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THE PROXIMAL POINT METHOD FOR SOLVING VARIATIONAL INEQUALITIES UNDER THE PRESENCE OF COMPUTATIONAL ERRORS

ALEXANDER J. ZASLAVSKI

Department of Mathematics, The Technion – Israel Institute of Technology, Haifa, Israel

Dedicated to Professor Richard Vinter

Abstract. In the present paper, we use a proximal point method with remotest set control for finding an approximate common zero of a finite family of maximal monotone operators in a Hilbert space under the presence of perturbations. We show that if the perturbations are sufficiently small, then the inexact proximal point method produces approximate solutions.

Keywords. Computational errors; Monotone operator; Proximal point method; Variational inequality;

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1. Introduction

Proximal point method is an important tool in solving optimization problems [1, 4, 5, 8, 11]. In particular, it was used for solving variational inequalities with monotone operators [2, 3, 6, 7, 12, 13, 14], which is an important topic of nonlinear analysis and optimization. In the present paper, we use a proximal point method with remotest set control for finding an approximate common zero of a finite family of maximal monotone operators in a Hilbert space under the presence of perturbations. We show that if the perturbations are sufficiently small, then the the inexact proximal point method produces approximate solutions.

Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$ which induces the norm $\|\cdot\|$.

A multifunction $T : X \to 2^X$ is called a monotone operator if and only f

$$\langle z-z', w-w'\rangle \geq 0 \quad \forall z, z', w, w' \in X$$
$$\text{such that } w \in T(z) \text{ and } w' \in T(z'). \tag{1}$$

It is called maximal monotone if, in addition, the graph $\{(z,w) \in X \times X : \ w \in T(z)\}$ is not properly contained in the graph of any other monotone operator $T' : X \to 2^X$.

*Corresponding author.

E-mail address: ajzasl@technion.ac.il

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Let $T : X \to 2^X$ be a maximal monotone operator. The proximal point algorithm generates, for any given sequence of positive real numbers and any starting point in the space, a sequence of points and the goal is to show the convergence of this sequence. Note that in a general infinite-dimensional Hilbert space this convergence is usually weak. The proximal algorithm for solving the inclusion $0 \in T(z)$ is based on the fact established by Minty [9], who showed that, for each $z \in X$ and each $c > 0$, there is a unique $u \in X$ such that $z \in (I + cT)(u)$, where $I : X \to X$ is the identity operator ($Ix = x$ for all $x \in X$).

The operator

$$P_{c,T} := (I + cT)^{-1} \tag{2}$$

is therefore single-valued from all of X onto X (where c is any positive number). It is also nonexpansive:

$$\|P_{c,T}(z) - P_{c,T}(z')\| \le \|z - z'\| \text{ for all } z, z' \in X \tag{3}$$

and

$$P_{c,T}(z) = z \text{ if and only if } 0 \in T(z). \tag{4}$$

Following the terminology of Moreau [10], $P_{c,T}$ is called the proximal mapping associated with cT.

The proximal point algorithm generates, for any given sequence $\{c_k\}_{k=0}^{\infty}$ of positive real numbers and any starting point $z^0 \in X$, a sequence $\{z^k\}_{k=0}^{\infty} \subset X$, where

$$z^{k+1} := P_{c_k,T}(z^k),\ k = 0, 1, \ldots.$$

It is not difficult to see that the

$$\operatorname{graph}(T) := \{(x, w) \in X \times X :\ w \in T(x)\}$$

is closed in the norm topology of $X \times X$.

Set

$$F(T) = \{z \in X :\ 0 \in T(z)\}. \tag{5}$$

Usually algorithms considering in the literature generate sequences which converge weakly to an element of $F(T)$. In this paper, for a given $\varepsilon > 0$, we are interested to find a point x for which there is $y \in T(x)$ such that $\|y\| \le \varepsilon$. This point x is considered as an ε-approximate solution.

For every point $x \in X$ and every nonempty set $A \subset X$, define $d(x, A) := \inf\{\|x - y\| :\ y \in A\}$. For every point $x \in X$ and every positive number r, put $B(x, r) = \{y \in X :\ \|x - y\| \le r\}$. We denote by $\mathrm{Card}(A)$ the cardinality of the set A. For each $x \in R^1$ set $\lfloor x \rfloor = \max\{i \text{ is an integer} : i \le x\}$.

