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INTERVAL FINITE ELEMENT METHOD FOR UNCERTAIN STRUCTURES WITH INCORPORATION OF DEPENDENCY INFORMATION

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Abstract. Interval analysis is often associated with a large overestimation. In this study, we provide a novel formulation that avoids overestimation provided that information is available on full dependence between some of the variables. In order to illustrate the efficacy of the suggested method, we solve an uncertain continuous beam problem using a finite element method in conjunction with a special modeling of dependency via parameterization by employing a linear representation of each uncertain interval. Two characteristic sets of parameters may be subjected to uncertainty, namely the geometric characteristics, specifically the width and the length of the cross sectional area, as well as the material property (Young's modulus). The results obtained with the parameterization approach are compared with other methods. The problem of Chen and Yang is revisited and compared within two interval analyses.

Keywords. Dependency of intervals; Finite element method; Internal analysis; Parametrization.

2020 Mathematics Subject Classification. 65N30, 65G40.

1. Introduction

As Levkovich and Zeheb [1] stress, "no matter how accurate one tries to mathematically model an an engineering system (e.g., by a transfer function of an active and even passive circuit), the model never describes exactly the system's behavior. Environmental changes as well as production tolerances affect the values of the system's parameters. Hence, it is more realistic to assume a model with uncertainties."

Uncertainty analysis can be performed by different means. Elishakoff and Ohsaki [2] discuss three major methods to analyze uncertainty: stochasticity, fuzzy sets, and anti-optimization analysis. The latter methodology includes the simplest possible treatment of uncertainty via interval analysis. In this context, uncertainty is represented as a variation within an interval, and

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the result of the analysis turns out to be an interval itself. However, the classical interval analysis tends to be associated with a large overestimation of the results. As an example, consider the interval variables X and Y. Lower bounds of X and Y are $\underline{x}$ and $\underline{y}$, respectively. Upper bounds of X and Y are $\bar{x}$ and $\bar{y}$, respectively. According to the classic definition, the difference between two intervals $X-Y$ is a third interval $Z=[\underline{z},\bar{z}]$ with $\underline{z}=\underline{x}-\bar{y}$ and $\bar{z}=\bar{x}-\underline{y}$. However, the interval analysis, as described by Moore et al. [3], and as it is evident from the above example, may yield large overestimation of response quantities. Indeed, if X and Y are totally independent variables that happen to have some respective bounds, i.e., if $\underline{x}=\underline{y}$ and $\bar{x}=\bar{y}$, then $\underline{z}=\underline{x}-\bar{x}$ and $\bar{z}=\bar{x}-\underline{x}$. If the width $\bar{x}-\underline{x}$ is large, the overestimation will be large too. However, if X and Y are totally dependent intervals, i.e., $X\equiv Y$, in other words if Y cannot exist without the existence of X, and bounds coincide, one would expect that $X-Y\equiv 0$. Here we suggest an *alternative* interval analysis that enables one to incorporate available information on possible dependency. To illustrate ideas, the simplest possible parameterization, namely the linear parameterization of intervals in the FEM context will be presented.

2. Main Idea

The uncertainty associated with a physical parameter can be efficiently represented by $a_1 < a < a_2$, where a_1 is the lowest possible value that the parameter a can take, whereas a_2 is its greatest possible value. In terms of interval analysis, the above inequality can be written as $a\in[a_1,a_2]$. According to Moore et al. [3], the lower and upper endpoints of an interval A are denoted as $a_1=\underline{A}$, $a_2=\bar{A}$. Crisp numbers are obtained as degenerative intervals, with coalescing endpoints $\underline{A}=\bar{A}$. The interval A can be written as $A\in[\underline{A},\bar{A}]$ with $\underline{A}=a_1$ and $\bar{A}=a_2$. The midpoint of interval A is identified as $A^c=\frac{\bar{A}+\underline{A}}{2}$. The width of an interval A is identified as $\Delta A=\bar{A}-\underline{A}$. In this book, the following expression that he referred to as a "useful formula" is used

$$A=A^c+\left[-\frac{1}{2}\Delta A,\frac{1}{2}\Delta A\right]=A^c+\frac{1}{2}\Delta A\,[-1,1].$$

From now on, we suggest parameterizing the interval by using a linear function as follows

$$A(t)=A^c+\frac{\Delta A}{2}t,\quad t\in[-1,1].$$

Thus, the interval A becomes a function $A(t)$, with the argument t varying in the interval $[-1,1]$. This seemingly simple substitution will be shown to have far-reaching consequences in evaluation of interval quantities if combined with available information on possible dependency between intervals. In the following part, we revisit the numerical example proposed by Chen and Yang [4] using the parameterization technique, and compare the result provided by that method and the one utilized in Ref. [4]. Note that there is a considerable body of literature dedicated to the interval FEM. The interested reader could consult Refs. [2, 5]. This paper is a continuation of the studies by Elishakoff and Miglis [6, 7] who proposed to use trigonometric functions for the parameterization purpose, as well as of Elishakoff and Ducreux [8] who resorted to a linear transformation.

3. Numerical Example

We study a continuous beam as described in Ref. [4], as shown in Figure 1

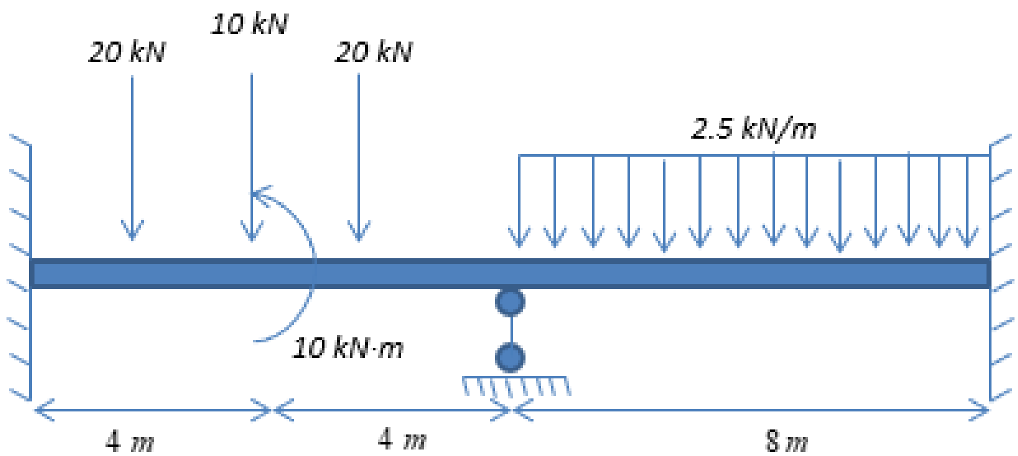

FIGURE 1. Clamped–clamped beam under concentrated and distributed loads

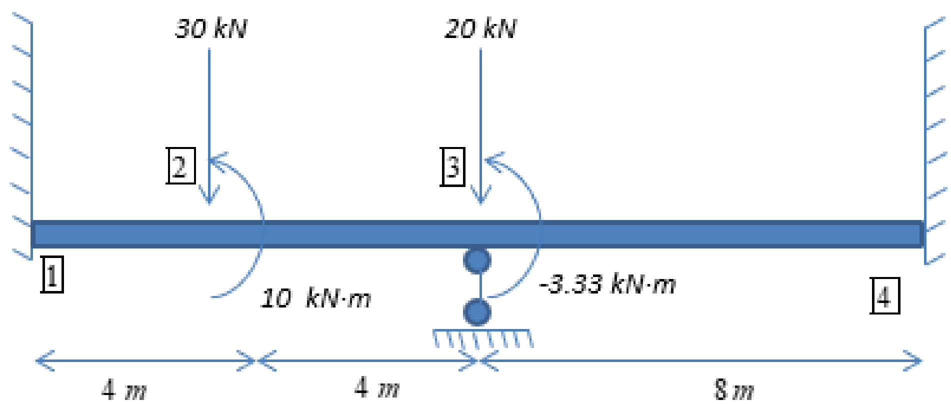

FIGURE 2. Clamped–clamped beam with the effect of the forces transferred at the nodes

In order to perform a finite element method analysis, one has to transfer the effect of the forces at the nodes. The four nodes are identified in Fig. 2. We dismiss voluntarily the forces and moment acting at nodes 1 and 4 taking into account that the displacements at these nodes vanish because they are clamped.

