\omega \in \Omega$ and consider the convex function $\varphi(\omega, \cdot)$ and its associated subgradient map $\partial\varphi(\omega, \cdot)$ (taken with respect the second variable). Since the epigraph of $\varphi(\omega, \cdot)$ is closed and convex, it can be shown by the separation theorem that $\varphi(\omega, \cdot)$ is *conically minorized*, that is, it enjoys the estimate

$$\varphi(\omega, z) \geq -c_\varphi(\omega)\,(1 + \|z\|)\,, z \in H$$

with some $c_\varphi(\omega) > 0$. Hence for any $z \in H, z^* \in \partial\varphi(\omega, z)$, $z_0 \in H$ fixed,

$$\langle z^*, z - z_0\rangle \geq \varphi(\omega, z) - \varphi(\omega, z_0) \geq -c_\varphi(\omega)\,(1 + \|z\|) - \varphi(\omega, z_0)\,.$$

Further by (3.1),

$$\langle B(\omega)z, z - z_0\rangle \geq c_0(\omega)\|z\|^2 - \|B(\omega)\|\|z\|\|z_0\|\,.$$

Thus the coercivity condition

$$\liminf_{\|z\| \to \infty} \left\{ \frac{\langle z^*, z - z_0\rangle}{\|z\|} : z^* \in B(\omega)z + \partial\varphi(\omega, z) \right\} = \infty$$

is satisfied and the fundamental measurable-existence theorem [1, Theorem 9] applies: There exists a measurable function $\omega \mapsto \hat{y}(\omega) \in K(\omega)$ that is a solution to the mixed random VI (1.1).

Here we want to go beyond measurable solvability to solvability in Bochner-Lebesgue space. To this end we strengthen the above assumptions. Now we impose that $B \in L^\infty(\Omega, \mathscr{L}(H,H))$, where $\mathscr{L}(H,H)$ denotes the space of bounded linear operators mapping H to H, and that B alias β is μ*-coercive*, that is, there exists a constant $\tilde{c}_0 > 0$ independent of $\omega \in \Omega$, such that

$$\tilde{c}_0 \, \|y\|^2 \leq \beta(\omega, y, y) = \langle B(\omega) y, y \rangle \quad \text{for } \mu - \text{almost all } \omega \in \Omega, \forall y \in H\,. \tag{3.2}$$

Next we require that φ is μ*-conically minorized*, that is, there exists a constant $\tilde{c}_\varphi > 0$ such that

$$\varphi(\omega, z) \geq -\tilde{c}_\varphi \, (1 + \|z\|) \quad \text{for } \mu - \text{almost all } \omega \in \Omega, \forall z \in H\,. \tag{3.3}$$

Concerning λ we demand that $\lambda \in L^\infty(\Omega, H)$. Finally we asume that there exists some $z_0 \in H$ that belongs to $K(\omega)$ for $\mu-$ almost all $\omega \in \Omega$.

Under these conditions we can derive an a priori estimate as follows. By (3.2), for shortly $y = \hat{y}(\omega)$,

$$\begin{aligned}
\tilde{c}_0 \, \|\hat{y}(\omega) - z_0\|^2 \; & \leq \beta(\omega, y, y - z_0) \, - \, \beta(\omega, z_0, y - z_0) \\
& \leq \varphi(\omega, z_0) - \varphi(\omega, y) \, + \, \lambda(\omega)(z_0 - y) \, + \, \beta(\omega, z_0, z_0 - y) \\
& \leq \|y - z_0\| \Big(\tilde{c}_\varphi + \|\lambda(\omega)\| + \|\beta(\omega)\|_{\mathscr{L}(H,H)} \|z_0\| \Big) \\
& \quad + \varphi(\omega, z_0) \, + \, c_\varphi(1 + \|z_0\|)\,.
\end{aligned}$$