We apply the proximal point algorithm with remotest set control in order to obtain a good approximation of a point which is a common zero of a finite family of maximal monotone operators and a common fixed point of a finite family of quasi-nonexpansive operators.

Let $\mathscr{L}_1$ be a finite set of maximal monotone operators $T : X \to 2^X$ and $\mathscr{L}_2$ be a finite set of mappings $T : X \to X$. We suppose that the set $\mathscr{L}_1 \cup \mathscr{L}_2$ is nonempty. (Note that one of the sets $\mathscr{L}_1$ or $\mathscr{L}_2$ may be empty.)

Let $\bar{c} \in (0, 1]$ and let $\bar{c} = 1$, if $\mathscr{L}_2 = \emptyset$. We suppose that

$$F(T) \ne \emptyset \text{ for any } T \in \mathscr{L}_1 \tag{6}$$

and that for every mapping $T \in \mathscr{L}_2$,

$$\mathrm{Fix}(T) := \{z \in X : \ T(z) = z\} \neq \emptyset, \tag{7}$$

$$\|z-x\|^2 \geq \|z-T(x)\|^2 + \bar{c}\|x-T(x)\|^2$$
$$\text{for all } x \in X \text{ and all } z \in \mathrm{Fix}(T). \tag{8}$$

Let $\bar{\lambda} > 0$ and let $\bar{\lambda} = \infty$ and $\bar{\lambda}^{-1} = 0$, if $\mathscr{L}_1 = \emptyset$. Define

$$F := (\cap_{T \in \mathscr{L}_1} F(T)) \cap (\cap_{Q \in \mathscr{L}_2} \mathrm{Fix}(Q)).$$

Let $\varepsilon > 0$. For every monotone operator $T \in \mathscr{L}_1$ define

$$F_\varepsilon(T) = \{x \in X : \ T(x) \cap B(0, \varepsilon) \neq \emptyset\} \tag{9}$$

and for every mapping $T \in \mathscr{L}_2$ set

$$\mathrm{Fix}_\varepsilon(T) = \{x \in X : \ \|T(x) - x\| \leq \varepsilon\}. \tag{10}$$

Define

$$F_\varepsilon = (\cap_{T \in \mathscr{L}_1} F_\varepsilon(T)) \cap (\cap_{Q \in \mathscr{L}_2} \mathrm{Fix}_\varepsilon(Q)),$$
$$\tilde{F}_\varepsilon = (\cap_{T \in \mathscr{L}_1} (B(0, \varepsilon) + F_\varepsilon(T))) \cap (\cap_{Q \in \mathscr{L}_2} \mathrm{Fix}_\varepsilon(Q)). \tag{11}$$

We are interested to find an approximate solutions belonging to the set $\tilde{F}_\varepsilon$ where ε is sufficiently small. Set $\mathscr{L} = \mathscr{L}_2 \cup \{P_{c,T} : \ T \in \mathscr{L}_1, \ c \in [\bar{\lambda}, \infty)\}$.

2. Auxiliary Results

It is easy to see that the following lemma holds.

Lemma 2.1. *Let $z, x_0, x_1 \in X$. Then*

$$2^{-1}\|z-x_0\|^2 - 2^{-1}\|z-x_1\|^2 - 2^{-1}\|x_0-x_1\|^2 = \langle x_0 - x_1, x_1 - z \rangle.$$

Lemma 2.2. *Let*

$$A \in \mathscr{L}, \ x \in X,$$
$$z \in F. \tag{12}$$

Then

$$\|z-x\|^2 - \|z-A(x)\|^2 - \bar{c}\|x-A(x)\|^2 \geq 0.$$

Proof. There are two cases:

(i) $T \in \mathscr{L}_2$;

(ii) there exist a mapping $T \in \mathscr{L}_1$,a number $c \in [\bar{\lambda}, \infty)$ such that $A = P_{c,T}$.