We suppose the height and the width of the cross section to be respectively 0.3 m and 0.2 m; the Young's modulus of the first element is set at $E_1 = 5 \times 10^6\,\text{kN/m}^2$ while in elements 2 and 3 the Young's moduli are $E_2 = E_3 = 10^7\,\text{kN/m}^2$. Following Chen and Yang [4], we study that system using a finite element method and taking into account the axial deformations. First, we deal with the problem without uncertainty in order to find the beam's nominal displacement. The elementary stiffness matrix reads

$$K_i = \begin{bmatrix} \frac{E_iB_iH_i}{L_i} & 0 & 0 & -\frac{E_iB_iH_i}{L_i} & 0 & 0 \\ 0 & \frac{E_iB_iH_i^3}{L_i^3} & \frac{1}{2}\frac{E_iB_iH_i^3}{L_i^2} & 0 & -\frac{E_iB_iH_i^3}{L_i^3} & \frac{1}{2}\frac{E_iB_iH_i^3}{L_i^2} \\ 0 & \frac{1}{2}\frac{E_iB_iH_i^3}{L_i^2} & \frac{1}{3}\frac{E_iB_iH_i^3}{L_i} & 0 & -\frac{1}{2}\frac{E_iB_iH_i^3}{L_i^2} & \frac{1}{6}\frac{E_iB_iH_i^3}{L_i} \\ -\frac{E_iB_iH_i}{L_i} & 0 & 0 & \frac{E_iB_iH_i}{L_i} & 0 & 0 \\ 0 & -\frac{E_iB_iH_i^3}{L_i^3} & -\frac{1}{2}\frac{E_iB_iH_i^3}{L_i^2} & 0 & \frac{E_iB_iH_i^3}{L_i^3} & -\frac{1}{2}\frac{E_iB_iH_i^3}{L_i^2} \\ 0 & \frac{1}{2}\frac{E_iB_iH_i^3}{L_i^2} & \frac{1}{6}\frac{E_iB_iH_i^3}{L_i} & 0 & -\frac{1}{2}\frac{E_iB_iH_i^3}{L_i^2} & \frac{1}{3}\frac{E_iB_iH_i^3}{L_i} \end{bmatrix}$$

E_i being the Young's modulus of element i, B_i and H_i its cross section's width and height, and L_i its length. The vector of displacements reads

$$U_i^T = \left[d_{i,x}\ \ d_{i,y}\ \ \varphi_i\ \ d_{i+1,x}\ \ d_{i+1,y}\ \ \varphi_{i+1}\right],$$

where d_i denotes the displacement in the indicated direction at node i, and φ_i the rotation (in radians) at node i. The vector of loads reads

$$F_i^T = \left[f_{i,x}\ \ f_{i,y}\ \ M_{z,i}\ \ f_{i+1,x}\ \ f_{i+1,y}\ \ M_{z,i+1}\right]$$

f_i being the force in the indicated direction at the node i, and $M_{z,i}$ the moment at node i. Then, after applying the standard finite-element assembly procedure, we obtain the global stiffness matrix K and the force vector F for the entire structure. In these circumstances, the equilibrium equation for the element i reads $KU = F$. Therefore, displacements equal

$$U = K^{-1}F. \tag{3.1}$$

At the nodes 1 and 4, the displacements and rotations are equal to 0. At node 3, the displacement along y is equal to 0. Then, the general stiffness matrix of the system reads

$$K = \begin{bmatrix} \frac{E_1B_1H_1}{L_1}+\frac{E_2B_2H_2}{L_2} & 0 & 0 & -\frac{E_2B_2H_2}{L_2} & 0 \\ 0 & \frac{E_1B_1H_1^3}{L_1^3}+\frac{E_2B_2H_2^3}{L_2^3} & -\frac{1}{2}\frac{E_1B_1H_1^3}{L_1^2}+\frac{1}{2}\frac{E_2B_2H_2^3}{L_2^2} & 0 & \frac{1}{2}\frac{E_2B_2H_2^3}{L_2^2} \\ 0 & -\frac{1}{2}\frac{E_1B_1H_1^3}{L_1^2}+\frac{1}{2}\frac{E_2B_2H_2^3}{L_2^2} & \frac{1}{3}\frac{E_1B_1H_1^3}{L_1}+\frac{1}{3}\frac{E_2B_2H_2^3}{L_2} & 0 & \frac{1}{6}\frac{E_2B_2H_2^3}{L_2} \\ -\frac{E_2B_2H_2}{L_2} & 0 & 0 & \frac{E_2B_2H_2}{L_2}+\frac{E_3B_3H_3}{L_3} & 0 \\ 0 & \frac{1}{2}\frac{E_2B_2H_2^3}{L_2^2} & \frac{1}{6}\frac{E_2B_2H_2^3}{L_2} & 0 & \frac{1}{3}\frac{E_2B_2H_2^3}{L_2}+\frac{1}{3}\frac{E_3B_3H_3^3}{L_3} \end{bmatrix}.$$

The vector of loads reads

$$F^T = \left[0 \quad -30000 \quad 10000 \quad 0 \quad -\frac{10000}{3}\right].$$

The product of the matrix K^{-1} and the vector F becomes

$$K^{-1}F = \left\{ \begin{array}{c} d_{2,x} \\ d_{2,y} \\ \varphi_2 \\ d_{3,x} \\ \varphi_3 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ -0.035744 \\ 0.003527 \\ 0 \\ 0.007266 \end{array} \right\}.$$

Now we want to study the parameterization of the system subjected to uncertainties. The following parameters range in the associated intervals

$$[E_i] = \left[E_i \left(1 - \frac{2}{1000} \right), E_i \left(1 + \frac{2}{1000} \right) \right],$$
$$[B_i] = \left[B_i \left(1 - \frac{2}{1000} \right), B_i \left(1 + \frac{2}{1000} \right) \right],$$

and

$$[H_i] = \left[H_i \left(1 - \frac{2}{1000} \right), H_i \left(1 + \frac{2}{1000} \right) \right],$$

as was adopted in Ref. [3]. Thus, the different terms of the rigidity matrix are

$$[K_i^{(j,k)}] = \int_0^{L_i} D_i(x) \frac{\partial^2 N_j(x)}{\partial x^2} \frac{\partial^2 N_k(x)}{\partial x^2} \, dx, \quad j,k \in [1,6] \tag{3.2}$$

with

$$D_i(x) = \begin{cases} [E_i A_i], & \text{for } j = (1,4) \text{ or } k = (1,4), \\ [E_i I_i], & \text{otherwise.} \end{cases}$$