Since $x^2 \leq Ax + B \, (A, B > 0) \Rightarrow x \leq 1 + A + B$, we obtain

$$\|\hat{y}(\omega)\| \leq \tilde{c} \Big(1 + |\varphi(\omega, z_0)| + \|\lambda(\omega)\| + \|\beta(\omega)\|_{\mathscr{L}(H,H)} \Big)\,, \tag{3.4}$$

where $\tilde{c} > 0$ depends on $\|z_0\|$, $\tilde{c}_0$ and $\tilde{c}_\varphi$. Since $B \in L^\infty(\Omega, \mathscr{L}(H,H))$ and $\lambda \in L^\infty(\Omega, H)$, we can exploit (3.4) and can conclude that the solution $\hat{V}$ given by $\hat{V}(\omega) := \hat{y}(\omega)$ belongs to the Bochner-Lebesgue space $L^\infty(\Omega, \mu, H)$, the Banach space of (classes of) μ-measurable, μ-essentially bounded maps $V : \Omega \to H$.

THEOREM 3.1. *Let H be a separable Hilbert space and $(\Omega, \mathscr{A}, \mu)$ be a complete measure space with finite measure μ. Suppose in addition, that β is μ - coercive, and φ is μ - conically minorized. Moreover suppose, that $B \in L^\infty(\Omega, \mathscr{L}(H,H))$ and $\lambda \in L^\infty(\Omega, H)$. Then the random variational inequality (1.1) admits a unique solution $\hat{V} \in L^\infty(\Omega, \mu, H)$.*

4. STABILITY OF RANDOM HEMIVARIATIONAL INEQUALITIES IN L^∞ BOCHNER-LEBESGUE SPACE

Here we specialize to a probability space $(\Omega, \mathscr{A}, P)$ and consider the above mixed random VI (1.1) in the Bochner-Lebesgue space $\mathscr{V} := L^\infty(\Omega, P, H)$ of all H - valued P - measurable random variables V such that

$$\|V\|_{\mathscr{V}} = \operatorname*{ess\,sup}_{\omega \in \Omega} \|V(\omega)\| < \infty\,.$$

Then under the above assumptions, due to Theorem 3.1, (1.1) admits the unique solution $\hat{V} \in \mathscr{V}$ via $\hat{V}(\omega) = \hat{y}_\omega$.

Let us now study stability with respect to the Carathéodory function φ and the right hand side λ, and moreover differently from [14], also with respect to the Carathéodory operator B. To do so, consider sequences $\{\varphi^{(\nu)}\}_{\nu \in \mathbb{N}}$, $\{\lambda^{(\nu)}\}_{\nu \in \mathbb{N}}$, $\{B^{(\nu)}\}_{\nu \in \mathbb{N}}$, where $\varphi^{(\nu)}$ are Carathéodory functions with $\varphi^{(\nu)}(\omega,\cdot)$ convex continuous on H, $\lambda^{(\nu)} \in \mathscr{V}$ and $B^{(\nu)}$ are Carathéodory μ -coercive operators in $\mathscr{W} := L^\infty(\Omega,P,\mathscr{L}(H,H))$.

Thus we are led to the perturbed mixed VI pathwisely formulated: For each $\omega \in \Omega$, find $y^{(\nu)} = y^{(\nu)}_\omega \in K(\omega)$ such that

$$\begin{aligned} &\langle B^{(\nu)}(\omega)y^{(\nu)}, z - y^{(\nu)}\rangle + \varphi^{(\nu)}(z) - \varphi^{(\nu)}(y^{(\nu)}) \\ &\quad \geq \langle \lambda^{(\nu)}(\omega), z - y^{(\nu)}\rangle\,, \quad \forall z \in K(\omega)\,. \end{aligned} \tag{4.1}$$

Likewise due to Theorem 3.1, the unique solution $y^{(\nu)}_\omega \in K(\omega)$, $\omega \in \Omega$ of (4.1) gives $V^{(\nu)} \in \mathscr{V}$ via $V^{(\nu)}(\omega) := y^{(\nu)}_\omega$.

