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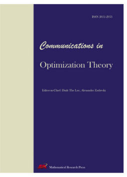

PERIODIC SOLUTIONS OF DISSIPATIVE SYSTEMS AND THE HAUSDORFF MEASURE OF NONCOMPACTNESS IN QUASI-UNIFORM SPACES

MARTIN VÄTH

Mathematical Institute, Academy of Sciences of the Czech Republic, Czech Republic

Abstract. In order to prove the existence of periodic solutions of dissipative systems in Banach spaces in the absence of uniqueness, a general notion of a Hausdorff measure of noncompactness is introduced in quasi-uniform spaces. A corresponding class of "condensifying" maps is defined which is shown to be equivalent to maps which have a "uniform" (pre)compact attractor. The results are used to prove that each dissipative and condensing on bounded subsets map has a compact attractor. In particular, the corresponding system has a periodic solution. Most results apply to single- and multivalued maps, even to monotone maps of power sets or not necessarily monotone sequences of sets, and also the connectedness of certain attractors is discussed.

Keywords. Browder's fixed point theorem; Condensing map; Dissipative system; Multivalued monotone map; Measure of noncompactness; Periodic solution.

2020 Mathematics Subject Classification. 34A12, 47N20, 47H04, 47H05.

1. INTRODUCTION

The main aim of this paper is to show the existence of compact attractors for "condensing" dissipative systems in Banach spaces (Theorem 8.6). In Section 2 (Theorem 2.4) we show how this leads to new results about periodic solutions of such systems. Although to the author's knowledge all these results are new even in the single-valued case, the results hold also in the multivalued case. In fact, our approach is to consider limits of sequences of sets which leads us to consider "uniform" attractors: Although it is not a direct hypothesis, the main ingredient for our proof of the main result (Theorem 8.6) are uniform attractors for which we obtain various characterizations in Theorem 7.20 in terms of measures of noncompactness. Hence, in the sense, Theorem 7.20 can be considered as the abstract main result of the paper.

In this connection, we point out that the classical notion of a measure of noncompactness and of "condensing" maps is closely tied to a metric on the space. For our point-set topological considerations, it is more natural to ignore the metric and to deal only with its induced uniform structure.

E-mail address: martin@mvath.de

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Hence, we introduce in Section 4 a new notion of a “Hausdorff measure of noncompactness” on quasi-uniform spaces (which is actually a family of sets): This new notion is more natural for our considerations and also more natural if one deals e.g. with locally convex spaces, even in the metric case. The author believes that this new notion is of independent interest and will turn out useful also in many other situations.

This new notion allows us to introduce a class of “condensifying” maps in a very natural manner (Section 5) which sheds some new light on the well-known class of condensing maps. In fact, we show in Section 6 that strictly condensing maps and even condensing maps have this property. (The reason for the terminology “condensifying” is that, roughly speaking, all maps which are compactifying [9, 28] also have the property that they are condensifying if restricted to a neighborhood of their fixed point set. In this sense, being “condensifying” is a weaker requirement than being condensing or than being compactifying.) The mentioned main result about uniform compact atttractors (Theorem 7.20) actually turns out to be a characterization of these “condensifying” maps.

Our approach allows us to consider not only attractors for iterations of single- or multivalued maps but even of monotone maps of power sets like $F(M) = \overline{\operatorname{conv}}(f(M) \cup \{0\})$ which yields compact attractors for f with additional invariance properties. Although such monotone maps of power sets seem to be a rater natural framework for attractors, it seems not to be widely studied in literature yet.

As a curious side result of our considerations, we obtain also nontrivial sufficient conditions for the connectedness of certain “canonical” attractors which generalize a classical result of Kuratowski for complete metric spaces. Although in the particular case of compact Hausdorff spaces this is well-known, it seems to be new even in case of compact (non-Hausdorff) spaces. However, the main novelty of our result arises of course in the noncompact case.

We point out once more that the main results (in particular Theorems 2.4 and 8.6) seem to be new even in case of a single-valued map F in a Banach space: Our apparently technical considerations of multivalued maps and uniformities actually simplify the formulation and proofs. For instance, one would have technical problems to speak about iterates of the restriction of a map $F\colon X \to X$ to a set $B \subseteq X$ which is not necessarily invariant under F while this is clear for a multivalued map $F\colon X \multimap X$ if empty values are allowed (cf. Theorem 8.5).

In order to motivate why results about the existence of compact attractors are of interest, we briefly sketch in the next Section 2 why our result implies that dissipative systems have periodic solutions (under natural additional hypotheses).

It is somewhat remarkable that, although most of our results are essentially of a point-set topological nature, we need never require any sort of continuity of the involved maps (except for the sketched sample application in Section 2 where we also want to apply a fixed point theorem).

2. PERIODIC SOLUTIONS OF DISSIPATIVE SYSTEMS

Consider in a Banach space $X = X_1 \times \cdots \times X_n$ the system

$$x' = f(t,x), \tag{2.1}$$

where $f\colon \mathbb{R} \times X \to X$ is continuous and τ-periodic with respect to the first variable. Such a system is called *dissipative* if there is some $D > 0$ such that for each solution x there is some t_x

with

$$\|x(t)\| < D \qquad (t > t_x). \tag{2.2}$$

The following result is classical, see e.g. [17, Theorem 39.3] or [29, p. 172-173].

Theorem 2.1. *Suppose $X = \mathbb{R}^n$ and that (2.1) is dissipative and has for each initial value at most one solution. Then (2.1) has a τ-periodic solution.*

The classical proofs are based on the fact that, due to the local compactness of $\mathbb{R}^n$ and the uniqueness, problem (2.1) is actually *uniformly dissipative* with some constant $D > 0$, i.e. such that the constant t_x in (2.2) can be chosen uniformly for all solutions with initial value of norm at most D. Then Browder's fixed point theorem is applied for the translation operator T_τ which associates to each initial value at time $t = 0$ the solution of (2.1) at time $t = \tau$.

In the lack of uniqueness, a classical result [5] implies that, under natural assumptions, the (multivalued) translation operator T_τ is the composition of an upper semicontinuous multivalued map with acyclic values (in $C([0,\tau],\mathbb{R}^n)$) and of a single-valued continuous (evaluation) map. Meanwhile, many generalizations of that result are known [12, 20], in particular also for an infinite-dimensional Banach space X and for more general problems like e.g.

$$x' \in f(t,x), \tag{2.3}$$

see, e.g., [1, 2, 16]. In particular, T_τ is admissible in the sense of Górniewicz [10]. Moreover, in case $X = \mathbb{R}^n$, the map T_τ is of course compact, and in case of infinite-dimensional X, the map T_τ is often at least condensing; see, e.g., [2, Theorem III.4.16]. Since a degree theory with the mod-p-property is known for such maps T_τ [7, 9, 11, 28] (this theory is based on the deep theory of chain approximations [26]), one could in principle prove some sort of Browder's fixed point theorem and thus a generalization of Theorem 2.1 for infinite-dimensional X without assuming uniqueness and for the more general problem (2.3), provided one *assumes* that (2.3) is uniformly dissipative.

However, a much simpler proof for Theorem 2.1 was suggested in [3]: Instead of using fixed point theorems of Browder type, one can use asymptotic fixed point theorems, i.e. for maps with a compact attractor. One such asymptotic fixed point theorem is the following from [2, p. 97] which we cite without going into details of the definitions:

Theorem 2.2. *Let X be an open subset of a Fréchet space and let $F\colon X \multimap X$ be an admissible condensing (on bounded sets) map with a compact attractor. Then the Lefschetz set $\Lambda(F)$ is well defined, and if $\Lambda(F) \neq \{0\}$ then F has a fixed point. The latter is in particular the case if X is the whole Fréchet space.*

Forerunners (and partial generalizations) of this result are contained in [24, 25] and in [9, 28] for single- and multivalued maps, respectively. Theorem 2.2 not only has the advantage that its proof is much more elementary than the proof of generalizations of Browder's theorem based on the sophisticated theory of chain approximations and the mod-p-property. It has also the advantage that, in our above considerations, we can relax the hypothesis of "uniform dissipativity" to "initial dissipativity":

Definition 2.3. Problem (2.1) or (2.3) is *initially dissipative* if there is a constant $D > 0$ such that for each initial value $x_0 \in X$ there is a constant t_{x_0} such that all solutions satisfying $x(0) = x_0$ satisfy $\|x(t)\| < D$ for all $t > t_{x_0}$.

Clearly, we have the implications

$$\text{uniformly dissipative} \implies \text{initially dissipative} \implies \text{dissipative}.$$

Moreover, in case of uniqueness (i.e. if for each x_0 there is at most one solution x satisfying $x(0) = x_0$) the last two properties are of course equivalent.

If $X = \mathbb{R}^n$, then the closed ball with radius $D > 0$ is for initially dissipative problems a compact attractor for $F := T_\tau$, and so Theorem 2.2 applies which shows that Theorem 2.1 holds also in the lack of uniqueness, provided one assumes that the problem is initially dissipative. However, using the result of this paper, we obtain a generalization also in infinite-dimensional X. We point out that this result seems to be new even in case of single-valued T_τ, since we require in this case only dissipativity without any uniformness.

Theorem 2.4. *Let X be a Banach space. Suppose that Problem (2.1) or (2.3) is such that the translation operator $T_\tau \colon X \multimap X$ is admissible and condensing on bounded sets. If the problem is initially dissipative, then it has a τ-periodic solution.*

Proof. By Theorem 8.6, $F := T_\tau \colon X \multimap X$ has a compact attractor and thus a fixed point by Theorem 2.2. ☐

3. NOTATIONS

Let X and Y be sets. By 2^X and 2^Y we denote the corresponding power sets. We are mainly interested in maps $F \colon 2^X \to 2^Y$ where we usually assume that F is *monotone*. By the latter, we will always mean monotonicity with respect to the subset relation, i.e.

$$A_1 \subseteq A_2 \subseteq X \implies F(A_1) \subseteq F(A_2) \subseteq Y.$$

If F is a self-map, the symbol F^n always denotes the n-th iterate, where we define $F^0 := id$, the identity map.