Thus, we treat $D(x) = E(x)I(x)$ as a function of axial coordinate with constant envelope functions. For example, if $j = 2$ and $k = 3$, we get

$$\frac{\partial^2 N_2(x)}{\partial x^2} \frac{\partial^2 N_3(x)}{\partial x^2} = \left(-\frac{6}{L_i^2} + \frac{12x}{L_i^3} \right) \left(-\frac{4}{L_i} + \frac{6x}{L_i^2} \right) = \frac{24}{L_i^3} - \frac{84x}{L_i^4} - \frac{72x^2}{L_i^5}.$$

Then,

$$\frac{\partial^2 N_2(x)}{\partial x^2} = 0 \quad \text{for } x = \frac{L_i}{2} \quad \text{and} \quad \frac{\partial^2 N_3(x)}{\partial x^2} = 0 \quad \text{for } x = \frac{2L_i}{3}.$$

So,

$$\frac{\partial^2 N_2(x)}{\partial x^2} \frac{\partial^2 N_3(x)}{\partial x^2} > 0, \quad \text{for } 0 < x < \frac{L_i}{2} \quad \text{and for} \quad \frac{2L_i}{3} < x < L_i, \tag{3.3}$$
$$\frac{\partial^2 N_2(x)}{\partial x^2} \frac{\partial^2 N_3(x)}{\partial x^2} < 0, \quad \text{for } \frac{L_i}{2} < x < \frac{2L_i}{3}.$$

Hence, we have to divide the integral above into three parts as follows, separating negative and positive terms,

$$[K_i^{(2,3)}] = \int_0^{\frac{L_i}{2}} D_i(x) \frac{\partial^2 N_2(x)}{\partial x^2} \frac{\partial^2 N_3(x)}{\partial x^2} \, dx + \int_{\frac{L_i}{2}}^{\frac{2L_i}{3}} D_i(x) \frac{\partial^2 N_2(x)}{\partial x^2} \frac{\partial^2 N_3(x)}{\partial x^2} \, dx$$
$$+ \int_{\frac{2L_i}{3}}^{L_i} D_i(x) \frac{\partial^2 N_2(x)}{\partial x^2} \frac{\partial^2 N_3(x)}{\partial x^2} \, dx.$$

Now, in view of Eq. (3.3), in order to find the upper bound of the first term in $[K_i^{(2,3)}]$ we need to take the upper bound of $D_i(x)$, i.e., $\overline{E_i I_i}=\overline{E_i}\,\overline{I_i}$ (because both E_i and I_i are positive), whereas to achieve the same goal on the second term one substitutes the lower bound $\underline{E_i I_i}=\underline{E_i}\,\underline{I_i}$; in the third term, one has to take $\overline{E_i}\,\overline{I_i}$ to obtain the contribution to the upper bound of $[K_i^{(2,3)}]$. Analogous reasoning for the lower bound leads to the final expression for the interval:

$$[K_i^{(2,3)}] = \frac{9\,[E_i I_i]}{2L_i^2} - \frac{[E_i I_i]}{18L_i^2} + \frac{14\,[E_i I_i]}{9L_i^2}$$

or, more simply,

$$[K_i^{(2,3)}] = \frac{109\,[E_i I_i] - [E_i I_i]}{18L_i^2}. \tag{3.4}$$

So, following the work of Köylüoğlu, Çakmak and Nielsen [9] and Köylüoğlu and Elishakoff [10], the bounds of $[K_i^{(2,3)}]$ are

$$\underline{K_i^{(2,3)}} = \frac{109\,\underline{E_i I_i} - \overline{E_i I_i}}{18L_i^2}, \qquad \overline{K_i^{(2,3)}} = \frac{109\,\overline{E_i I_i} - \underline{E_i I_i}}{18L_i^2}$$

within the classical interval analysis. However, in this study, we will not utilize the classical but the parameterized interval analysis. So, if one takes into account that the intervals $[E_i I_i]$ in Eq. (3.4) appear twice and are fully dependent, we obtain

$$[K_i^{(2,3)}] = \frac{108\,[E_i I_i]}{18L_i^2} = \frac{6\,[E_i I_i]}{L_i^2}.$$

Thus, using the above analysis for all j and k, we obtain, with incorporation of dependence, the following stiffness matrix,

$$[K_i] = \begin{bmatrix} \frac{[E_i][B_i][H_i]}{L_i} & 0 & 0 & -\frac{[E_i][B_i][H_i]}{L_i} & 0 & 0 \\ 0 & \frac{[E_i][B_i][H_i]^3}{L_i^3} & \frac{1}{2}\frac{[E_i][B_i][H_i]^3}{L_i^2} & 0 & -\frac{[E_i][B_i][H_i]^3}{L_i^3} & \frac{1}{2}\frac{[E_i][B_i][H_i]^3}{L_i^2} \\ 0 & \frac{1}{2}\frac{[E_i][B_i][H_i]^3}{L_i^2} & \frac{1}{3}\frac{[E_i][B_i][H_i]^3}{L_i} & 0 & -\frac{1}{2}\frac{[E_i][B_i][H_i]^3}{L_i^2} & \frac{1}{6}\frac{[E_i][B_i][H_i]^3}{L_i} \\ -\frac{[E_i][B_i][H_i]}{L_i} & 0 & 0 & \frac{[E_i][B_i][H_i]}{L_i} & 0 & 0 \\ 0 & -\frac{[E_i][B_i][H_i]^3}{L_i^3} & -\frac{1}{2}\frac{[E_i][B_i][H_i]^3}{L_i^2} & 0 & \frac{[E_i][B_i][H_i]^3}{L_i^3} & -\frac{1}{2}\frac{[E_i][B_i][H_i]^3}{L_i^2} \\ 0 & \frac{1}{2}\frac{[E_i][B_i][H_i]^3}{L_i^2} & \frac{1}{6}\frac{[E_i][B_i][H_i]^3}{L_i} & 0 & -\frac{1}{2}\frac{[E_i][B_i][H_i]^3}{L_i^2} & \frac{1}{3}\frac{[E_i][B_i][H_i]^3}{L_i} \end{bmatrix}. \tag{3.5}$$

In Ref. [4], Chen and Yang studied this problem but by using Taylor expression and interval arithmetic. This method does not allow them to take into account the possible dependency between the variables. Hence, we will study the continuous beam with the case where all variables are independent of each other to compare with the results found in Ref. [4]. We will also study some other interesting cases of uncertainty with different parameterizations. These cases are more likely to be realized because they appear to reflect reality, as explained later. For example, the three segments of the continuous beam with a constant width will probably be made in the same plant and with the same manufacturing process. Thus, the widths of these three segments will constitute dependent interval variables rather than independent ones. The additional cases will also be treated in forthcoming sections compared to the results exposed in. [4], even though Chen and Yang did not directly treat the dependency problem amongst the intervals. We can compare these additional cases to results exposed in [4] since in all cases the problem will be treated under various scenarios of the dependency information.