Then we can directly obtain from Corollary 2.6 stability for the pathwise formulation.

To derive stability in the Bochner-Lebesgue space $\mathscr{V}$ we need the subsequent hypotheses.

$\mathrm{H}(\varphi,\varphi^{(\nu)})$: There holds the convergence condition (CC):

$$|(\varphi^{(\nu)}(\omega,x) - \varphi(\omega,x)) - (\varphi^{(\nu)}(\omega,y) - \varphi(\omega,y))| \leq \varepsilon_\nu \|x-y\| \quad \forall \omega \in \Omega; x,y \in H\,,$$

where $\varepsilon_\nu \to 0$ for $\nu \to \infty$.

$\mathrm{H}(\lambda,\lambda^{(\nu)})$: There holds $\lambda^{(\nu)} \to \lambda$ in $\mathscr{V}$ for $\nu \to \infty$.

$\mathrm{H}(B,B^{(\nu)})$: There holds $B^{(\nu)} \to B$ in $\mathscr{W}$ for $\nu \to \infty$. In addition $B^{(\nu)}$ ($\nu \in \mathbb{N}$) are uniformly μ-coercive (analogously to (3.2)), that is,

$$\tilde{c}_0\,\|y\|^2 \leq \langle B^{(\nu)}(\omega)y,y\rangle \quad \text{for } \mu - \text{almost all } \omega \in \Omega, \forall y \in H, \nu \in \mathbb{N}\,.$$

THEOREM 4.1. *Let $B, B^{(\nu)}, \varphi, \varphi^{(\nu)}, \lambda, \lambda^{(\nu)}$ and K be given as above. Suppose that for any $\omega \in \Omega$, $\varphi^{(\nu)}(\omega,\cdot) \xrightarrow{\mathrm{M}} \varphi(\omega,\cdot)$, and $\lambda^{(\nu)}(\omega,\cdot) \to \lambda(\omega,\cdot)$. in H for $\nu \to \infty$. Then for each $\omega \in \Omega$, $y^{(\nu)}(\omega) \to \hat{y}(\omega)$ in H for $\nu \to \infty$.*
If moreover $\mathrm{H}(\varphi,\varphi^{(\nu)})$, $\mathrm{H}(\lambda,\lambda^{(\nu)})$, and $\mathrm{H}(B,B^{(\nu)})$ hold, then $\lim_{\nu\to\infty} \|V^{(\nu)} - \hat{V}\|_{\mathscr{V}} = 0$ in $\mathscr{V} = L^\infty(\Omega,P,H)$.

Proof. The claimed convergence in the pathwise formulation is obvious from Corollary 2.6.

To prove convergence in $\mathscr{V}$, test (1.1) with $y = y^{(\nu)}_\omega$ and test (4.1) with $y = \hat{y}_\omega$, add, and obtain

$$\begin{aligned} &\langle B^{(\nu)}(\omega)y^{(\nu)}_\omega - B^{(\nu)}(\omega)\hat{y}_\omega, y^{(\nu)}_\omega - \hat{y}_\omega\rangle \\ &+\varphi^{(\nu)}(\omega,y^{(\nu)}_\omega) - \varphi^{(\nu)}(\omega,\hat{y}_\omega) - (\varphi(\omega,y^{(\nu)}_\omega) - \varphi(\omega,\hat{y}_\omega)) \\ &\leq \langle (B(\omega) - B^{(\nu)}(\omega))\hat{y}_\omega, y^{(\nu)}_\omega - \hat{y}_\omega\rangle \\ &+\langle \lambda^{(\nu)}(\omega) - \lambda(\omega), y^{(\nu)}_\omega - \hat{y}_\omega\rangle\,. \end{aligned}$$