We use the symbol $F \colon X \multimap Y$ to denote a multivalued map from X to Y, i.e. F associates to each $x \in X$ a (possibly empty) subset of Y, denoted by $F(x)$. For a multivalued map $F \colon X \multimap Y$, it is usual to define the *image* and the *large preimage* by

$$F(A) := \bigcup_{x \in A} F(x) \qquad (A \subseteq X),$$

$$F^+(B) := \{x \in X : F(x) \cap B \neq \emptyset\} \qquad (B \subseteq Y),$$

respectively. Note that, by the first of these conventions (which is actually a slight misuse of notation, because we use the same symbol F), each single- or multivalued map $F \colon X \multimap Y$ can be understood as a monotone map $F \colon 2^X \to 2^Y$. However, not each monotone map $F \colon 2^X \to 2^Y$ is generated in such a way. For example, if X is a topological vector space and $F_0 \colon X \multimap Y$, then the monotone map $F \colon 2^X \to 2^Y$, defined by

$$F(M) := \overline{\operatorname{conv}}(F_0(M) \cup \{y_0\}) \tag{3.1}$$

with a fixed $y_0 \in Y$, is not generated from any map $G \colon X \multimap Y$ (except in some pathological cases). Actually, this example is the main reason why we consider such general monotone maps $F \colon 2^X \to 2^Y$ as we explain now. If $F_0 \colon X \multimap X$ is a multi-valued map of a closed convex subset of a topological vector space and if the monotone map (3.1) has the property that

$K := \bigcap_{n=0}^{\infty} F^n(X)$ is relatively compact (and thus compact) then K is a *nonempty compact convex* subset satisfying $F_0(K) \subseteq K$, and so standard variants of Schauder's fixed point theorem can be applied for $F_0 \colon K \multimap K$. Note that the precompactness of K can be shown if F_0 is condensing. This argument proves in particular the celebrated fixed point theorem of Darbo [6] (and extensions thereof, e.g. in locally convex spaces). More general, in the context of degree theory (or so-called 0-*epi maps*), one frequently uses similar considerations concerning iterates of maps of the form

$$F(M) := \overline{\operatorname{conv}}(F_0(M \cap \Omega) \cup V) \tag{3.2}$$

with certain fixed sets $\Omega \subseteq X$, $V \subseteq Y$; see, e.g., [27]. This motivates why we are interested also to study iterates (and attractors) of general monotone $F \colon 2^X \to 2^X$ and not only of multivalued maps $F \colon X \multimap X$ or even only of single-valued maps $F \colon X \to X$. For $F \colon X \multimap X$, we use of course the symbols $id = F^0$ and F^n to denote the multivalued maps $F^n \colon X \multimap X$ which generate the iterates of $F^n \colon 2^X \to 2^X$ (i.e., we keep the slight ambiguity in notation).

Clearly, one can identify $F \colon X \multimap Y$ with its *graph*

$$\operatorname{Graph}(F) := \{(x,y) \in X \times Y : y \in F(x)\}.$$

Note that, conversely, to each subset $U \subseteq X \times Y$ there is a multivalued function $F_U \colon X \multimap Y$ with $\operatorname{Graph}(F_U) = U$. For this reason, we will tacitly identify F with its graph without any further distinction, i.e. $F \colon X \multimap Y$ means also $F \subseteq X \times Y$ and vice versa. To each $U \subseteq X \times Y$, one associates the *inverse*

$$U^{-1} := \{(y,x) \in Y \times X : (x,y) \in U\}.$$

For example, with the above identification, we have $F^+(B) = F^{-1}(B)$ (for $F \colon X \multimap Y$ and $B \subseteq Y$).

It will be convenient for us to describe (quasi-)uniformities in terms of multivalued functions.

Definition 3.1. Let X be some set. A family $\mathscr{U}$ of multivalued maps $U \colon X \multimap Y$ is a *quasi-uniformity* on X (and the pair $(X, \mathscr{U})$ a *quasi-uniform space*) if $\mathscr{U}$ is a *filter* on $X \times X$, i.e.

(1) $U, V \in \mathscr{U}$ implies $U \cap V \in \mathscr{U}$, and
(2) $U \in \mathscr{U}$ and $U \subseteq V \subseteq X \times X$ imply $V \in \mathscr{U}$,

and if each $U \in \mathscr{U}$ has the following properties:

(1) $x \in U(x)$ for each $x \in X$.
(2) There is some $V \in \mathscr{U}$ with $V^2 \subseteq U$.

If additionally $U^{-1} \in \mathscr{U}$ for each $U \in \mathscr{U}$, then $\mathscr{U}$ is a *uniformity* (and $(X, \mathscr{U})$ a *uniform space*).

It is well-known that each quasi-uniform space is in particular a topological space when one defines the neighborhoods of a point $x \in X$ as the sets $U(x)$ $(U \in \mathscr{U})$. Since we are mainly interested in metric spaces, we are also mainly interested in uniformities. However, our reason to consider quasi-uniformities instead of uniformities is that every topology can be generated in such a way, see [8, Section 2.1]. In contrast, topological spaces which can be generated by uniformities are precisely the T_{3a} spaces. This advantage of quasi-uniformities will become relevant in particular in Corollaries 7.16 and 7.17.

Recall that each topological group (and thus in particular each topological vector space with the addition as group operation) is a uniform space (with the same topology) where $\mathscr{U}$ consists of all sets $U \subseteq X \times X$ which contain a set of the form $\{(x,y) \in X \times X : y^{-1}x \in O\}$, where O is a neighborhood of the neutral element. It is also known that $(X,\mathscr{U})$ is a uniform space if and only if there is a family D of pseudometrics on X (i.e., in contrast to a metric, it is not required that d is finite or that $d(x,y)=0$ implies $x=y$) such that D generates $\mathscr{U}$, i.e., such that $U\subseteq X\times X$ belongs to $\mathscr{U}$ if and only if there are $\varepsilon>0$ and finitely many $d_1,\dots,d_n\in D$ with

$$\{(x,y)\in X\times X: d_i(x,y)<\varepsilon \text{ for } i=1,\dots,n\}\subseteq U.$$

Actually, it can be arranged that each $d\in D$ assumes only finite values. If the filter $\mathscr{U}$ has a countable base, D can clearly be chosen to consist of only one element. An analogous result holds for quasi-uniform spaces if we consider a so-called quasi-metric which is not required to be symmetric [8, Section 1.5].

Of particular interest for us is the case of a metric space, but it will make no difference to consider the more general case of a *q-quasi-pseudometric*, i.e. $d\colon X\times X\to[0,\infty]$ is assumed to satisfy only (for all $x,y,z\in X$)

(1) $d(x,x)=0$
(2) $d(x,z)\le q\cdot(d(x,y)+d(y,z))$

where q is a finite constant. If additionally $d(x,y)=d(y,x)$, we speak of a *q-pseudometric*. An example where the choice $q=1$ is not possible in general is $L_p(S)$ $(0<p<1)$ with

$$d(x,y):=\Big(\int_S |x(s)-y(s)|^p\,ds\Big)^{1/p}.$$

In this example, d is a $2^{1/p}$-pseudometric. It is easily checked that $D:=\{d\}$ induces a quasi-uniformity as above, and that the topology induced by d corresponds with the topology of that quasi-uniformity. By the above remarks, we have in particular:

Proposition 3.2. For each q-(quasi-)pseudometric there is a 1-(quasi-)pseudometric assuming only finite values inducing the same (quasi-)uniformity and thus the same topology.

In the above example, such an equivalent 1-pseudometric is given by

$$d(x,y):=\int_S |x(s)-y(s)|^p\,ds.$$

In the case of a q-quasi-pseudometric d there is a certain correspondence between the quasi-uniformity and $(0,\infty]$. For simplicity, we use the same letter d to denote this correspondence:

Definition 3.3. Let d be a q-quasi-pseudometric on X. For $\varepsilon\in[0,\infty]$, we use the notation

$$U_{d,\varepsilon}:=\{(x,y)\in X\times X: d(x,y)<\varepsilon\},$$

and we denote by $d(\varepsilon)$ the family of all sets $U\subseteq X\times X$ satisfying $U_{d,\varepsilon}\subseteq U$.

Conversely, for $U\subseteq X\times X$, we put

$$d(U):=\sup\{\varepsilon\in[0,\infty]: U\in d(\varepsilon)\}=\sup\{\varepsilon\in[0,\infty]: U_{d,\varepsilon}\subseteq U\}.$$

Note that for $\varepsilon=0$, we always have $U\in d(0)$, i.e. the supremum is defined.

It follows from the very definition:

Proposition 3.4. The uniformity $\mathscr{U}$ generated by a q-quasi-pseudometric d is the union of the sets $d(\varepsilon)$ $(\varepsilon>0)$. In particular, $U\subseteq X\times X$ belongs to $\mathscr{U}$ if and only if $d(U)>0$.

It will turn out that the Hausdorff measure of noncompactness will be in a natural correspondence with precompact sets:

Definition 3.5. Let $(X,\mathscr{U})$ be a quasi-uniform space. Then $M\subseteq X$ is *precompact* (with respect to $\mathscr{U}$) if and only if for each $U\in\mathscr{U}$ there is a finite set $N\subseteq M$ with $M\subseteq U(N)$.