3.1. **Parameterization of only the elastic modulus E_i.** Here, we study the system when B_i and H_i are fixed values whereas the values E_i are uncertain.

3.1.1. *Taking axial deformations into account.* We study here three cases: in the first one, we treat the uncertainties in E_i as totally independent. In the second one, we consider that the uncertainties in E_2 and E_3 are dependent, i.e., they have been designed according to the same manufacturing processes and the same plant but in different ones than the material with elastic modulus E_1. In the third case, we treat the uncertainties E_i as totally dependent (i.e., produced in some plant with the same manufacturing process). When the uncertainties are independent, we introduce three different parameters t_1, t_2, and t_3 to describe the situation

$$\begin{cases} [E_1] = E_1\left(1+\frac{2t_1}{1000}\right), & t_1 \in [-1,1], \\ [E_2] = E_2\left(1+\frac{2t_2}{1000}\right), & t_2 \in [-1,1], \\ [E_3] = E_3\left(1+\frac{2t_3}{1000}\right), & t_3 \in [-1,1]. \end{cases} \tag{3.6}$$

We replace $[E_i]$ by the quantities in Eq. (3.5) to get

$$[K] = \begin{bmatrix} 2.25\times10^8+1.5\times10^5t_1+3\times10^5t_2 & 0 & 0 & -1.5\times10^8-3\times10^5t_2 & 0 \\ 0 & 1.2656\times10^6+843.75t_1+1687.5t_2 & 8.4375\times10^5-1687.5t_1+3375t_2 & 0 & 1.6875\times10^6+3375t_2 \\ 0 & 8.4375\times10^5-1687.5t_1+3375t_2 & 6.75\times10^6+4500t_1+9000t_2 & 0 & 2.25\times10^6+4500t_2 \\ -1.5\times10^8-3\times10^5t_2 & 0 & 0 & 2.25\times10^8+4.5\times10^5t_2 & 0 \\ 0 & 1.6875\times10^6+3375t_2 & 2.25\times10^6+4500t_2 & 0 & 6.75\times10^6+9000t_2+4500t_3 \end{bmatrix} \tag{3.7}$$

Solving the system in Eq. (3.1) in conjunction with Eq. (3.7) yields parameterized displacements. These displacements are then minimized or maximized to determine the global extrema, depending on whether the minimum or maximum value is sought. The corresponding results are presented in Table 1.

TABLE 1. Maximum uncertainty compared to results in Ref. [4]

	Interval using parameterization	Parameterization		Ref. [3]	
		U^c	ΔU	U^c	ΔU
$d_{1,y}$	$[-0.0358153, -0.0356723]$	-0.0357438	1.43×10^{-4}	-0.0357437	4.395×10^{-4}
φ_2	$[0.00350453, 0.00355017]$	0.00352734	4.564×10^{-5}	0.00352734	8.114×10^{-5}
φ_3	$[0.00725172, 0.00728096]$	0.00726634	2.924×10^{-5}	0.00726632	1.894×10^{-4}

As seen from the comparison, although the corresponding values of U^c are nearly identical, the predicted widths of the output intervals differ. In particular, the estimate of ΔU is significantly smaller in the present study than in Ref. [4].

In Ref. [4], Chen and Yang described the problem as follows: "the Young's modulus of the first element is $E_1 = 5 \times 10^6\,\mathrm{kN/m^2}$, while in elements 2 and 3 the Young's moduli are $E_2 = E_3 = 10^7\,\mathrm{kN/m^2}$." It is therefore reasonable to assume that their analysis implicitly postulates the coincidence of the intervals E_2 and E_3, i.e., $[E_2] = [E_3]$. This allows one to introduce the same parameter t_2 to describe the uncertainty in E_2 and E_3, while retaining a separate parameter t_1 for E_1:

$$\begin{cases} [E_1] = E_1\left(1 + \frac{2t_1}{1000}\right), & t_1 \in [-1,1], \\ [E_2] = E_2\left(1 + \frac{2t_2}{1000}\right), & t_2 \in [-1,1], \\ [E_3] = E_3\left(1 + \frac{2t_2}{1000}\right), & t_2 \in [-1,1]. \end{cases}$$

In these circumstances, by replacing E_1, E_2, and E_3 with $[E_1]$, $[E_2]$, and $[E_3]$ in Eq. (3.5), we obtain

$$[K] = \begin{bmatrix} 2.25\times10^8 + 1.5\times10^5 t_1 + 3\times10^5 t_2 & 0 & 0 & -1.5\times10^8 - 3\times10^5 t_2 & -1.5\times10^8 - 3\times10^5 t_2 \\ 0 & 1.2656\times10^6 + 843.75 t_1 + 1687.5 t_2 & 8.4375\times10^5 - 1687.5 t_1 + 3375 t_2 & 0 & 1.6875\times10^6 + 3375 t_2 \\ 0 & 8.4375\times10^5 - 1687.5 t_1 + 3375 t_2 & 6.75\times10^6 + 4500 t_1 + 9000 t_2 & 0 & 2.25\times10^6 + 4500 t_2 \\ -1.5\times10^8 - 3\times10^5 t_2 & 0 & 0 & 2.25\times10^8 + 4.5\times10^5 t_2 & 0 \\ 0 & 1.6875\times10^6 + 3375 t_2 & 2.25\times10^6 + 4500 t_2 & 0 & 6.75\times10^6 + 13500 t_2 \end{bmatrix} \tag{3.8}$$

Solving the system in Eq. (3.1) in conjunction with Eq. (3.8) yields parameterized displacements. These displacements are then minimized or maximized to determine the global extrema, depending on whether the minimum or maximum value is sought. The corresponding results are presented in Table 2.

TABLE 2. Displacement and uncertainties in the fully dependent case

	Interval using parameterization	Parameterization		Ref. [3]	
		U^c	ΔU	U^c	ΔU
$d_{1,y}$	$[-0.0358153, -0.0356723]$	-0.0357438	1.43×10^{-4}	-0.0357437	4.395×10^{-4}
φ_2	$[0.00352030, 0.00353441]$	0.00352735	1.411×10^{-5}	0.00352734	8.114×10^{-5}
φ_3	$[0.00725181, 0.00728088]$	0.00726634	2.907×10^{-5}	0.00726632	1.894×10^{-4}

Now we consider the special case in which the uncertainties E_1, E_2, and E_3 are fully dependent on each other. This situation is described by setting $t_1 = t_2 = t_3 = t$ in Eq. (3.6).

$$\begin{cases} [E_1] = E_1\left(1 + \frac{2t}{1000}\right) \\ [E_2] = E_2\left(1 + \frac{2t}{1000}\right), \quad t \in [-1,1]. \\ [E_3] = E_3\left(1 + \frac{2t}{1000}\right) \end{cases}$$

Replacing E_i by $[E_i]$ in Eq. (3.5), and subsequently minimizing or maximizing the displacements, we obtain the numerical results summarized in Table 3.