If (i) holds, then our result follows from (8) and (12). Assume that (ii) holds. Then by Lemma 2.1,

$$2^{-1}\|z-x\|^2 - 2^{-1}\|z-A(x)\|^2 - 2^{-1}\|x-A(x)\|^2 = \langle x - A(x), A(x) - z \rangle. \tag{13}$$

By (ii) and (2), there exist $T \in \mathscr{L}_1$, $c \geq \bar{\lambda}_1$ such that $A(x) = P_{c,T}(x)$, $x \in (I + cT)(A(x))$, and $x - A(x) \in cT(A(x))$. By (1), (5), (12), (13), and the relation above, our result holds. Lemma 2.2 is proved. $\square$

3. THE MAIN RESULT

In [13, 14], we assumed that $F \neq \emptyset$. In the following result we do not make this assumption. We merely assume that there exist a point z_* which is closed to the sets $\mathrm{Fix}(T), T \in \mathscr{L}_2$, $F(T)$, $T \in \mathscr{L}_1$.

Theorem 3.1. *Let $M \geq 1$, $\varepsilon \in (0, 1/4]$,*

$$\varepsilon_0 = 2^{-1}\varepsilon \min\{\bar{\lambda},\ 1\}, \tag{14}$$

$$0 < \delta, \delta_* \leq 64^{-1}(M+1)^{-1}\bar{c}\varepsilon_0^2, \tag{15}$$

$$z_* \in B(0, M), \tag{16}$$

$$B(z_*, \delta_*) \cap Fix(T) \neq \emptyset,\ T \in \mathscr{L}_2, \tag{17}$$

$$B(z_*, \delta_*) \cap F(T) \neq \emptyset,\ T \in \mathscr{L}_1, \tag{18}$$

a natural number

$$n_0 = \lfloor 8(4M+1)^2\bar{c}^{-1}\varepsilon_0^{-2} \rfloor + 1. \tag{19}$$

Assume that

$$\{S_k\}_{k=0}^{\infty} \subset \mathscr{L},\ \{x_k\}_{k=0}^{\infty} \subset X,$$

$$\|x_0\| \leq M, \tag{20}$$

for each integer $k \geq 0$,

$$\|x_k - S_k(x_k)\| \geq \|x_k - S(x_k)\|,\ S \in \mathscr{L}_2 \tag{21}$$

for each $T \in \mathscr{L}_1$ there is $c_T \geq \bar{\lambda}$ such that

$$\|x_k - S_k(x_k)\| \geq \|x_k - P_{c,T}(x_k)\| \tag{22}$$

and that

$$\|x_{k+1} - S_k(x_k)\| \leq \delta. \tag{23}$$

Then there exists an integer $q \in [1, n_0]$ such that

$$||x_k|| \leq 3M+1 \text{ for all integers } k = 1, \dots, q,$$

$$\|x_q - x_{q+1}\| \leq 2^{-1}\varepsilon_0. \tag{24}$$

Moreover, if an integer $q \geq 0$ and (24) holds, then $x_q \in \tilde{F}_\varepsilon$.

Proof. Let $T \in \mathscr{L}_1$, $c > 0$,

$$z \in F(T). \tag{25}$$

Lemma 2.2 implies that

$$\|z - x\|^2 - \|z - P_{c,T}(x)\|^2 - \|x - P_{c,T}(x)\|^2 = 2\langle x - P_{c,T}(x), P_{c,T}(x) - z\rangle. \tag{26}$$

By (1), (2) and (5), one has

$$x \in (I + cT)(P_{c,T}(x)),\ x - P_{c,T}(x) \in cT(P_{c,T}(x)),$$

$$\langle x - P_{c,T}(x), P_{c,T}(x) - z\rangle \geq 0.$$

Together with (26), this implies that

$$\|z - x\|^2 - \|z - P_{c,T}(x)\|^2 \geq \|x - P_{c,T}(x)\|^2.$$

Thus we have shown that the following property holds:

(i) for each $T \in \mathscr{L}_1$, each $c > 0$ and each $x \in X$, $\|z - x\|^2 - \|z - P_{c,T}(x)\|^2 \geq \|x - P_{c,T}(x)\|^2$.