In the case of a q-quasi-pseudometric space X this means that for each $\varepsilon>0$ there is some finite ε-net $N\subseteq M$. The latter means, by definition, that for each $y\in M$ there is some $x\in N$ such that $d(x,y)<\varepsilon$.

Recall that a filter $\mathscr{F}$ on a quasi-uniform space $(X,\mathscr{U})$ is a *Cauchy filter* if for each $U\in\mathscr{U}$ there is some $x\in X$ with $U(x)\in\mathscr{F}$. $(X,\mathscr{U})$ is called *complete* if each Cauchy filter converges. If the quasi-uniformity is induced by a q-quasi-pseudometric (and thus is actually induced by a quasi-metric) this is the case if and only if each Cauchy sequence converges, see [8, Section 7.33]. Recall in this connection that if $\mathscr{U}$ is a quasi-uniformity, a sequence $x_n\in X$ is called a *Cauchy sequence* if for each $U\in\mathscr{U}$ there is some $x\in X$ such that $x_n\in U(x)$ for all except finitely many n.

A set $M\subseteq X$ is compact if and only if M is precompact and complete with respect to the induced quasi-uniformity

$$\mathscr{U}_M:=\{U\cap(M\times M):U\in\mathscr{U}\}.$$

In particular, if X is complete, then a closed subset is compact if and only if it is precompact.

Note, however, that the nontrivial implication of the just mentioned two equivalences requires the (uncountable) axiom of choice.

Proposition 3.6. In ZF (without AC) the following statements are equivalent:

1. PI (Boolean prime ideal theorem: “every Boolean algebra contains a prime ideal”).
2. Every filter is contained in an ultrafilter.
3. Each product of compact Hausdorff spaces is compact.
4. Each precompact complete quasi-uniform space is compact.
5. Each precompact complete uniform Hausdorff space is compact.

Proof. The equivalence of the first three statements is well-known, see [21] (for the formulation in terms of ultrafilters see e.g. the discussion in [22]). If every filter is contained in some ultrafilter, then the proof of [8, Section 3.24] shows that every precompact complete quasi-uniform space is compact. Conversely, suppose that each precompact complete uniform Hausdorff space is compact. If $X=\prod_i X_i$ with compact Hausdorff spaces X_i, then the topology on each X_i is induced by a uniquely determined uniformity. Hence, there is a corresponding product uniformity on X which induces the product topology. Since each X_i is precompact and Hausdorff, a standard calculation shows that X is precompact and Hausdorff. Similarly, each Cauchy filter $\mathscr{F}$ on X induces a Cauchy filter $\mathscr{F}_i$ on each X_i; since $\mathscr{F}_i$ has a *unique* limit point x_i in X_i (because X_i is *Hausdorff* and complete), we can define a point $x\in X$ by $x:=(x_i)_i$ without using AC. A standard calculation shows that $\mathscr{F}$ has the limit point x. Hence, $X=\prod_i X_i$ is complete (and precompact) and thus compact, by hypothesis. □

Although it is known that PI is logically strictly weaker than AC [13], PI is (like AC) in the author’s opinion still an unnatural axiom for applications of analysis in physics. Thus, unless stated explicitly, we will only work in ZF with the so-called axiom of dependent choices (DC) which allows only countably many recursive choices and thus allows to prove all “standard” results of analysis. In particular, we will not use any of the (equivalent) statements of Proposition 3.6 without further remark.

4. THE HAUSDORFF MEASURE OF NONCOMPACTNESS IN QUASI-UNIFORM SPACES

Throughout this section, let $(X,\mathscr{U})$ be a quasi-uniform space, and $Y \subseteq X$.

Definition 4.1. For $M \subseteq X$, we define the *fine quasi-uniform Hausdorff measure of noncompactness* of $M \subseteq X$ with respect to Y as the family

$$\mathscr{H}_Y^\circ(M) := \{U \in \mathscr{U} : \text{There is no finite set } N \subseteq Y \text{ with } M \subseteq U(N)\}.$$

The smaller the family $\mathscr{H}_Y^\circ(M)$ is, the more precompact is M:

Proposition 4.2. (1) If M is precompact then $\mathscr{H}_M^\circ(M) = \emptyset$.
(2) Conversely, if $\mathscr{H}_Y^\circ(M) = \emptyset$ for some $Y \subseteq M$ then M is precompact.
(3) If $\mathscr{U}$ is a uniformity and $\mathscr{H}_Y^\circ(M) = \emptyset$ for some $Y \subseteq X$ then M is precompact.
(4) If $M_1 \subseteq M_2$ and $Y_1 \supseteq Y_2$ then $\mathscr{H}_{Y_1}^\circ(M_1) \subseteq \mathscr{H}_{Y_2}^\circ(M_2)$.
(5) In particular, if $\mathscr{U}$ is a uniformity then $\mathscr{H}_M^\circ(M) = \emptyset$ if and only if $\mathscr{H}_Y^\circ(M) = \emptyset$ $(M \subseteq Y \subseteq X)$.
(6) If $U \in \mathscr{H}_Y^\circ(M)$ and $V \in \mathscr{U}$ satisfy $V \subseteq U$ then $V \in \mathscr{H}_Y^\circ(M)$.

Proof. Only the third claim is not immediate. Thus, assume $\mathscr{H}_Y^\circ(M) = \emptyset$, and let $U \in \mathscr{U}$. Choose some $V \in \mathscr{U}$ with $V^2 \subseteq U$. Since $\mathscr{U}$ is a uniformity, we have $W := V \cap V^{-1} \in \mathscr{U}$. Since $W \in \mathscr{U} \setminus \mathscr{H}_Y^\circ(M)$, there is a finite set $N_0 \subseteq Y$ with $M \subseteq W(N_0)$. Let $N_1 = \{y_1,\dots,y_n\}$ be the collection of all $y \in N_0$ which satisfy $W(y) \cap M \neq \emptyset$. Since $W(N_0 \setminus N_1) \cap M = \emptyset$, we have $M = W(N_0) \cap M \subseteq W(N_1)$. For $k = 1,\dots,n$, choose some $x_k \in W(y) \cap M$. Then $N := \{x_1,\dots,x_n\} \subseteq M$ satisfies $N_1 \subseteq W^+(N) = W^{-1}(N) = W(N)$, and so $M \subseteq W(N_1) \subseteq W^2(N) \subseteq V^2(N) \subseteq U(N)$. Hence, M is precompact. □

In view of the last property of Proposition 4.2, it might be tempting to consider the complement $\mathscr{F}_Y(M) := \mathscr{U} \setminus \mathscr{H}_Y^\circ(M)$ as a filter which measures the noncompactness of M (i.e. with the inverse order as is usual for filters: the larger the filter the smaller the measure). Indeed, by the above observation and since $\mathscr{F}_Y(M) \subseteq \mathscr{U}$, one clearly has

(1) $X \times X \in \mathscr{F}_Y(M)$.
(2) $U_1,\dots,U_n \in \mathscr{F}_Y(M)$ implies $U_1 \cap \cdots \cap U_n \neq \emptyset$.
(3) $U \in \mathscr{F}_Y(M)$ and $U \subseteq V \subseteq X \times X$ implies $V \in \mathscr{H}_Y^\circ(M)$.

Unfortunately, $\mathscr{F}_Y(M)$ is in general not a filter, even if $\mathscr{U}$ is a uniformity. Although the above properties imply that $\mathscr{F}_Y(M)$ is a base of a filter, this filter contains no information about M as the following disappointing observation shows. (Note in this connection that all our definitions are only useful if Y is infinite.)

Example 4.3. Let Y contain at least two points. Then for each $M \subseteq X$ the filter generated by the set $\mathscr{F}_Y(M) := \mathscr{U} \setminus \mathscr{H}_Y^\circ(M)$ is $\mathscr{U}$.

Proof. Let $y_1, y_2 \in Y$ satisfy $y_1 \neq y_2$. For $U \in \mathscr{U}$ define $U_i := U \cup (\{y_i\} \times X)$ $(i = 1,2)$. Then $U_i(y_i) \supseteq M$, and so $U_i \in \mathscr{F}_Y(M)$. Hence, $U = U_1 \cap U_2$ (recall $y_1 \neq y_2$) belongs to the filter generated by $\mathscr{F}_Y(M)$. □

Hence, it does not seem appropriate to consider any other concepts from the theory of filters, but (as we do) to consider the family $\mathscr{H}_Y^\circ(M)$ (or the "almost-filter" $\mathscr{F}_Y(M)$) as the measure itself.

For the case that $\mathscr{U}$ is induced by a q-quasi-pseudometric d, one can of course measure this family by the (possibly infinite) number

$$\chi_Y(M) := \sup\{d(U) : U \in \mathscr{H}_Y^\circ(M)\}. \tag{4.1}$$

Here and in the following we use the conventions $\sup\emptyset := 0$ and $\inf\emptyset := \infty$. By this convention and Proposition 3.4 it is clear that

$$\chi_Y(M) = 0 \iff \mathscr{H}_Y^\circ(M) = \emptyset.$$

The quantity χ_Y is indeed what is usually called the Hausdorff measure of noncompactness with respect to Y (with respect to the q-quasi-pseudometric d):

Proposition 4.4. $\chi_Y(M) = \inf\{\varepsilon > 0 : M \text{ has a finite } \varepsilon\text{-net in } Y\}$ $(\in [0,\infty])$.