TABLE 3. Displacement and uncertainties in the fully dependent case

	Interval using parameterization	Parameterization		Ref. [3]	
		U^c	ΔU	U^c	ΔU
$d_{1,y}$	$[-0.0358153, -0.0356723]$	-0.0357438	1.43×10^{-4}	-0.0357437	4.395×10^{-4}
φ_2	$[0.00352030, 0.00353441]$	0.00352735	1.411×10^{-5}	0.00352734	8.114×10^{-5}
φ_3	$[0.00725181, 0.00728088]$	0.00726634	2.907×10^{-5}	0.00726632	1.894×10^{-4}

3.1.2. *Neglecting axial deformations.* First, we consider the uncertain displacements with three different parameters t_1, t_2, and t_3 for the uncertainties. In this case, the stiffness matrix becomes

$$[K]=\begin{bmatrix} 1.256\times10^6+843.75t_1+1687.5t_2 & 8.4375\times10^5-1687.5t_1+3375t_2 & 1.6875\times10^6+3375t_2 \\ 8.4375\times10^5-1687.5t_1+3375t_2 & 6.75\times10^6+4500t_1+9000t_2 & 2.25\times10^6+4500t_2 \\ 1.6875\times10^6+3375t_2 & 2.25\times10^6+4500t_2 & 6.75\times10^6+9000t_2+4500t_3 \end{bmatrix} \tag{3.9}$$

The vector of load reads

$$F=\begin{bmatrix} -30000N \\ 10000N\cdot m \\ -\dfrac{10000}{3}N\cdot m \end{bmatrix}$$

The parameterized displacements read

$$d_{1,y}=\frac{N_1}{D},\ c_{\varphi_2}=\frac{N_2}{D},\ c_{\varphi_3}=\frac{N_3}{D}, \tag{3.10}$$

where the respective numerators N_j and the denominator D are

$$\begin{aligned} N_1 &= -2.53125\times10^9\Big(4.75\times10^7+30000t_1+1.225\times10^5t_2 \\ &\qquad +37500t_3+39t_1t_2+76t_2^2+21t_3t_1+54t_3t_2\Big), \\ N_2 &= -6.667\times\Big(-1.780\times10^{15}+4.034\times10^{12}t_1-8.590\times10^{12}t_2-2.563\times10^{12}t_3 \\ &\qquad +5.221\times10^9t_1t_2-7.214\times10^9t_2^2+2.848\times10^9t_3t_1-7.973\times10^9t_3t_2\Big), \\ N_3 &= -10\times\Big(-2.444\times10^{15}-2.468\times10^{12}t_1-7.309\times10^{12}t_2 \\ &\qquad +4.999\times10^9t_2t_1-4.809\times10^9t_2^2+3.164\times10^7t_2^2\Big), \\ D &= \Big(3.364\times10^{18}+6.621\times10^{15}t_1+1.004\times10^{16}t_2+3.524\times10^{15}t_3+6.407\times10^{11}t_1^2 \\ &\quad +6.834\times10^{12}t_2^2+1.879\times10^{13}t_2t_1+6.407\times10^{12}t_3t_1+7.689\times10^{12}t_3t_2 \\ &\quad -10t_2^3+8.543\times10^8t_1^2t_2+1.196\times10^{10}t_2^2t_1+4.271\times10^8t_1^2t_3 \\ &\quad +1.709\times10^{10}t_2^2t_3+1.196\times10^{10}t_1t_2t_3\Big). \end{aligned} \tag{3.11}$$

The numerical results are summarized in Table 4.

TABLE 4. Maximum uncertainty compared to results in Ref. [4]

	Interval using parameterization	Parameterization		Ref. [3]	
		U^c	ΔU	U^c	ΔU
$d_{1,y}$	$[-0.0358153, -0.0356723]$	-0.0357438	1.43×10^{-4}	-0.0357437	4.395×10^{-4}
φ_2	$[0.00350453, 0.00355017]$	0.00352734	4.564×10^{-5}	0.00352734	8.114×10^{-5}
φ_3	$[0.00725172, 0.00728096]$	0.00726634	2.924×10^{-5}	0.00726632	1.894×10^{-4}

We observe that the results are indistinguishable from those obtained when axial deformations are taken into account (see Table 1). This indicates that the effect of axial deformation is negligible for all three displacements $d_{1,y}$, φ_2, and φ_3. Therefore, axial deformations need not be considered in the analysis of this transversely loaded beam. We also observe that the bounds obtained for the independent case (with different t_j) are consistently sharper than those derived using the Taylor expansion method. In particular, there is approximately an order-of-magnitude difference between the values of ΔU reported in Ref. [4] and those obtained using the parameterization approach. The corresponding percentage differences are about 207% for $d_{1,y}$, 78% for φ_2, and 548% for φ_3.

$$[K]=\begin{bmatrix} 1.256\times10^6+843.75t_1+1687.5t_2 & 8.4375\times10^5-1687.5t_1+3375t_2 & 1.6875\times10^6+3375t_2\\ 8.4375\times10^5-1687.5t_1+3375t_2 & 6.75\times10^6+4500t_1+9000t_2 & 2.25\times10^6+4500t_2\\ 1.6875\times10^6+3375t_2 & 2.25\times10^6+4500t_2 & 6.75\times10^6+13500t_2 \end{bmatrix}.$$

The parameterized displacements read

$$C_{d_{1,y}}=\frac{N_1^{(2)}}{D^{(2)}},\ C_{\varphi_2}=\frac{N_2^{(2)}}{D^{(2)}},\ C_{\varphi_3}=\frac{N_3^{(2)}}{D^{(2)}},$$

$$\begin{aligned} N_1^{(2)}=-1.667\times\Big(&3.943\times10^{31}t_1+3.956\times10^{26}t_2^3+2.720\times10^{29}t_1t_2+3.130\times10^{29}t_2^2\\ &+6.909\times10^{26}t_2^2t_1+2.174\times10^{28}t_1^2+1.250\times10^{26}t_1^2t_2+1.122\times10^{32}t_2^2\\ &+2.052\times10^{23}t_2^4+1.825\times10^{24}t_1^3+2.108\times10^{19}t_2^5+7.619\times10^{23}t_2^3t_1\\ &+2.390\times10^{23}t_2t_1^2+7.298\times10^{21}t_1^3t_2+3.049\times10^{20}t_2^4t_1+1.520\times10^{20}t_1^3t_2\\ &+7.298\times10^{18}t_1^3t_2^2+1.517\times10^{34}\Big), \end{aligned}$$

$$\begin{aligned} N_2^{(2)}=-0.167\times\Big(&-1.780\times10^{16}-1.115\times10^{14}t_2+4.034\times10^{13}t_1\\ &+8.068\times10^{10}t_2t_1-1.519\times10^{11}t_2^2\Big), \end{aligned}$$

$$\begin{aligned} N_3^{(2)}=-0.167\times\Big(&-3.666\times10^{16}-1.096\times10^{14}t_2-3.702\times10^{13}t_1\\ &-7.499\times10^{10}t_2t_1-7.214\times10^{10}t_2^2+4.476\times10^{8}t_2^3\Big), \end{aligned}$$

$$D^{(2)} = \Big(1.655\times10^{15}t_1 + 4.271\times10^{8}t_3^2 + 8.409\times10^{17} + 6.300\times10^{12}t_2t_1 + 3.631\times10^{12}t_2^2 + 5.980\times10^{9}t_2^2t_1 + 1.602\times10^{11}t_1^2 + 3.204\times10^{8}t_1^2t_2 + 3.390\times10^{15}t_2^2\Big)^2.$$

The numerical results are summarized in Table 5.