There are two cases:

(1) There exists $T \in \mathscr{L}_2$ such that $S_0 = T$;
(2) There exist $c \geq \bar{\lambda}$ and $T \in \mathscr{L}_1$ such that $S_0 = P_{c,T}$.
Assume that case (1) holds. By (17), there exists

$$z_0 \in B(z_*, \delta_*) \cap \mathrm{Fix}(T). \tag{27}$$

In view of (27), one asserts

$$\|z_0\| \leq \|z_*\| + \delta_*. \tag{28}$$

We have $S_0 = T$. It follows from (8), (16), (20), (23), (27) and (28) that

$$\begin{aligned}\|z_* - x_1\| &\leq \|z_* - z_0\| + \|z_0 - T(x_0)\| + \|T(x_0) - x_1\| \\ &\leq \delta_* + \|z_0 - x_0\| + \delta \leq 2M + 2\delta_* + \delta \leq 2M + 1.\end{aligned}$$

Assume that case (2) holds and that

$$c \geq \bar{\lambda},\ T \in \mathscr{L}_1,\ S_0 = P_{c,T}. \tag{29}$$

By (18), there exists

$$z_0 \in B(z_*, \delta_*) \cap F(T). \tag{30}$$

By (16), (20), (23), (29), (30) and property (i), one obtains

$$\begin{aligned}\|z_* - x_1\| &\leq \|z_* - z\| + \|z - P_{c,T}(x_0)\| + \|P_{c,T}(x_0) - x_0\| \\ &\leq \delta_* + \|z - x_0\| + \delta \leq \delta_* + \delta + 2M + \delta_* \leq 2M + 1.\end{aligned}$$

Thus, in the both cases,

$$\|z_* - x_1\| \leq 2M + 1. \tag{31}$$

Assume that q is a natural number such that, for each integer $p \in [1, q]$,

$$\|x_p - x_{p+1}\| > \varepsilon_0. \tag{32}$$

Assume that an integer $p \in [1, q]$ and that

$$\|x_p - z\| \leq 2M + 1. \tag{33}$$

(In view of (31), equation (33) holds for $p = 1$.) There are two cases:
(3) There exists $T \in \mathscr{L}_2$ such that $S_p = T$.
(2) There exist $c \geq \bar{\lambda}$ and $Q \in \mathscr{L}_1$ such that $S_p = P_{c,Q}$.
Assume that case (3) holds. By (18), there exists

$$z \in B(z_*, \delta_*) \cap \mathrm{Fix}(T). \tag{34}$$

In view of (8) and (34), one has

$$\|z - x_p\|^2 - \|z - S_p(x_p)\|^2 \geq \bar{c}\|x_p - S_p(x_p)\|^2. \tag{35}$$

Consider case (4). By (18), there exists

$$z \in B(z_*, \delta_*) \cap F(T). \tag{36}$$

By (36) and property (i), one obtains

$$\|z - x_p\|^2 - \|z - S_p(x_p)\|^2 \geq \|x_p - S_p(x_p)\|^2.$$

Thus in the both cases the point z is defined and (35) holds. In view of (34) and (36), one has

$$|\|z_* - x_p\| - \|z - x_p\|| \leq \|z - z_*\| \leq \delta_*. \tag{37}$$

Equations (33) and (37) imply that

$$\begin{aligned}&|\|z_* - x_p\|^2 - \|z - x_p\|^2| \\ &\leq |\|z_* - x_p\| - \|z - x_p\||(\|z_* - x_p\| + \|z - x_p\|) \\ &\leq \delta_*(2\|z_* - x_p\| + \delta_*) \leq \delta_*(4M + 3). \qquad (38)\end{aligned}$$

It follows from (15), (23) and (32) that

$$\begin{aligned}\|x_p - S_p(x_p)\| &\geq \|x_{p+1} - x_p\| - \|S_p(x_p) - x_{p+1}\| \\ &> \varepsilon_0 - \delta \geq \varepsilon_0/2, \\ \|x_p - S_p(x_p)\|^2 &\geq \varepsilon_0^2/4. \qquad (39)\end{aligned}$$

Property (i) and (8), (33), (34) and (36) imply that

$$\|z - S_p(x_p)\| \leq \|z - x_p\| \leq \|z - z_*\| + \|z_* - z_p\| \leq 2M + 1 + \delta_* \leq 2M + 2. \qquad (40)$$

By (23), (34) and (36), one obtains that

$$|\|z_* - x_{p+1}\| - \|z - S_p(x_p)\|| \leq \|z - z_*\| + \|x_{p+1} - S_p(x_p)\| \leq \delta_* + \delta. \qquad (41)$$