Proof. Let S denote the right-hand side. Assume first $\chi_Y(M) > S$. Let $S < \varepsilon_0 < \varepsilon < \chi_Y(M)$. Then M has a finite ε_0-net $N \subseteq Y$. Hence, for each $U \in \mathscr{U}$ satisfying $d(U) \geq \varepsilon$, we have $U(N) \subseteq M$ and thus $U \notin \mathscr{H}_Y^\circ(M)$. Consequently, $\chi_Y(M) \leq \varepsilon$, a contradiction

Conversely, assume $S > \chi_Y(M)$. Let $\chi_Y(M) < \varepsilon < S$. Then $U := U_{d,\varepsilon}$ does not belong to $\mathscr{H}_Y^\circ(M)$, since otherwise $\chi_Y(M) \geq d(U) \geq \varepsilon$. Hence, there is a finite set $N \subseteq Y$ with $M \subseteq U(N)$. By the definition of U this means that N is a finite ε-net. Hence, $S \leq \varepsilon$, a contradiction. □

In view of (4.1), one might be tempted to think that, in the case of q-quasi-pseudometric spaces, the Hausdorff and the fine quasi-uniform Hausdorff measure of noncompactness share practically the same properties. However, this is not true, since the formula (4.1) contains a "sup". Roughly speaking, $\mathscr{H}^\circ$ is a much finer "measure" than χ. For example, in contrast to the Hausdorff measure of noncompactness χ_X the fine quasi-uniform Hausdorff measure $\mathscr{H}_X^\circ$ can "measure the difference in noncompactness" of the open and closed unit ball in a normed space X.

Example 4.5. Let B be the open unit ball in an infinite-dimensional normed space X, and let $0 \in Y \subseteq X$. Then $\mathscr{H}_Y^\circ(B) \neq \mathscr{H}_Y^\circ(\overline{B})$. More precisely, $U := \{(x,y) \in X \times X : \|x-y\| < 1\}$ belongs to $\mathscr{H}_X^\circ(\partial B) \subseteq \mathscr{H}_Y^\circ(\overline{B})$ but not to $\mathscr{H}_{\{0\}}^\circ(B) \supseteq \mathscr{H}_Y^\circ(B)$.

Proof. The second claim follows from $B \subseteq U(0)$. For the first claim, assume by contradiction that there is a finite set $N \subseteq X$ such that $U(N)$ contains the unit sphere ∂B. Then $X_0 := \operatorname{span} N$ is a finite-dimensional subspace, and so a well-known variant of Riesz's Lemma implies that the unit sphere ∂B contains some "orthogonal" element x_0 with $\operatorname{dist}(x_0, X_0) = 1$. In particular, $x_0 \notin U(N)$, a contradiction. □

In particular, $\mathscr{H}_X^\circ(M) \neq \mathscr{H}_X^\circ(\overline{M})$ in general, although it is well-known that $\chi_X(M) = \chi_X(\overline{M})$ for each $M \subseteq X$. Similarly, $\mathscr{H}^\circ$ is lacking another useful property which χ has in case of an 1-quasi-pseudometric space: The definition of $\mathscr{H}^\circ$ is not equivalent to the definition

$$\mathscr{H}_Y^\infty(M) := \{U \in \mathscr{U} : \text{There is no precompact set } N \subseteq Y \text{ with } M \subseteq U(N)\},$$

as the following example shows.

Example 4.6. Consider Example 4.5 with $X = c_0$. Then there is a countable compact set $N \subseteq B$ with $\overline{B} \subseteq U(N)$. In particular, $U \in \mathscr{H}_Y^\circ(\overline{B}) \setminus \mathscr{H}_Y^\infty(\overline{B})$ for each Y with $B \subseteq Y \subseteq X$.

Proof. Each element of c_0 assuming only values in $\{0,\pm 1\}$ is an eventually 0 sequence, and so the set of these elements is countable. Let $\{x_k : k = 1,2,\dots\} \subseteq \overline{B}$ be an enumeration of this set, and put

$$N := \{\frac{x_k}{2k} : k = 1,2,\dots\} \subseteq B.$$

Then N is the union of a null sequence with $\{0\}$ and thus compact. For each $x = (\xi_n)_n \in \overline{B}$ there is some index n_0 with $|\xi_n| < 1$ for all $n > n_0$. There is some k such that $x_k = (\eta_n)_n$ satisfies $\eta_n = \operatorname{sgn}\xi_n$ for $n \le n_0$ and $\eta_n := 0$ for $n > n_0$. Hence, $\|x - x_k/(2k)\| < 1$, i.e. $x \in U(x_k)$. Thus, $\overline{B} \subseteq U(N)$. □

Remark 4.7. Although the same construction works also in $X = \ell_p$ $(1 \le p < \infty)$, it is not clear whether Example 4.6 holds for each separable Banach space, because for an arbitrary dense set $\{x_k : k = 1,2,\dots\} \subseteq \overline{B}$ it is not necessarily true that $\overline{B} \subseteq U(N)$.

To get rid of the problems of Examples 4.5 and 4.6, we choose the following variant of $\mathscr{H}^\circ$ to define the Hausdorff measure of noncompactness in a uniform space:

Definition 4.8. The *left/right quasi-uniform Hausdorff measure of noncompactness* and the *quasi-uniform Hausdorff measure of noncompactness* of $M \subseteq X$ with respect to $Y \subseteq X$ are defined by

$$\mathscr{H}_Y^-(M) := \{U \in \mathscr{U} : \text{There is } V \in \mathscr{U} \text{ with } V \circ U \in \mathscr{H}_Y^\circ(M)\},$$

$$\mathscr{H}_Y^+(M) := \{U \in \mathscr{U} : \text{There is } W \in \mathscr{U} \text{ with } U \circ W \in \mathscr{H}_Y^\circ(M)\}, \tag{4.2}$$

$$\mathscr{H}_Y(M) := \{U \in \mathscr{U} : \text{There are } V, W \in \mathscr{U} \text{ with } V \circ U \circ W \in \mathscr{H}_Y^\circ(M)\}, \tag{4.3}$$

respectively.

The definition immediately implies

$$\mathscr{H}_Y(M) \subseteq \mathscr{H}_Y^\pm(M) \subseteq \mathscr{H}_Y^\circ(M). \tag{4.4}$$

The above definitions do not have the disadvantages of Examples 4.5, 4.6, or both, respectively, although we still have the connection with the usual Hausdorff measure of noncompactness:

Proposition 4.9. For each $M, Y \subseteq X$, we have

$$\mathscr{H}_Y^-(M) = \mathscr{H}_Y^-(\overline{M}), \quad \mathscr{H}_Y(M) = \mathscr{H}_Y(\overline{M}), \tag{4.5}$$

and in (4.2) and (4.3) one may equivalently replace $\mathscr{H}_Y^\circ(M)$ by $\mathscr{H}_Y^\infty(M)$. In particular,

$$\mathscr{H}_Y(M) \subseteq \mathscr{H}_Y^+(M) \subseteq \mathscr{H}_Y^\infty(M) \subseteq \mathscr{H}_Y^\circ(M). \tag{4.6}$$

Moreover, in Proposition 4.2 and, if X is an 1-quasi-pseudometric space also in (4.1), one can equivalently replace $\mathscr{H}^\circ$ by any of $\mathscr{H}^\pm$, $\mathscr{H}^\infty$, or $\mathscr{H}$.

In particular, $\mathscr{H}$ actually measures the non-precompactness and shares the above mentioned nice properties with χ. For this reason, $\mathscr{H}$ can indeed be considered as the "quasi-uniform Hausdorff measure of noncompactness".

Proof. For the first claim, note that for each $V \in \mathscr{U}$ there is some $V_0 \in \mathscr{U}$ with $V_0^2 \subseteq V$. Hence, if $M \subseteq (V_0 \circ U)(N)$ then $\overline{M} \subseteq V_0((V_0 \circ U)(N)) \subseteq (V \circ U)(N)$. Thus $V \circ U \in \mathscr{H}_Y^\circ(M)$ implies $V_0 \circ U \in \mathscr{H}_Y^\circ(\overline{M})$, and the same statement holds of course if one replaces U by $U \circ W$.

For the second claim, choose similarly for $W\in\mathscr{U}$ some $W_0\in\mathscr{U}$ with $W_0^2\subseteq W$ and note that for each precompact set $N\subseteq Y$ there is some finite set $N_0\subseteq N\subseteq Y$ with $N\subseteq W_0(N_0)$. Hence, if $M\subseteq(U\circ W_0)(N)$ then $M\subseteq(U\circ W_0)(W_0(N_0))\subseteq(U\circ W)(N_0)$. Thus, $U\circ W\in\mathscr{H}_Y^\circ(M)$ implies $U\circ W_0\in\mathscr{H}_Y^\infty(M)$, and the same statement holds of course if one replaces U by $V\circ U$.

The first inclusion of (4.6) is part of (4.4), and the second inclusion follows from the second claim which we had proved above. The last inclusion follows from the fact that each finite set is precompact.

Now we prove the claim about Proposition 4.2. If $M\subseteq X$ is precompact then $\mathscr{H}_M^\circ(M)=\emptyset$, and so (4.4)/(4.6) implies that also $\mathscr{H}_M(M),\mathscr{H}_M^\pm(M),\mathscr{H}_Y^\infty(M)=\emptyset$. Conversely, let some of the sets $\mathscr{H}_M(M),\mathscr{H}_M^\pm(M),\mathscr{H}_Y^\infty(M)$ be empty; by (4.4)/(4.6) we have $\mathscr{H}_Y(M)=\emptyset$. For each $U\in\mathscr{U}$, there are $V,W\in\mathscr{U}$ with $V^2\subseteq U$ and $W^2\subseteq V$. Since $W\notin\mathscr{H}_Y(M)$, we have $V\circ W\circ W\notin\mathscr{H}_Y^\circ(M)$, and since $V\circ W\circ W\subseteq V^2\subseteq U$, this implies $U\notin\mathscr{H}_Y^\circ(M)$. Hence Proposition 4.2 implies that U is precompact. The proof of the other claims of Proposition 4.2 for $\mathscr{H},\mathscr{H}^\pm,\mathscr{H}^\infty$ is straightforward.