Hence by $\mathrm{H}(\varphi,\varphi^{(\nu)})$, $\mathrm{H}(\lambda,\lambda^{(\nu)})$, and $\mathrm{H}(B,B^{(\nu)})$,

$$\tilde{c}_0\,\|\hat{y}_\omega - y^{(\nu)}_\omega\|^2 \leq (\varepsilon_\nu + \|B^{(\nu)} - B\|_{\mathscr{W}}\|\hat{y}_\omega\| + \|\lambda^{(\nu)} - \lambda\|_{\mathscr{V}})\,\|\hat{y}_\omega - y^{(\nu)}_\omega\|$$

proving the claim. □

5. A SCALAR NONSMOOTH UNILATERAL BOUNDARY VALUE PROBLEM UNDER STOCHASTIC UNCERTAINTY

As an application of the previous theory we recall from [14] the non-smooth random boundary value problem (and corect some misprints), which stems from [12] in the deterministic description. This scalar boundary value problem captures all the difficulties of a Tresca frictional unilateral contact problem that result in unilateral boundary conditions and a non-smooth convex sublinear functional, see [14, Remark 4.1] for details.

Let Q, R_b, R_d, S, T be real-valued random variables on a probability space $(\Omega, \mathscr{A}, P)$. Further let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded smooth domain with outer unit normal n and with its boundary $\Gamma := \partial D$ which splits in mutually disjoint open parts, namely the Dirichlet part Γ_D, the Neumann part Γ_N, the Signorini part Γ_S, and the Tresca part Γ_T, such that $\partial D = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_S \cup \overline{\Gamma}_T$.

With given data $g \geq 0, h_d, h_b, \chi$ specified later, we consider the following non-smooth boundary value problem in its strong form: Find a function u smooth enough on the domain D depending on $\omega \in \Omega$ such that almost surely (a.s.)

$$
(BVP) \quad \left\{ \begin{array}{lll}
 & -\nabla \cdot (S \nabla u) = R_d h_d & \text{in } D, \\
 & q_n := S\nabla u \cdot n & \text{on } \partial D \\
 & u = \chi & \text{on } \Gamma_D, \\
 & q_n = R_b h_b & \text{on } \Gamma_N, \\
 & u \leq T\chi, q_n \leq 0, (u - T\chi) q_n = 0 & \text{on } \Gamma_S, \\
(FC) & |q_n| \leq Q g, u\, q_n + Q g |u| = 0 & \text{on } \Gamma_T.
\end{array} \right.
$$

Note that without any restriction of generality, we can assume in (FC) that $g > 0$ on Γ_T; otherwise the part where g vanishes and hence q_n is required to vanish can be subsumed to Γ_N with an obvious modification of h_b.

In accordance with the standard case of uniformly strongly elliptic operators, we assume that P-almost surely, S is contained in a compact interval that is included in $(0, +\infty)$. Hence S belongs to $L^\infty(\Omega)$. Similarly we assume that the random variables Q, T are nonnegative a.s. and in $L^\infty(\Omega)$ and also the right hand sides R_d, R_b in $L^\infty(\Omega)$.

Let us proceed to the weak formulation of the above random nonsmooth boundary value problem in the Hilbert space $H := H^1(D)$ with the data

$$
h_d \in L^2(D), \chi \in H^{1/2}(\Gamma_D \cup \Gamma_S), h_b \in H^{-1/2}(\Gamma_N), g \in L^\infty(\Gamma_T),
$$

where $g > 0$. Define the convex closed set depending on $\omega \in \Omega$

$$
K(\omega) := \{ v \in H^1(D) : v|\Gamma_D = \chi \text{ and } v|\Gamma_S \leq T(\omega)\chi \}, \tag{5.1}
$$

the continuous bilinear form associated to the Laplacian

$$
a(v,w) := \int_D \nabla v \cdot \nabla w \, dx, \quad v, w \in H^1(D),
$$

the continuous linear trace maps $\gamma_N : H^1(D) \to L^2(\Gamma_N)$ and $\gamma_T : H^1(D) \to L^2(\Gamma_T)$, the continuous linear forms