TABLE 5. Maximum uncertainty compared to results in Ref. [4]

	Interval using parameterization	Parameterization		Ref. [3]	
		U^c	ΔU	U^c	ΔU
$d_{1,y}$	$[-0.0358153, -0.0356723]$	-0.0357438	1.43×10^{-4}	-0.0357437	4.395×10^{-4}
φ_2	$[0.00350453, 0.00355017]$	0.00352735	4.564×10^{-5}	0.00352734	8.114×10^{-5}
φ_3	$[0.00725181, 0.00728088]$	0.00726634	2.907×10^{-5}	0.00726632	1.894×10^{-4}

Here, we also conclude that the effect of axial deformation is negligible. We observe that these results are very close to those obtained in the independent case (with three parameters). The discrepancy with the results reported in Ref. [4] remains the same, namely about 207% for $d_{1,y}$, 78% for φ_2, and 552% for φ_3. If we now consider the case where information is available indicating that the three uncertainties E_1, E_2, and E_3 are fully dependent, we set $t_1 = t_2 = t_3 = t$ in Eq. (3.9) and obtain

$$\begin{aligned} C_{d_{1,y}} &= \frac{-0.133\left(1.804\times10^{16} + 7.214\times10^{13}t + 7.214\times10^{10}t^2\right)}{6.728\times10^{16} + 4.037\times10^{14}t + 8.073\times10^{11}t^2 + 5.382\times10^{8}t^3} \\ C_{\varphi_1} &= \frac{0.033\left(7.119\times10^{15} + 2.848\times10^{13}t + 2.848\times10^{10}t^2\right)}{6.728\times10^{16} + 4.037\times10^{14}t + 8.073\times10^{11}t^2 + 5.382\times10^{8}t^3} \\ C_{\varphi_3} &= \frac{0.033\left(1.467\times10^{16} + 5.866\times10^{13}t + 5.866\times10^{10}t^2\right)}{6.728\times10^{16} + 4.037\times10^{14}t + 8.073\times10^{11}t^2 + 5.382\times10^{8}t^3}. \end{aligned} \tag{3.12}$$

In Table 6, we display the extrema of Eq. (3.12), found by using MATLAB®.

TABLE 6. Uncertainties with no axial deformations and only one uncertain parameter

	Interval using parameterization	Parameterization		Ref. [4]	
		U^c	ΔU	U^c	ΔU
$d_{1,y}$	$[-0.0358153, -0.0356723]$	-0.0357438	1.43×10^{-4}	-0.0357437	4.395×10^{-4}
φ_2	$[0.00352030, 0.00353441]$	0.00352735	1.411×10^{-5}	0.00352734	8.114×10^{-5}
φ_3	$[0.00725181, 0.00728088]$	0.00726634	2.907×10^{-5}	0.00726632	1.894×10^{-4}

Here too, we arrive at the conclusion that the effect of the axial deformation is negligible. We also notice that, when the uncertainties are fully dependent, the uncertain rotation φ_2 turns out to be smaller, but the values of the displacement $d_{1,y}$ and of the rotation φ_3 stay approximately the same as in the independent case. This result should be attributed to the increased level of information: indeed, in this case more connections appear between uncertainties than in the independent case. As these bounds are sharper than the one obtained with the independent case, they are also sharper than the ones obtained with the Taylor's expansion method. There is about an order of magnitude difference between the results found in Ref. [4] for ΔU and those found with the parameterization, with percentagewise errors constituting 207% for $d_{1,y}$, 475% for φ_2 and 552% for φ_3.

3.2. **Parameterization of only the width B_i.** Now we perform a similar analysis, with B_i being the varying parameter but E_i and H_i are fixed. We will first study the fully independent case with three parameters p_1, p_2 and p_3

$$\begin{cases} [B_1] = B_1\left(1+\frac{2p_1}{1000}\right), & p_1 \in [-1,1], \\ [B_2] = B_2\left(1+\frac{2p_2}{1000}\right), & p_2 \in [-1,1], \\ [B_3] = B_3\left(1+\frac{2p_3}{1000}\right), & p_3 \in [-1,1]. \end{cases}$$

We replace $[B_i]$ by the quantity in Eq. (3.5) to get

$$[K] = \begin{bmatrix} 1.256\times10^6+843.75\,p_1+1687.5\,p_2 & 8.4375\times10^5-1687.5\,p_1+3375\,p_2 & 1.6875\times10^6+3375\,p_2 \\ 8.4375\times10^5-1687.5\,p_1+3375\,p_2 & 6.75\times10^6+4500\,p_1+9000\,p_2 & 2.25\times10^6+4500\,p_2 \\ 1.6875\times10^6+3375\,p_2 & 2.25\times10^6+4500\,p_2 & 6.75\times10^6+9000\,p_2+4500\,p_3 \end{bmatrix}$$

which is the same matrix as the one obtained when E_i was treated as uncertain, for the fully independent case. In those circumstances, the vector of displacements is the same as the one described in Eqs. (3.10)–(3.11).

The extrema of the displacements are summarized in Table 7.

TABLE 7. Displacement and uncertainties with independent uncertainties

	Interval using parameterization	Parameterization		Ref. [4]	
		U^c	ΔU	U^c	ΔU
$d_{1,y}$	$[-0.0358153, -0.0356723]$	-0.0357438	1.43×10^{-4}	-0.0357437	4.395×10^{-4}
φ_2	$[0.00350453, 0.00355017]$	0.00352734	4.564×10^{-5}	0.00352734	8.114×10^{-5}
φ_3	$[0.00725172, 0.00728096]$	0.00726634	2.924×10^{-5}	0.00726632	1.894×10^{-4}

As expected, and as found in Ref. [4], the displacements are coinciding with the ones found for the parameterization of E_i when all the uncertainties are independent. So, the difference with the results found in Ref. [4] is the same: 207% for $d_{1,y}$, 78% for φ_2 and 548% for φ_3.

In the description of the problem, B_i is assumed to be the same in three sections; therefore, it is justifiable to assume that if there is an uncertainty associated with the manufacturing process, it will be reflected in each part of the beam in the like manner. Hereinafter, for this second case, we can use only one uncertain parameter p. We obtain

$$[B_i] = B_i\left(1+\frac{2p}{1000}\right), \qquad t\in[-1,1].$$

We replace $[B_i]$ by the quantity in Eq. (3.5) and we obtain

$$[K] = \begin{bmatrix} 1.256\times10^6+2531.3\,p & 8.4375\times10^5+1687.5\,p & 1.6875\times10^6+3375\,p \\ 8.4375\times10^5+1687.5\,p & 6.75\times10^6+13500\,p & 2.25\times10^6+4500\,p \\ 1.6875\times10^6+3375\,p & 2.25\times10^6+4500\,p & 6.75\times10^6+13500\,p \end{bmatrix}.$$

In those circumstances, the vector of displacements is the same than the one described in Eq. (3.12).

$$U = \begin{bmatrix} \dfrac{-0.133\left(1.804\times10^{16}+7.214\times10^{13}p+7.214\times10^{10}p^2\right)}{6.728\times10^{16}+4.037\times10^{14}p+8.073\times10^{11}p^2+5.382\times10^8p^3} \\ \dfrac{0.033\left(7.119\times10^{15}+2.848\times10^{13}p+2.848\times10^{10}p^2\right)}{6.728\times10^{16}+4.037\times10^{14}p+8.073\times10^{11}p^2+5.382\times10^8p^3} \\ \dfrac{0.033\left(1.467\times10^{16}+5.866\times10^{13}p+5.866\times10^{10}p^2\right)}{6.728\times10^{16}+4.037\times10^{14}p+8.073\times10^{11}p^2+5.382\times10^8p^3} \end{bmatrix}. \qquad (3.13)$$

The extrema of the quantities in Eq. (3.13) are summarized in Table 8.