The claim about (4.1) follows in view of (4.4)/(4.6) from the fact that for each $U_0\in\mathscr{H}_Y^\circ(M)$ and each $\varepsilon<d(U_0)$ there are $U,V,W\in\mathscr{U}$ with $d(U)\geq\varepsilon$ and $V\circ U\circ W\subseteq U_0$ and thus $U\in\mathscr{H}_Y(M)$; hence also $U\in\mathscr{H}_Y^\pm(M),\mathscr{H}_Y^\infty(M)$ by (4.6). ☐

If $\mathscr{U}$ is a uniformity or if $Y\subseteq M$, the following result is an immediate consequence of Proposition 4.2/4.9. However, in general, a separate proof is needed.

Proposition 4.10. For each $M,Y\subseteq X$ either none or all of the 10 sets $\mathscr{H}_Y(M)=\mathscr{H}_Y(\overline{M})$, $\mathscr{H}_Y^-(M)=\mathscr{H}_Y^-(\overline{M})$, $\mathscr{H}_Y^+(M)$, $\mathscr{H}_Y^+(\overline{M})$, $\mathscr{H}_Y^\infty(M)$, $\mathscr{H}_Y^\infty(\overline{M})$, $\mathscr{H}_Y^\circ(M)$, and $\mathscr{H}_Y^\circ(\overline{M})$ are empty.

Although an elementary proof might be given, we postpone it until we have shown the more general Propositions 5.2 and 5.6.

We show now that all of the above definitions yield different sets, in general, even for normed spaces. Since (as we already have seen) $\mathscr{H}_Y^\circ(M)\neq\mathscr{H}_Y^\circ(\overline{M})$ in general, (4.5) implies $\mathscr{H}^\circ\neq\mathscr{H}^-,\mathscr{H}$. The following example shows that $\mathscr{H}_Y^+(M)\neq\mathscr{H}_Y^+(\overline{M})$ in general, and thus $\mathscr{H}^+\neq\mathscr{H}^-,\mathscr{H}$.

Example 4.11. Consider in Example 4.5 the set

$$U_0:=\{(x,y)\in X\times X:\|x-y\|<1/2\}\cup(\{0\}\times B).$$

Then U_0 belongs to $\mathscr{H}_Y^+(\partial B)\subseteq\mathscr{H}_Y^+(\overline{B})\subseteq\mathscr{H}_Y^\circ(\overline{B})$ but not to $\mathscr{H}_Y^\circ(B)\supseteq\mathscr{H}_Y^+(B)$.

Proof. The second statement follows from $B\subseteq U_0(0)$. For the first statement, observe that $U_0\in\mathscr{U}$ and put

$$W_0:=\{(x,y)\in X\times X:\|x-y\|<1/2\}. \tag{4.7}$$

Then $U_0\circ W_0\subseteq U\in\mathscr{H}_Y^\circ(\partial B)$ by Example 4.5, and so $U_0\in\mathscr{H}_Y^-(\partial B)$. ☐

The following examples imply the remaining inequalities $\mathscr{H}^\circ\neq\mathscr{H}^+$ and $\mathscr{H}^-\neq\mathscr{H}$:

Example 4.12. The set U in Example 4.5 satisfies $U\in\mathscr{H}_Y^\circ(\overline{B})\setminus\mathscr{H}_Y^\pm(\overline{B})$.

Proof. $U\in\mathscr{H}_Y^\circ(\overline{B})$ has been shown in Example 4.5. For each $V\in\mathscr{U}$ the sets $(V\circ U)(0)$ and $(U\circ V)(0)$ both contain

$$\{x\in X:\|x\|<d(U)+d(V)\}\supseteq\overline{B},$$

and so $V\circ U,U\circ V\notin\mathscr{H}_Y^\circ(\overline{B})$. Hence, $U\notin\mathscr{H}_Y^\pm(\overline{B})$. ☐

Example 4.13. In Example 4.5, the set

$$U_1 := \{(x,y) \in X \times X : \|x-y\| < 1/2\} \cup \bigcup_{x \neq 0} (\{x\} \times \operatorname{span}\{x\})$$

belongs to $\mathscr{H}_Y^-(\partial B) \subseteq \mathscr{H}_Y^-(\overline{B})$ but not to $\mathscr{H}_Y^+(X) \supseteq \mathscr{H}(X) \supseteq \mathscr{H}_Y(\overline{B})$.

Proof. For each $W \in \mathscr{U}$ the set $W(0)$ is a neighborhood of 0, and so $U_1(W(0)) = X$, i.e. $U_1 \circ W \notin \mathscr{H}_Y^\circ(X)$. Hence, $U_1 \notin \mathscr{H}_Y^+(X)$.

Let $W_0 \in \mathscr{U}$ be defined by (4.7). Assume by contradiction that there is a finite set $N \subseteq Y$ with $\partial B \subseteq (W_0 \circ U_1)(N)$. As in Example 4.5, we find for $X_0 := \operatorname{span}(N)$ by a variant of Riesz's Lemma some $x_0 \in \partial B$ with $\operatorname{dist}(x_0, X_0) = 1$. Since $x_0 \in W_0(U_1(N))$, there is some $y_0 \in U_1(N)$ with $\|x_0 - y_0\| < 1/2$. Since in particular $\operatorname{dist}(y_0, N) > \operatorname{dist}(x_0, N) - 1/2 \geq 1/2$, the definition of U_1 implies $y_0 \in \operatorname{span} N = X_0$ and thus $\operatorname{dist}(x_0, X_0) \leq \|x_y - y_0\| < 1/2$, a contradiction. Since $U_1 \in \mathscr{U}$, we thus have shown $W_0 \circ U_1 \in \mathscr{H}_Y^\circ(\partial B)$, hence $U_1 \in \mathscr{H}_Y^-(\partial B)$. □

It is well-known that if $X = Y$ is a normed space (and even in more general situations) the formula

$$\chi_Y(\operatorname{conv} M) = \chi_Y(M) \qquad (M \subseteq X)$$

holds. The reader should be warned that an analogous result fails for each of the "quantities" $\mathscr{H}^\circ$, $\mathscr{H}^\infty$, $\mathscr{H}$ and $\mathscr{H}^\infty$. For example, if M is the unit sphere in an infinite-dimensional normed space, then $\mathscr{H}_X^\circ(M)$ does not contain the set

$$U := \{(x,y) : \|x-y\| < 1/2\} \cup (\{0\} \times M),$$

because $M \subseteq U(0)$ while a similar argument as in Example 4.5 shows that $U \in \mathscr{H}_X^\circ(\operatorname{conv} M)$.

5. CONDENSIFYING MAPS

As announced in the introduction, we are interested in iterates of monotone maps $F \colon 2^X \to 2^X$ (which are considered as generalizations of maps $F \colon X \to X$ or $F \colon X \multimap X$). The crucial role will be played by the sets $X_n := F^n(X)$ which in view of the monotonicity of F are monotonically decreasing. However, as far as possible, we work with arbitrary sequences of sets $X_n \subseteq X$ which are not necessarily decreasing.

5.1. **Condensifying Sequences.**

Throughout this section, let $(X, \mathscr{U})$ be a quasi-uniform space, $Y \subseteq X$, and $(X_n)_n$ be a sequence of subsets of X. Actually the following definition was the motivation to define $\mathscr{H}_Y(M)$ as a family of sets. Although we will soon give a characterization also in terms of the "classical" Hausdorff measure of noncompactness, it appears that this definition is much more natural and convenient (even if X is a metric space).

Definition 5.1. The sequence $(X_n)_n$ is called *condensifying* (with respect to Y) if

$$\bigcap_n \bigcup_{k=n}^\infty \mathscr{H}_Y(X_k) = \emptyset,$$

almost condensifying (with respect to Y) if

$$\bigcup_n \bigcap_{k=n}^\infty \mathscr{H}_Y(X_k) = \emptyset,$$

and *weakly condensifying* (with respect to Y) if

$$\bigcap_n \mathscr{H}_Y(X_n) = \emptyset.$$

Clearly, each condensifying sequence is almost condensifying, and each almost condensifying sequence is weakly condensifying.

For decreasing sequences (i.e. $X_{n+1} \subseteq X_n$) the three definitions coincide. Indeed, Proposition 4.2/4.9 implies $\mathscr{H}_Y(X_{n+1}) \subseteq \mathscr{H}_Y(X_n)$ for all n. However, in general, it seems to be a good idea to consider (at least) the three cases of Definition 5.1; the reason will become clearer in Proposition 7.4. Our first observation is that the particular choice of the Hausdorff measure plays no role here.

Proposition 5.2. In Definition 5.1, one may equivalently replace $\mathscr{H}$ by any of $\mathscr{H}^{\pm}$, $\mathscr{H}^{\infty}$, or $\mathscr{H}^{\circ}$.

Proof. In view of (4.4) and (4.6), it suffices to show that any (weakly/almost) condensifying sequence satisfies

$$\bigcap_n \bigcup_{k=n}^{\infty} \mathscr{H}_Y^{\circ}(X_k) = \emptyset, \quad \bigcup_n \bigcap_{k=n}^{\infty} \mathscr{H}_Y^{\circ}(X_k) = \emptyset, \quad \text{or} \quad \bigcap_n \mathscr{H}_Y^{\circ}(X_n) = \emptyset, \tag{5.1}$$

respectively. Thus, let $(X_n)_n$ be (weakly/almost) condensifying, and let $U \in \mathscr{U}$. Choose $V \in \mathscr{U}$ and $W \in \mathscr{U}$ with $V^2 \subseteq U$ and $W^2 \subseteq V$. By hypothesis, we find some index n with $W \notin \mathscr{H}_Y(X_k)$ for $k = n$, infinitely many k, or for all $k \geq n$, respectively. In particular, $V \circ W \circ W \notin \mathscr{H}_Y^{\circ}(X_k)$. Since $V \circ W \circ W \subseteq V^2 \subseteq U$, this implies $U \notin \mathscr{H}_Y^{\circ}(X_k)$ for all the corresponding k. This shows the respective property of (5.1). ☐

In case of a q-quasi-pseudometric space, the following result shows that one can rewrite the above definition in terms of the usual Hausdorff measure of noncompactness. However, the above definition will be more convenient for our purposes (and is possible even if $\mathscr{U}$ is not induced by a quasi-pseudometric).