$$
l_d(v) := \int_D h_d \, v \, dx; \qquad l_b(v) := \int_{\Gamma_N} h_b \, \gamma_N v \, ds, \quad v \in H^1(D),
$$

and the continuous, positively homogeneous and sublinear, hence convex functional

$$
f(v) := \int_{\Gamma_T} g |\gamma_T v| \, ds, \quad v \in H^1(D).
$$

Then it can be shown, see [14], that the boundary value problem (BVP) is equivalent in the sense of distributions to the following concrete variational inequality problem which is of the form (1.1): For any fixed $\omega \in \Omega$ find $u = u_\omega \in K(\omega)$ such that for all $v \in K(\omega)$,

$$S(\omega)\, a(u, v-u) + Q(\omega)\, [f(v) - f(u)] \geq R_d\, l_d(v-u) + R_b\, l_b(v-u)\,. \tag{5.2}$$

By the Poincaré inequality the bilinear form a is coercive, provided Γ_D has positive measure - what we assume in what follows. Thus, since S is bounded away from zero, the Carathéodory operator B, defined by

$$\langle B(\omega)v, w\rangle = \beta(\omega, v, w) := S(\omega)\, a(u, v)\,, \quad \omega \in \Omega; v, w \in H^1(D)$$

is $P-$coercive. Further, since $R_d, R_b \in L^\infty(\Omega, P)$, the linear Carathéodory operator λ, defined by

$$\lambda(\omega)v := R_d(\omega)\, l_d(v) + R_b(\omega)\, l_b(v)\,, \quad \omega \in \Omega; v \in H^1(D)\,,$$

belongs to $L^\infty(\Omega, P, H^1(D))$. Since $Q \in L^\infty(\Omega, P)$ is nonnegative, the Carathéodory function, defined by

$$\varphi(\omega, v) := Q(\omega)\, f(v)\,, \quad \omega \in \Omega; v \in H^1(D)\,,$$

is $P-$conically minorized.

To examine the set-valued map K, which is defined in (5.1), abstract from the random framework and first consider the closed convex set

$$C := \{x \in X : x(s) \leq \chi(s) \text{ for almost all } s \in \Gamma_S\}$$

in the Hilbert space $X := L^2(\Gamma_S)$ with scalar product $(\cdot,\cdot)$. Then using the representation $z = z^+ - z^-$ with positive part $z^+ = \max(z, 0)$ and negative part $z^- = \max(-z, 0)$ for any fixed $z \in X$, we find that

$$\hat{z} := \text{proj}_C z = \chi - (z - \chi)^-$$

Indeed, for any $x \in C$,

$$\begin{aligned}
&(z - \hat{z}, x - \hat{z}) \\
&= (z - \chi + (z-\chi)^-, x - \chi + (z-\chi)^-) \\
&= ((z-\chi)^+, x - \chi) + (z-\chi)^+, (z-\chi)^-) \\
&\leq 0\,.
\end{aligned}$$

Coming back to the random framework it results that for any $v \in H^1(D)$, the distance function

$$d(v, K(\omega)) = \|v|\Gamma_D - \chi\|_{L^2(\Gamma_D)} + \|(v|\Gamma_S - T(\omega)\,\chi)^+\|_{L^2(\Gamma_S)}\,,$$

is measurable. Therefore due to [22, Prop. 2.7.5] the set-valued map $K : \omega \in \Omega \mapsto K(\omega) \subset H^1(D)$ is measurable, a "random set". We assume that the gap function $\chi|\Gamma_S$ is nonnegative, then a harmonic extension $w_0 \in H^1(D)$ $(-\Delta w_0 = 0)$ with the boundary values $w_0|\Gamma_D = \chi, w_0|\Gamma_S = 0$ belongs to all $K(\omega)$. Hence Theorem 3.1 applies and (5.2) has a unique solution $U \in L^\infty(\Omega, P, H^1(D))$ with $U(\omega) := u_\omega$.