In view of (40) and (41), one sees that

$$\begin{aligned}&|\|z_* - x_{p+1}\|^2 - \|z - S_p(x_p)\|^2| \\ &\leq (\delta_* + \delta)(2\|z - S_p(x_p\| + \delta_* + \delta) \leq (\delta_* + \delta)(4M + 5). \qquad (42)\end{aligned}$$

It follows from (15), (35), (38), (39) and (42) that

$$\begin{aligned}\|z_* - x_p\|^2 &\geq \|z - x_p\|^2 - \delta_*(4M + 3) \\ &\geq \|z - S_p(x_p)\|^2 + \bar{c}\|x_p - S_p(x_p)\|^2 - \delta_*(4M + 3) \\ &\geq \|z_* - x_{p+1}\|^2 - (\delta_* + \delta)(4M + 5) + \bar{c}\varepsilon_0^2/4 - \delta_*(4M + 3) \\ &\geq \|z_* - x_{p+1}\|^2 + \bar{c}\varepsilon_0^2/8. \qquad (43)\end{aligned}$$

By (33) and (43), one concludes that

$$\begin{aligned}\|x_{p+1} - z_*\| &\leq 2M + 1, \\ \|z_* - x_p\|^2 &\geq \|z_* - x_{p+1}\|^2 + \bar{c}\varepsilon_0^2/8. \qquad (44)\end{aligned}$$

By induction and (44), we conclude that, for all integers $p \in [1, q+1]$, $\|x_p - z\| \leq 2M + 1$ and (44) holds each $p \in \{1, \ldots, q\}$. By (31) and (44), we arrive at

$$\begin{aligned}(4M + 1)^2 &\geq \|z_* - x_1\|^2 \geq \|z_* - x_1\|^2 - \|z_* - x_{q+1}\|^2 \\ &= \sum_{p=1}^{q} (\|z_* - x_p\|^2 - \|z_* - x_{p+1}\|^2) \geq q\bar{c}\varepsilon_0^2/8\end{aligned}$$

and $q \leq 8(4M + 1)^2 \bar{c}^{-1} \varepsilon_0^{-2} < n_0$. Thus we have shown that t he following property holds:

(i) if $q \geq 1$ is an integer such that, for each integer $p \in [1, q]$, $\|x_p - x_{p+1}\| > \varepsilon_0$, then

$$\|x_p\| \leq 3M + 1,\ p = 1, \ldots, q + 1$$

and $q < n_0$. This implies that there exists an integer $q \in [1, n_0]$ such that for every integer $p \in [1, q]$,

$$\begin{aligned}\|x_p\| &\leq 3M + 1, \\ \|x_q - x_{q+1}\| &\leq \varepsilon_0. \qquad (45)\end{aligned}$$

Assume that $q \geq 0$ is an integer and that (45) holds. In view of (23) and (45), we have

$$\|x_q - S_q(x_q)\| \leq \|x_q - x_{q+1}\| + \|x_{q+1} - S_q(x_q)\| \leq \varepsilon_0 + \delta. \tag{46}$$

It follows from (21) and (46) that, for each $T \in \mathscr{L}_2$,

$$\varepsilon \geq \varepsilon_0 + \delta \geq \|x_q - S_q(x_q)\| \geq \|x_q - T(x_q)\|,$$
$$x_q \in \mathrm{Fix}_\varepsilon(T).$$

Assume that $T \in \mathscr{L}_1$. By (22) and (46), there exists $c \geq \bar{\lambda}$ such that

$$\varepsilon_0 + \delta \geq \|x_q - S_q(x_q)\| \geq \|x_q - P_{c,T}(x_q)\|,$$
$$x_q \in (I + cT)(P_{c,T}(x_q)),$$
$$x_q - P_{c,T}(x_q) \in cT(P_{c,T}(x_q)),$$
$$c^{-1}(x_q - P_{c,T}(x_q)) \in T(P_{c,T}(x_q))$$

and

$$\|c^{-1}(x_q - P_{c,T}(x_q))\| \leq \bar{\lambda}^{-1}(\varepsilon_0 + \delta) \leq \varepsilon,$$
$$P_{c,T}(x_q) \in F_\varepsilon(T),$$
$$x_q \in \tilde{F}_\varepsilon$$

Theorem 3.1 is proved. □

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