Proposition 5.3. Let X be a q-quasi-pseudometric space. A sequence $(X_n)_n$ is condensifying with respect to Y if and only if

$$\lim_{n\to\infty} \chi_Y(X_n) = 0, \tag{5.2}$$

almost condensifying with respect to Y if and only if

$$\liminf_{n\to\infty} \chi_Y(X_n) = 0, \tag{5.3}$$

and weakly condensifying with respect to Y if and only if

$$\inf_n \chi_Y(X_n) = 0. \tag{5.4}$$

Proof. We apply Proposition 5.2. If $(X_n)_n$ is not (almost/weakly) condensifying, then there is some $U \in \mathscr{U}$ such that $U \in \mathscr{H}_Y^{\circ}(X_n)$ for infinitely many n (almost all n/all n). Put $\varepsilon := d(U) > 0$ (Proposition 3.4). For each n with $U \in \mathscr{H}_Y^{\circ}(X_n)$, we have $\chi_Y(X_n) \geq \varepsilon$ by (4.1). Hence, $\limsup \chi_Y(X_n)$ (or (5.3)/(5.4), respectively), is bounded from below by $\varepsilon > 0$.

Conversely, suppose that (5.2) (or (5.3)/(5.4)) is false. Then there is some $\varepsilon > 0$ such that $\chi_Y(X_n) > \varepsilon$ for infinitely many n (almost all n/all n). For each such n, we find by (4.1) (and

Proposition 4.9) some $U_n \in \mathscr{H}_Y(X_n)$ with $d(U_n) > \varepsilon$, i.e. $U_{d,\varepsilon} \subseteq U_n$. Hence, Proposition 4.2 implies $U_{d,\varepsilon} \in \mathscr{H}_Y^\circ(X_n)$ all these n, i.e. $(X_n)_n$ is not (almost/weakly) condensifying. □

For later reference, we write down two simple but useful properties.

Proposition 5.4. $(X_n)_n$ is condensifying with respect to Y if and only if $(\overline{X}_n)_n$ is condensifying with respect to Y.

Proof. This follows immediately from (4.5). □

Proposition 5.5. If $Y_n \subseteq X_n$ and if $(X_n)_n$ is (almost/weakly) condensifying with respect to Y then so is $(Y_n)_n$.

Proof. $\mathscr{H}_Y(Y_n) \subseteq \mathscr{H}_Y(X_n)$ by Proposition 4.2/4.9. □

As one might expect, there are relations between (weakly/almost) condensifying sequences and the precompactness/empty Hausdorff measure of noncompactness of the respective topological limit sets

$$\overline{\lim_n} X_n := \bigcap_n \overline{\bigcup_{k=n}^{\infty} X_k}, \tag{5.5}$$

$$\underline{\lim_n} X_n := \bigcup_n \overline{\bigcap_{k=n}^{\infty} X_k}, \tag{5.6}$$

and $\bigcap_n \overline{X}_n$. The precompactness of the limit sets $\underline{\lim}_n X_n \subseteq \overline{\lim}_n X_n$ will be obtained in Propositions 5.9 and 7.8, respectively, under some additional requirements which are sufficient or necessary for condensifying sequences, respectively (the latter under a further hypothesis). However, the precompactness of the third set is rather immediate.

Proposition 5.6. Let $(X_n)_n$ be weakly condensifying with respect $Y \subseteq X$. Then $\mathscr{H}_Y^\circ(\bigcap_n \overline{X}_n) = \emptyset$. In particular, if $\mathscr{U}$ is a uniformity then $\bigcap_n \overline{X}_n$ is precompact.

Proof. For each $U \in \mathscr{U}$ there is some $V \in \mathscr{U}$ with $V^2 \subseteq U$. By Proposition 5.2, there is some n_U with $X_{n_U} \notin \mathscr{H}_Y^\circ(V)$. Hence, there is a finite set $N \subseteq Y$ with $X_{n_U} \subseteq V(N)$, and so

$$\bigcap_n \overline{X}_n \subseteq \overline{X}_{n_U} \subseteq V(X_{n_U}) \subseteq V(V(N)) \subseteq U(N),$$

i.e. $U \notin \mathscr{H}_Y^\circ(\bigcap_n \overline{X}_n)$. The last claim follows from Proposition 4.2. □

Proposition 5.2/5.6 contains the earlier announced Proposition 4.10 as a special case.

Proof of Proposition 4.10. If one of the sets mentioned in Proposition 4.10 is empty, then Propositions 5.2 and 5.4 imply that the constant sequence $X_n := M$ is condensifying. By Proposition 5.6, we obtain $\mathscr{H}_Y^\circ(\overline{M}) = \emptyset$ (and thus also $\mathscr{H}_Y^\circ(M) = \emptyset$). Hence, (4.4) and (4.6) imply that each of the sets mentioned in Proposition 4.10 is empty. □

It will become clear later on why it is sometimes useful to restrict our considerations to countable subsets.

Lemma 5.7. *Let $\mathscr{U}$ be a uniformity. $(X_n)_n$ is (almost/weakly) condensifying with respect to Y if and only if for each choice of countable subsets $C_n \subseteq X_n$ the sequence $(C_n)_n$ is (almost/weakly) condensifying with respect to Y.*

Proof. One implication follows by Proposition 5.5. For the converse implication, suppose that $(X_n)_n$ is not (almost/weakly) condensifying. We find by Proposition 5.2 some $U \in \mathscr{U}$ which belongs to $\mathscr{H}_Y^\circ(X_n)$ for almost all n (some n/infinitely many n). Choose $V_1, V_2 \in \mathscr{U}$ with $V_1^2 \subseteq U$, $V_2^2 \subseteq V_1$, and put $W := V_2 \cap V_2^{-1}$. For each n with $U \in \mathscr{H}_Y^\circ(X_n)$, we will show that there is an (at most) countable set $C_n \subseteq X_n$ such that $W \in \mathscr{H}_Y^\circ(C_n)$. Hence, the claim follows from Proposition 5.2.

If there is some $x \in X_n$ such that there is no $y \in Y$ with $x \in W(y)$, then the set $C_n := \{x\}$ already has the required property $W \in \mathscr{H}_Y^\circ(C_n)$. Thus, assume that for each $x \in X_n$ there is some $y \in Y$ with $x \in W(y)$.

Now we choose inductively sequences x_k, y_k according to the following rule. If $x_1, \dots, x_{k-1} \in X_n$ and $y_1, \dots, y_{k-1} \in Y$ are already chosen, note that $N_k := \{y_1, \dots, y_{k-1}\} \subseteq Y$ is finite (or empty). Since $U \in \mathscr{H}_Y^\circ(X_n)$ (and thus X_n is not precompact by Proposition 4.2, in particular nonempty), we find some $x_k \in X_n \setminus U(N_k)$. By our assumption we also find $y_k \in X$ such that $x_k \in W(y_k)$.

We claim that $C_n := \{x_1, x_2, \dots\}$ has the required property $W \in \mathscr{H}_Y^\circ(C_n)$. Assume by contradiction that there is a finite set $N \subseteq Y$ with $C_n \subseteq W(N)$. Then there is some $y \in N$ such that $W(y)$ contains infinitely many of the elements x_k, in particular there are $k_1 < k_2$ with $x_{k_i} \in W(y)$ $(i = 1, 2)$. Since $x_{k_1} \in W(y_{k_1})$ and $W \subseteq V_2 \subseteq V_2^2 \subseteq V_1$, we have

$$x_{k_2} \in W(y) \subseteq W(W^{-1}(x_{k_1})) \subseteq V_2(V_2(x_{k_1})) \subseteq V_2^2(W(y_{k_1})) \subseteq V_1(V_1(y_{k_1})) \subseteq U(y_{k_1}),$$

and so $x_{k_2} \in U(N_{k_1}) \subseteq U(N_{k_2-1})$, contradicting our choice of the sequence $(x_k)_k$. ☐

Corollary 5.8. *Let $(X, \mathscr{U})$ be a uniform space and $M \subseteq X$. If $\overline{M}$ is not precompact then there is a countable subset $C \subseteq M$ which is not precompact.*

Proof. Apply Lemma 5.7 with $Y := X$ and the constant sequence $X_n := M$, and observe that Propositions 4.2 and 4.9 imply that $A \subseteq X$ is precompact if and only if $\mathscr{H}_Y(A) = \emptyset$ and that $\mathscr{H}_Y(M) = \mathscr{H}_Y(\overline{M})$. ☐

Another sufficient criterion is sometimes convenient. To formulate it, we introduce the further notation

$$\mathrm{Fin}_Y\,(X_n)_n := \Big\{\bigcup_n N_n : N_n \subseteq X_n \cap Y \text{ is finite for every } n\Big\}. \tag{5.7}$$

Proposition 5.9. Let $\mathscr{U}$ be a uniformity and assume that all sets in $\mathrm{Fin}_Y\,(X_n)_n$ are precompact. Then the following holds:

(1) If $X_n \subseteq Y \subseteq X$ for infinitely many n, then $(X_n)_n$ is condensifying with respect to Y.
(2) $\underline{\lim}_n(Y \cap X_n)$ is precompact.