Next we intend to give a novel stability result for the nonsmooth boundary value problem under uncertainty described above. We shall consider the dependence of the solution $u_\omega \in C$, $\omega \in \Omega$ of (5.2) and of the associated random distributed solution U with respect to the operator B, the right hand side λ and to the function φ. By the existence and uniqueness discussed above we have the solution map $(B, \lambda, \varphi) \mapsto S_\omega(B, \lambda, \varphi) = u_\omega \in K(\omega)$ $(\forall \omega \in \Omega)$, the solution of (5.2).

To describe explicitly the dependence on the right hand side, consider $R_{d,n}, R_{b,n}, S_n$ all in $L^\infty(\Omega)$ and $h_{d,n} \in L^2(D), h_{b,n} \in L^2(\Gamma_N)$ for $n \in \mathbb{N}$. Then $R_{d,n} \to R_d$ in $L^\infty(\Omega)$, $R_{b,n} \to R_b$ in $L^\infty(\Omega)$, $h_{d,n} \to h_d$ in $L^2(D)$, $h_{b,n} \to h_b$ in $L^2(\Gamma_N)$ for $n \to \infty$ implies $\lambda_n \to \lambda$ in $\mathscr{V}$, where

$$\lambda_n(\omega, u) := R_{d,n}(\omega) \int_\Omega h_{d,n}\, u\, dx + R_{b,n}(\omega) \int_{\Gamma_N} h_{b,n}\, \gamma_N u\, ds \qquad \omega \in \Omega; u \in H^1(D).$$

Similarly, $S_n \to S$ in $L^\infty(\Omega)$, for $n \to \infty$, implies $B_n \to B$ in $\mathscr{W}$, where

$$\langle B_n(\omega)v, w\rangle = \beta_n(\omega, v, w) := S_n(\omega)\, a(v, w), \quad \omega \in \Omega; v, w \in H^1(D).$$

Further introduce the Carathéodory functions φ_n, defined by

$$\varphi_n(\omega, v) := Q_n(\omega)\, f_n(v), \quad \omega \in \Omega; v \in H^1(D),$$

where $Q_n \in L^\infty(\Omega)$ are nonnegative and

$$f_n(w) := \int_{\Gamma_T} g_n |\gamma_T w|$$

with $g_n \in L^\infty(\Gamma_T)$ and $g_n > 0$ a.e.

To describe also explicitly the dependence on the convex function f, we insert the following

PROPOSITION 5.1. *Suppose $g_n \to g$ in $L^\infty(\Gamma_T)$ for $n \to \infty$. Then $f_n \xrightarrow{\mathrm{M}} f$ on $H^1(D)$.*

Proof. Let $w_n \rightharpoonup w$ in H, then by the Trace Theorem, see e.g. [3, Theorem 5.6.1], $\gamma_T w_n \rightharpoonup \gamma_T w$ in $L^2(\Gamma_T)$. This means

$$\langle \ell, \gamma_T w_n - \gamma_T w\rangle_{L^2(\Gamma_T)\times L^2(\Gamma_T)} \to 0 \quad \forall \ell \in L^2(\Gamma_T).$$

We claim that $g_n\, \gamma_T w_n \rightharpoonup g\, \gamma_T w$ in $L^2(\Gamma_T)$. By the Uniform Boundedness Principle, $\gamma_T w_n$ is bounded in $L^2(\Gamma_T)$, say by $C > 0$. Further, for all $\ell \in L^2(\Gamma_T)$,

$$\begin{aligned} &|\langle \ell, g_n\, \gamma_T w_n\rangle - \langle \ell, g\, \gamma_T w\rangle| \\ &\leq |\langle g\,\ell, (\gamma_T w_n - \gamma_T w)\rangle| + |\langle \ell, (g_n - g)\gamma_T w_n\rangle|. \end{aligned}$$

Both summands above converge to zero for $n \to \infty$; the first because of weak convergence $\gamma_T w_n \rightharpoonup \gamma_T w$, the second because of convergence $g_n \to g$ in $L^\infty(\Gamma_T)$ and because of the estimate

$$|\langle \ell, (g_n - g)\gamma_T w_n\rangle| \leq C\, \|g_n - g\|_{L^\infty(\Gamma_T)} \|\ell\|_{L^2(\Gamma_T)}.$$

Thus the claim is proven.