TABLE 8. Displacement and uncertainties with dependent uncertainties

	Interval using parameterization	Parameterization		Ref. [4]	
		U^c	ΔU	U^c	ΔU
$d_{1,y}$	$[-0.0358153, -0.0356723]$	-0.0357438	1.43×10^{-4}	-0.0357437	4.395×10^{-4}
φ_2	$[0.00352030, 0.00353441]$	0.00352735	1.411×10^{-5}	0.00352734	8.114×10^{-5}
φ_3	$[0.00725181, 0.00728088]$	0.00726634	2.907×10^{-5}	0.00726632	1.894×10^{-4}

These displacements are coinciding with the ones found for the parameterization of E_i when all the uncertainties are dependent. So, the difference with the results found in Ref. [3] is the same: 207% for $d_{1,y}$, 475% for φ_2 and 552% for φ_3.

3.3. **Parameterization of only the height H_i.** Here, H_i is treated as the varying parameter (E_i and B_i are fixed). We will first study the fully independent case with three parameters q_1, q_2 and q_3

$$\begin{cases} [H_1] = H_1\left(1+\frac{2q_1}{1000}\right), & q_1\in[-1,1], \\ [H_2] = H_2\left(1+\frac{2q_2}{1000}\right), & q_2\in[-1,1], \\ [H_3] = H_3\left(1+\frac{2q_3}{1000}\right), & q_3\in[-1,1]. \end{cases}$$

Replacing $[H_i]$ by the quantity in Eq. (3.5), we obtain

$$\begin{cases} [K_{1,1}] & = 1.5625\times 10^7(0.3+0.0006q_1)^3+3.125\times 10^7(0.3+0.0006q_2)^3, \\ [K_{1,2}] & = -3.125\times 10^7(0.3+0.0006q_1)^3+6.25\times 10^7(0.3+0.0006q_2)^3, \\ [K_{1,3}] & = 6.25\times 10^7(0.3+0.0006q_2)^3, \\ [K_{2,1}] & = -3.125\times 10^7(0.3+0.0006q_1)^3+6.25\times 10^7(0.3+0.0006q_2)^3, \\ [K_{2,2}] & = 8.333\times 10^7(0.3+0.0006q_1)^3+1.667\times 10^8(0.3+0.0006q_2)^3, \\ [K_{2,3}] & = 8.333\times 10^7(0.3+0.0006q_2)^3, \\ [K_{3,1}] & = 6.25\times 10^7(0.3+0.0006q_2)^3, \\ [K_{3,2}] & = 8.333\times 10^7(0.3+0.0006q_2)^3, \\ [K_{3,3}] & = 1.667\times 10^8(0.3+0.0006q_2)^3+8.333\times 10^7(0.3+0.0006q_3)^3. \end{cases} \tag{3.14}$$

Solving the system in Eq. (3.1), Eq. (3.14) yields parameterized displacements. These displacements are minimized or maximized to find global extrema. The results are listed in Table 9.

TABLE 9. Displacement and uncertainties with independent uncertainties

	Interval using parameterization	Parameterization		Ref. [3]	
		U^c	ΔU	U^c	ΔU
$d_{1,y}$	$[-0.0359590, -0.0355301]$	-0.0357445	4.289×10^{-4}	-0.0357437	1.321×10^{-3}
φ_2	$[0.00345892, 0.00359584]$	0.00352738	1.369×10^{-4}	0.00352734	2.439×10^{-4}
φ_3	$[0.00722260, 0.00731034]$	0.00726647	8.773×10^{-5}	0.00726632	5.693×10^{-4}

The bounds obtained are sharper than the ones obtained with Taylor's expansion method. Indeed, there is about an order of magnitude difference between the results found in Ref. [4] for ΔU and those found with the parameterization, with percentagewise errors constituting 208% for $d_{1,y}$, 78% for φ_2 and 549% for φ_3.

In the description of the problem, H_i is assumed to be the same on the three sections; therefore, it is justifiable to treat an uncertainty in the manufacturing process, as equally reflected in each part of the beam. Hereinafter, for this second case, we consider that the three uncertainties are fully dependent and treated by using the same parameter t.

In these circumstances,

$$[H_i] = H_i\left(1+\frac{2q}{1000}\right), \qquad q\in[-1,1]. \tag{3.15}$$

We introduce the quantity in Eq. (3.15) into Eq. (3.5) and solve the system in Eq. (3.1). We obtain the following vector of displacements:

$$U = \begin{bmatrix} 0 \\ \dfrac{-9.650793649 \times 10^{-4}}{(0.3 + 6 \times 10^{-4} q)^3} \\ \dfrac{9.523809531 \times 10^{-5}}{(0.3 + 6 \times 10^{-4} q)^3} \\ 0 \\ \dfrac{1.961904761 \times 10^{-4}}{(0.3 + 6 \times 10^{-4} q)^3} \end{bmatrix}.$$

The extrema of the quantities in Eq. (3.11) are summarized in Table 10.

TABLE 10. Displacement and uncertainties with dependent uncertainties

	Interval using parameterization	Parameterization		Ref. [4]	
		U^c	ΔU	U^c	ΔU
$d_{1,y}$	$[-0.0359590, -0.0355301]$	-0.0357445	4.289×10^{-4}	-0.0357437	1.321×10^{-3}
φ_2	$[0.00350626, 0.00354859]$	0.00352742	4.233×10^{-5}	0.00352734	2.439×10^{-4}
φ_3	$[0.00722289, 0.00731009]$	0.00726649	8.72×10^{-5}	0.00726632	5.693×10^{-4}

Once again, the results found with the parameterization of interval applicable for the fully dependent variable are sharper than the ones derived in Ref. [4] as they should be. We notice that the percentagewise errors are approximately the same as when B_i is the varying problem (in the fully dependent case).

3.4. **Parameterization of E_i, B_i and H_i.** Here, we treat E_i, B_i and H_i as the varying parameters; we consider particular case of dependence. Only the first case will be treated with E_2, E_3, the B_i and the H_i which are independent, in the other cases, $E_2 = E_3 = E'$, $B_1 = B_2 = B_3 = B$ and $H_1 = H_2 = H_3 = H$. Hence, we have now

$$[E_1] = E_1 \left(1 + \frac{2s_1}{1000}\right), \qquad s_1 \in [-1, 1],$$

$$[E_2] = E_2 \left(1 + \frac{2s_2}{1000}\right), \qquad s_2 \in [-1, 1],$$

$$[E_3] = E_3 \left(1 + \frac{2s_3}{1000}\right), \qquad s_3 \in [-1, 1],$$

$$[B_1] = B_1 \left(1 + \frac{2s_4}{1000}\right), \qquad s_4 \in [-1, 1],$$

$$[B_2] = B_2 \left(1 + \frac{2s_5}{1000}\right), \qquad s_5 \in [-1, 1],$$

$$[B_3] = B_3\left(1+\frac{2s_6}{1000}\right), \qquad s_6 \in [-1,1],$$
$$[H_1] = H_1\left(1+\frac{2s_7}{1000}\right), \qquad s_7 \in [-1,1],$$
$$[H_2] = H_2\left(1+\frac{2s_8}{1000}\right), \qquad s_8 \in [-1,1],$$

and

$$[H_3] = H_3\left(1+\frac{2s_9}{1000}\right), \qquad s_9 \in [-1,1].$$

So, we can study five possible cases:

(1) when all variables are independent, so there are 9 parameters: s_i with $i \in \{1,\dots,9\}$;
(2) when $E_2 = E_3 = E'$ ($s_2 = s_3 = s$), $B_1 = B_2 = B_3 = B$ ($s_4 = s_5 = s_6 = u$) and $H_1 = H_2 = H_3 = H$ ($s_7 = s_8 = s_9 = v$), so there are 4 parameters: s_1, s, u and v;
(3) when B and H are dependent of each other $(u = v = U)$ but rest of the parameters are independent, so there are 3 parameters: s_1, s and U;
(4) when E_1 and E_2 are dependent $(s_1 = s = S)$ but rest of the parameters are independent, so there are 3 parameters: S, u and v;
(5) when E_1 and E_2 are dependent $(s_1 = s = S)$ and B and H are dependent $(u = v = U)$, so there are 2 parameters: S and U.

Note that we do not consider the case when all variables are fully dependent $(s_1 = s = u = v = R)$ because it would be difficult to justify the utilization of the same parameter to describe vastly different characteristics: elastic moduli and geometric characteristics apparently cannot be dependent of each other, in usual circumstances, although one can visualize some extremely special cases when this does take place.

Each of the above cases has been treated. The results for case 1 are summarized in Table 11.

TABLE 11. Displacement and uncertainties in case 1

Case 1	Interval using parameterization	Parameterization		Ref. [4]	
		U^c	ΔU	U^c	ΔU
$d_{1,y}$	$[-0.0361033, -0.0353884]$	-0.0357458	7.149×10^{-4}	-0.0357437	2.206×10^{-3}
φ_2	$[0.00341333, 0.00364154]$	0.00352743	2.282×10^{-4}	0.00352734	4.073×10^{-4}
φ_3	$[0.00719359, 0.00733981]$	0.00726670	1.462×10^{-4}	0.00726632	9.508×10^{-4}

The results found for case 1 are sharper than the ones derived in Ref. [4]. There is about an order of magnitude difference between the results found in Ref. [4] for ΔU and those found with the parameterization, with percentagewise errors constituting 209% for $d_{1,y}$, 78% for φ_2 and 550% for φ_3. We notice that the results of cases 2 and 3 are the same. Indeed, this is because the maximum and minimum responses in case 2 are respectively obtained with $u = v = -1$ and with $u = v = 1$.

The results for these cases are summarized in Table 12.

TABLE 12. Displacement and uncertainties in cases 2 and 3

Cases 2 and 3	Interval using parameterization	Parameterization		Ref. [4]	
		U^c	ΔU	U^c	ΔU
$d_{1,y}$	[−0.0361033, −0.0353884]	−0.0357458	7.149×10^{-4}	−0.0357437	2.206×10^{-3}
φ_2	[0.00347663, 0.00357871]	0.00352767	1.021×10^{-4}	0.00352734	4.073×10^{-4}
φ_3	[0.00719408, 0.00733942]	0.00726675	1.453×10^{-4}	0.00726632	9.508×10^{-4}

The results found for cases 2 and 3 are sharper than the ones for case 1. The difference with the results found in Ref. [4] is: 207% for $d_{1,y}$, 299% for φ_2 and 554% for φ_3.

The results of cases 4 and 5 are also the same (for the same reason that the results for cases 2 and 3 are the same) and are given in Table 13.

TABLE 13. Displacement and uncertainties in cases 4 and 5

Cases 4 and 5	Interval using parameterization	Parameterization		Ref. [4]	
		U^c	ΔU	U^c	ΔU
$d_{1,y}$	[−0.0361033, −0.0353884]	−0.0357458	7.149×10^{-4}	−0.0357437	2.206×10^{-3}
φ_2	[0.00349227, 0.00356282]	0.00352755	7.054×10^{-5}	0.00352734	4.073×10^{-4}
φ_3	[0.00719408, 0.00733942]	0.00726675	1.453×10^{-4}	0.00726632	9.508×10^{-4}

The results found for cases 4 and 5 are sharper than the ones for cases 1, 2 and 3. The difference with the results found in Ref. [4] is: 207% for $d_{1,y}$, 477% for φ_2 and 554% for φ_3.

We notice that the displacement $d_{1,y}$ and the rotation φ_3 are the same for cases 2, 3, 4 and 5. This is because in case 2, the maximum and minimum values of $d_{1,y}$ and of φ_3 are respectively obtained with $s_1 = s = u = v = -1$ and with $s_1 = s = u = v = 1$. But the maximum and minimum values of φ_2 are not obtained with $s_1 = s$. So, in cases 4 and 5, the bounds of φ_2 are not the same as in cases 2 and 3.

4. CONCLUSION

The efficiency of introducing information about the dependency of intervals via parameterized modeling of interval variables has been demonstrated. By solving a simple problem, we observe that the method used in Ref. [4] for the fully independent case may yield a displacement that is more than six times larger than when the variables are partially dependent, justifying the use of the parameterization technique. Therefore, the overestimation associated with interval analysis is largely reduced if, instead of using standard interval analysis on a problem, the possible dependencies are identified and incorporated using parameterization or another suitable technique. It should be noted that this work treats the full dependency between two intervals when addressing the dependency problem. It does not address the question of partial dependence between two intervals. It appears interesting to pursue the notion of partial dependence, of which the current study will constitute a particular case. This work, on the other hand, deals with full dependency between parts of uncertain variables, creating a bridge between totally independent and fully dependent variables (see also Refs. [11, 12, 13, 14])

5. Dedication

One of us (Isaac Elishakoff) dedicated the original version of this article to the 85th Birth Anniversary of Ezra Zeheb, the Dean of Science at Holon Institute of Technology and an Emeritus Gerard Swope Professor of Electrical Engineering at the Technion - Israel Institute of Technology. Production of the issue necessitated more time, and in the meantime, Professor Ezra Zaheb passed away on 12 September 2025. Before I. E. found himself in the United States, Ezra and he served at the same Institute, Technion-Israel Institute of Technology, although in different faculties. Ezra was associated with Electrical Engineering, whereas Isaac worked at the Aerospace Faculty. Once, Isaac prepared a research report, and while visiting the printing facility, he was told to wait so the work could be executed immediately. While waiting for the job to be completed, Isaac discovered a pamphlet listing courses offered by the Electrical Engineering faculty the next semester. He found a course titled ”Uncertain Systems” and decided to audit it. Isaac unfailingly attended all lectures on beautiful Kharitonov theorems, their veracious derivatives, and generalizations, including those of Ezra himself. Not only was Ezra an outstanding researcher and educator, but also an exciting and inspiring one, with lectures being full of nonstandard observations and superb presentations. Since then, Isaac rightfully considers himself Ezra’s student and dedicates this article to the blessed memory of his great educator and friend, Professor Ezra Zeheb.

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