Proof. To see the first claim, assume by contradiction that $(X_n)_n$ fails to be condensifying. Then there is some $U \in \mathscr{U}$ and a sequence $n_k \to \infty$ such that $X_{n_k} \subseteq Y$ and $U \in \mathscr{H}_Y^\circ(X_{n_k})$ for all k. By induction on $k = 1, 2, \dots$, we define a sequence $x_k \in X_{n_k} \subseteq Y$. as follows. If $x_1, \dots, x_{k-1}$ are already defined, we note that $M_k := \{x_1, \dots, x_{k-1}\}$ $(M_1 := \emptyset)$ is a finite subset of Y. Since $U \in \mathscr{H}_Y^\circ(X_{n_k})$, it follows that the set $X_{n_k} \setminus U(M_k)$ is not empty and thus contains some x_k.

For the thus defined sequence, we put $C := \{x_1, x_2, \dots\}$. Since C is the union of the sets $\{x_k\} \subseteq X_{n_k} \subseteq Y$, we have $C \in \mathrm{Fin}_Y\,(X_n)_n$. Hence, our hypothesis implies that C is precompact. Choose some $V \in \mathscr{U}$ with $V^2 \subseteq U$. Then $W := V \cap V^{-1} \in \mathscr{U}$. Since C is precompact, we find some finite $N \subseteq Y$ with $C \subseteq W(N)$. However, for each $y \in N$ there is at most one k with

$x_k \in W(y)$. (Hence, $W(N)$ contains x_k for at most finitely many k which is a contradiction.) Indeed, assume that there are $k_1 < k_2$ and $y \in N$ with $x_{k_1}, x_{k_2} \in W(y)$. Then $y \in W^{-1}(x_{k_1})$, and so

$$x_{k_2} \in W(y) \subseteq W(W^{-1}(x_{k_1})) \subseteq V^2(x_{k_1}) \subseteq U(x_{k_1}).$$

Since $x_{k_1} \in M_{k_1}$, this implies $x_{k_2} \in U(M_{k_1})$ which contradicts our choice of x_{k_2}.

To see the second claim, assume that

$$\underline{\lim_n}(Y \cap X_n) \subseteq \overline{\bigcup_n \bigcap_{k=n}^{\infty} (Y \cap X_k)}$$

fails to be precompact. By Corollary 5.8, we find a countable $C \subseteq Y \cap \bigcup_n \bigcap_{k=n}^{\infty} X_k$ which is not precompact. However, since for each $x \in C$ and each n, we find some $k > n$ with $x \in X_k$, we can by induction write C as a union of singletons $N_{k_n} = \{x_n\}$ where $x_n \in Y \cap X_{k_n}$ and $k_1 < k_2 < \cdots$, and so $C \in \mathrm{Fin}_Y\,(X_n)_n$, contradicting our hypothesis. $\square$

6. CONDENSING MAPS

In connection with fixed point theory, condensing maps have a long history. On a metric space X, one essentially distinguishes between *strictly* condensing maps $F\colon X \to X$ which satisfy

$$\chi_X(F(M)) \le k\chi_X(M) \qquad (M \subseteq X) \tag{6.1}$$

with some $k < 1$ (typical examples are compact perturbations of contractions), and condensing maps which satisfy

$$\chi_X(F(M)) < \chi_X(M) \qquad (M \subseteq X,\ \chi_X(M) > 0). \tag{6.2}$$

If $\chi_X(X) < \infty$ (i.e. if $\operatorname{diam} X < \infty$) then the latter is considerably less restrictive as it can be shown that, generically, condensing maps fail to be strictly condensing. It is easily seen in case $\chi_X(X) < \infty$ that every strictly condensing map satisfies

$$\lim_{n\to\infty} \chi_X(F^n(X)) = 0$$

and thus Proposition 5.3 implies that its induced map of power-sets is condensifying in the following sense (with $Y = X$).

Definition 6.1. A map $F\colon 2^X \to 2^X$ is (weakly/almost) *condensifying with respect to* Y if the sequence $(F^n(X))_n$ is (weakly/almost) condensifying with respect to Y.

However, in many situations (e.g. if F involves an integral operator of vector-valued functions), one can usually obtain good estimates of type (6.1) only for countable sets M, see e.g. [4, 14, 16, 18, 23]. Moreover, one can often obtain such estimates only if one replaces χ_X by another measure of noncompactness, e.g. by the Kuratowski measure α of noncompactness (recall that $\alpha(M)$ is the infimum of all $\varepsilon > 0$ such that M can be covered by finitely many sets of diameter at most ε). Finally, if e.g. X is a locally convex space, it is more natural to consider *families* of measures of noncompactness which are induced by an appropriate family of seminorms. Sometimes one might also want to replace the function $\varphi(r) := kr$ occurring in (6.1) by another function with $\varphi(r) < r$.

We will show now that, roughly speaking, in all these situations maps which are only strictly condensing in such a generalized sense are condensifying. Moreover, we will show that, surprisingly, even maps which are only condensing (in a certain generalized sense) are condensifying.

Since the details of the generalizations differ, these results do not strictly contain each other, and we treat them in separate sections.

6.1. **Strictly Condensing Maps.** In order to deal with Fréchet spaces or, more general, with locally convex spaces, we recall that the uniformity on such a space is generated by a family of (usually much simpler) uniformities in the following way.

Definition 6.2. Let I be some nonempty index set. A family $(\mathscr{U}_i)_{i\in I}$ of (quasi-)uniformities $\mathscr{U}_i$ on X generates a (quasi-)uniformity $\mathscr{U}$ by the following definition: A set $U\subseteq X\times X$ belongs to $\mathscr{U}$ if and only if there are a finite set $I_0\subseteq I$ and $U_i\in\mathscr{U}_i$ $(i\in I_0)$ such that

$$\bigcap_{i\in I_0} U_i\subseteq U. \tag{6.3}$$

The family $(\mathscr{U}_i)_{i\in I}$ is *almost directed* if for each finite set $I_0\subseteq I$ and each choice $U_i\in\mathscr{U}_i$ $(i\in I_0)$ the set $\bigcap_{i\in I_0}U_i$ belongs to $\mathscr{U}_j$ for some $j\in I$.

Remark 6.3. The family $(\mathscr{U}_i)_{i\in I}$ is almost directed if it is totally ordered with respect to set inclusion. This is typically the case in examples of locally convex spaces X which occur in applications (if $\mathscr{U}_i$ denotes uniformities generated by appropriate seminorms). We will give a typical example later on (Example 6.8).

Note that in Definition 6.2 one clearly always has

$$\bigcup_{i\in I}\mathscr{U}_i\subseteq\mathscr{U}. \tag{6.4}$$

The inverse inclusion is equivalent to the fact that $(\mathscr{U}_i)_{i\in I}$ is almost directed:

Lemma 6.4. *The family $(\mathscr{U}_i)_{i\in I}$ is almost directed if and only if it generates the quasi-uniformity $\mathscr{U}=\bigcup_{i\in I}\mathscr{U}_i$.*

Proof. If $(\mathscr{U}_i)_{i\in I}$ is almost directed, then for each $U\in\mathscr{U}$ there is some $V\in\mathscr{U}_j$ with $V\subseteq U$ for some $j\in I$. Since $\mathscr{U}_j$ is a filter, this means $U\in\mathscr{U}_j$. Hence, equality holds in (6.4). Conversely, if equality holds in (6.4) and if $I_0\subseteq I$ is finite and $U_i\in\mathscr{U}_i$ $(i\in I_0)$, then $U:=\bigcap_{i\in I_0}U_i$ belongs to $\mathscr{U}$, and so there is some $j\in I$ with $U\in\mathscr{U}_j$. ☐

Proposition 6.5. Let a quasi-uniformity $\mathscr{U}$ on X be generated by a family $(\mathscr{U}_i)_{i\in I}$. Let $Y\subseteq X$, and let $(X_n)_n$ be a sequence of subsets of X.

(1) If $(X_n)_n$ is (almost/weakly) condensifying with respect to Y in $(X,\mathscr{U})$ then $(X_n)_n$ is (almost/weakly) condensifying with respect to Y in each $(X,\mathscr{U}_i)$ $(i\in I)$.
(2) If $(X_n)_n$ is (almost/weakly) condensifying with respect to Y in each $(X,\mathscr{U}_i)$ $(i\in I)$ and if $(\mathscr{U}_i)_{i\in I}$ is almost directed, then $(X_n)_n$ is (almost/weakly) condensifying with respect to Y in $(X,\mathscr{U})$.

Proof. Let $\mathscr{H}^\circ_{i,Y}$ and $\mathscr{H}_{i,Y}$ denote the (fine) quasi-uniform Hausdorff measure of noncompactness with respect to Y in $(X,\mathscr{U}_i)$.

Let $(X_n)_n$ be (almost/weakly) condensifying with respect to Y in $(X,\mathscr{U})$, and let $i\in I$. Let $U\in\mathscr{U}_i$. By (6.4), we have $U\in\mathscr{U}$, and by Proposition 5.2, we have $U\notin\mathscr{H}^\circ(X_n)$ for almost all n (infinitely many n/some n). For each such n, we thus find a finite set $N\subseteq X$ with $X_n\subseteq U(N)$, and so $U\notin\mathscr{H}^\circ_{i,Y}(X_n)\supseteq\mathscr{H}_{i,Y}(X_n)$. Hence, $(X_n)_n$ is (almost/weakly) condensifying with respect to Y in $(X,\mathscr{U}_i)$.