Since $L^2(\Gamma_T)$ embeds continuously in $L^1(\Gamma_T)$, $g_n\, \gamma_T w_n \rightharpoonup g\, \gamma_T w$ in $L^1(\Gamma_T)$, too. Further the L^1 norm is convex and continous, hence weakly lower semicontinuous. Thus

$$f(w) \leq \liminf_{n\to\infty} f_n(w_n)$$

and (M1) is shown. (M2) is obvious from choosing $w_n := w$, the stationary sequence. So $f_n \xrightarrow{\mathrm{M}} f$ on H. □

Finally we arrive at the following stability result.

THEOREM 5.2. *Let $Q_n \to Q, R_{d,n} \to R_d, R_{b,n} \to R_b, S_n \to S$ all in $L^\infty(\Omega)$, $h_{d,n} \to h_d$ in $L^2(D)$, $h_{b,n} \to h_b$ in $L^2(\Gamma_N)$, and $g_n \to g$ in $L^2(\Gamma_T)$ for $n \to \infty$. Then there holds $S_\omega(B_n, \lambda_n, \varphi_n) \to S_\omega(B, \lambda, \varphi), \forall \omega \in \Omega$ and for the associated random distributed solutions $U_n, \hat{U}$ there holds $U_n \to U$ in $\mathscr{V}$.*

Proof. We only have to show (CC) for φ_n, φ defined above. We simply omit the argument ω, set $y := \gamma_T u, z := \gamma_T v$ for $u, v \in H^1(D)$ and since $Q_n \leq \|Q\|_{L^\infty(\Omega)} + 1$ for n large enough, estimate

$$\begin{aligned}
&\Big|(Q_n \int_{\Gamma_T} g_n|y| - Q\int_{\Gamma_T} g|y|) - (Q_n \int_{\Gamma_T} g_n|z| - Q\int_{\Gamma_T} g|z|)\Big| \\
&\leq Q_n \int_{\Gamma_T} |g_n - g|\,||y| - |z|| + |Q_n - Q| \int_{\Gamma_T} g\,||y| - |z|| \\
&\leq (\|Q\|_{L^\infty(\Omega)} + 1)\|g_n - g\|_{L^2(\Gamma_T)}\|y - z\|_{L^2(\Gamma_T)} + \|Q_n - Q\|_{L^\infty(\Omega)}\|g\|_{L^2(\Gamma_T)}\|y - z\|_{L^2(\Gamma_T)}\,.
\end{aligned}$$

Since the trace map γ_T is bounded, (CC) is satisfied and the conclusion follows from Theorem 4.1 . □

6. Some Concluding Remarks - An Outlook

In this paper we focused to linear random variational inequalities. By more involved analysis one can generalize some of the presented results to more general classes of random nonlinear variational inequalities and hemivariational inequalities, (see e.g. [9, 10, 20] for such nonlinear problems in the deterministic case). Moreover, for the existence analysis of considerably more general variational-hemivariational inequalities by an equilibrium approach we can refer to [24, 13] albeit in the deterministic case. One can expect that this theory can help in the analysis of more involved problems in stochastic continuum mechanics; see [2] for the treatment of stochastic contact and plasticity problems in the framework of VI of the first kind.

Finally let us point out that this paper is devoted to the *direct problems* of random/stochastic VIs while the *inverse problem* of parameter identification in random/stochastic VIs is a largely unexplored field. In this context we can only refer to [6] for the identification of the random coefficients in elliptic stochastic pdes up to order four. The efficient identification of a random friction parameter and random Lamé coefficients in frictional contact problems using boundary element methods [17] for discretization would be a challenging task.

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