Conversely, let $(X_n)_n$ be (almost/weakly) condensifying with respect to Y in each $(X,\mathscr{U}_i)$ $(i\in I)$, and let $(\mathscr{U}_i)_{i\in I}$ be almost directed. Let $U\in\mathscr{U}$. By Lemma 6.4, there is some index j with $U\in\mathscr{U}_j$. By Proposition 5.2, we have $U\notin\mathscr{H}^{\circ}_{j,Y}(X_n)$ for almost all n (infinitely many n/some n). For each such n, we thus find a finite set $N\subseteq X$ with $X_n\subseteq U(N)$, and so $U\notin\mathscr{H}^{\circ}_Y(X_n)\supseteq\mathscr{H}_Y(X_n)$. Hence, $(X_n)_n$ is (almost/weakly) condensifying with respect to Y in $(X,\mathscr{U}_i)$. $\square$

Definition 6.6. Let Γ be a class of maps $\gamma\colon 2^X\to[0,\infty]$, and let $F\colon 2^X\to 2^X$, and $Y\subseteq X$. We write $\Gamma\succeq_F\mathscr{H}_Y$ if for each sequence of sets $X_n\subseteq F^n(X)$ with $X_{n+1}\subseteq X_n$ for all n we have

$$\Bigl(\lim_{n\to\infty}\gamma(X_n)=0\text{ for each }\gamma\in\Gamma\Bigr)\implies\bigcap_n\mathscr{H}_Y(X_n)=\emptyset.$$

We write $\Gamma\succeq^c_F\mathscr{H}_Y$ if the above holds under the additional hypothesis that each X_n is a countable set.

Clearly,

$$\Gamma\succeq_F\mathscr{H}_Y\implies\Gamma\succeq^c_F\mathscr{H}_Y.$$

The notation is explained by the following test which for all practical purposes suffices to verify the above property.

Theorem 6.7. *Let the quasi-uniformity $\mathscr{U}$ on $X\supseteq Y$ be generated by an almost directed family $(\mathscr{U}_i)_{i\in I}$, and let $F\colon 2^X\to 2^X$ and $Y\subseteq X$. Let $\Gamma=\{\gamma_i : i\in I\}$ where $\gamma_i\colon 2^X\to[0,\infty]$ are such that for each $i\in I$ the following holds.*

$\mathscr{U}_i$ is induced by a q_i-quasi-pseudometric d_i such that for the corresponding Hausdorff measure of noncompactness $\chi_{i,Y}$ with respect to Y the implication

$$\lim_{n\to\infty}\gamma_i(X_n)=0\implies\inf_n\chi_{i,Y}(X_n)=0\tag{6.5}$$

holds for each sequence of (countable) sets $X_n\subseteq F^n(X)$ with $X_{n+1}\subseteq X_n$ for all n.

Then $\Gamma\succeq_F\mathscr{H}_Y$ (or $\Gamma\succeq^c_F\mathscr{H}_Y$, respectively).

Note that if F is monotone then (6.5) holds in particular if there is a finite constant C_i and a number n_i with

$$\chi_{i,Y}(M)\le C_i\gamma_i(M)\tag{6.6}$$

for each (countable) $M\subseteq F^{n_i}(X)$. This explains our notation $\Gamma\succeq_F\mathscr{H}_Y$.

Note that (6.6) holds e.g. in case $Y\supseteq F^n(X)$ if γ_i denotes the Kuratowski measure of noncompactness with respect to the q_i-quasi-pseudometric d_i.

Proof of Theorem 6.7. Let $(X_n)_n$ be a family of (countable) sets with $X_n\subseteq F^n(X)$, $X_{n+1}\subseteq X_n$, and

$$\lim_{n\to\infty}\gamma_i(X_n)=0\qquad(i\in I).$$

We are to show that $(X_n)_n$ is weakly condensifying with respect to Y with respect to the quasi-uniformity $\mathscr{U}$. Since $(\mathscr{U}_i)_{i\in I}$ is almost directed, it suffices by Proposition 6.5 to show that, for each $i\in I$, $(X_n)_n$ is weakly condensifying with respect to Y with respect to the quasi-uniformity $\mathscr{U}_i$. Since (6.5) implies (5.4), this follows from Proposition 5.3. $\square$

The following typical example shows that Fréchet spaces but also spaces which are not even locally convex fall under our consideration with very natural choices for Γ. In particular, in case of Fréchet spaces, the family Γ in this example is defined in terms of a generating family of seminorms and not in terms of the metric (in contrast to the usual Hausdorff measure of noncompactness).

Example 6.8. For $0 < p < \infty$, let $I \subseteq (0,\infty)$ be unbounded, and let

$$X := L_p^{\mathrm{loc}}(\mathbb{R}) = \{x \mid x\colon \mathbb{R} \to \mathbb{R} \text{ satisfies } x|_{[-i,i]} \in L_p([-i,i])\ (i \in I)\},$$

endowed with the uniformity $\mathscr{U}$ generated in the obvious way by the family of q-quasi-pseudo-norms

$$\|x\|_i := \Big(\int_{-i}^{i} |x(s)|^p\, ds\Big)^{1/p} \qquad (i \in I)$$

or, equivalently, by the q-pseudometrics

$$d(x,y) := \int_{-i}^{i} |x(s) - y(s)|^p\, ds \qquad (i \in I).$$

Let $\Gamma = \{\gamma_i : i \in I\}$ where γ_i is either the Hausdorff measure of noncompactness (with respect to d_i or with respect to $\|\cdot\|_i$) or some other equivalent measure of noncompactness (e.g. the Kuratowski measure of noncompactness).

Then $\Gamma := \{\gamma_i : i \in I\} \succeq_F \mathscr{H}_Y$ and thus also $\Gamma \succeq_F^c \mathscr{H}_Y$ whenever $F\colon 2^X \to 2^X$ is monotone and $Y \subseteq F^n(X)$ for some n.

Proof. The claim is an immediate consequence of Theorem 6.7. Note that the exact definition of γ_i (e.g. whether it is an equivalent measure of noncompactness or defined with respect to d_i or with respect to $\|\cdot\|_i$) plays no role, since we only need the property (6.5) which is obviously satisfied in all these cases.

Observe that the corresponding family $(\mathscr{U}_i)_{i\in I}$ of uniformities is indeed almost directed by Remark 6.3, because

$$i \le j \implies \mathscr{U}_i \subseteq \mathscr{U}_j \qquad (i,j \in I).$$

□

Definition 6.9. Let Γ be a class of maps $\gamma\colon 2^X \to [0,\infty]$. A map $F\colon 2^X \to 2^X$ is *strictly monotonically* Γ*-condensing* if for each $\gamma \in \Gamma$ there is a function $\varphi_\gamma\colon [0,\infty] \to [0,\infty]$ which is continuous from the right and satisfies $\varphi_\gamma(r) < r$ on $(0,\infty)$ and

$$\gamma(M \cap F(M)) \le \varphi_\gamma(\gamma(M)) \qquad (M \subseteq X). \tag{6.7}$$

F is *strictly monotonically countably* Γ*-condensing* if instead of (6.7) we require

$$\gamma(C \cap C_0) \le \varphi_\gamma(\gamma(C)) \qquad (C \subseteq X, C_0 \subseteq F(C) \text{ countable}).$$

As remarked earlier, in most applications one will actually have $\varphi_\gamma(r) = k_\gamma r$ with some $k_\gamma \in [0,1)$.

In order to deal with countable subsets in case of monotone maps $F\colon 2^X \to 2^Y$ which do *not* stem from a multivalued map $F\colon X \multimap Y$, we need another definition:

Definition 6.10. $F\colon 2^X \to 2^Y$ is *of finite/countable type* if for each $M \subseteq X$ and each $y \in F(M)$ there is a finite/countable set $C_y \subseteq M$ with $y \in F(C_y)$.

Maps of countable type are general enough to contain all multivalued maps $F\colon X \multimap Y$ and, moreover, all maps of the form (3.1) or (3.2):

Example 6.11. (1) If F is of finite type, then F is of countable type.

(2) Each multivalued map $F\colon X \multimap Y$ is of finite type (even if it assumes empty values).
In fact, for each $y \in F(M)$ there is some $x \in M$ with $y \in F(x)$, i.e. one can choose the singleton $C_Y := \{x\}$.

(3) If $F\colon 2^X \to 2^Y$ and $G\colon 2^Y \to 2^Z$ are of finite/countable type and F is monotone, then also $G \circ F$ is of finite/countable type. In particular, if $F\colon 2^X \to 2^X$ is of finite/countable type and monotone, then so is F^n for $n = 0, 1, \ldots$

(4) If X is a vector space, then $\operatorname{conv}\colon 2^X \to 2^X$ is of finite type.
In fact, each $y \in \operatorname{conv} M$ is a convex combination of finitely many elements $x_1, \ldots, x_n \in M$, i.e. one can choose $C_y := \{x_1, \ldots, x_n\}$.

(5) If X is a q-quasi-pseudometric space then $\overline{id}\colon X \multimap X$, defined by $\overline{id}(M) := \overline{M}$, is of countable type.
In fact, for each $y \in \overline{M}$ there is a sequence $x_n \in M$ with $x_n \to y$, and so one can choose $C_y := \{x_1, x_2, \ldots\}$.

(6) If Y is a subset of a vector space and $F_0\colon X \multimap Y$, $\Omega \subseteq X$, and $V \subseteq Y$ are such that
$$F(M) := \operatorname{conv}(F_0(M \cap \Omega) \cup V) \qquad (M \subseteq X)$$
satisfies $F(X) \subseteq Y$, then $F\colon 2^X \to 2^Y$ is of finite type (because it is a composition of conv with a multivalued map).

(7) If Y is a quasi-pseudometrizable subset of a topological vector space and $F_0\colon X \multimap Y$, $\Omega \subseteq X$, and $V \subseteq Y$ are such that
$$F(M) := \overline{\operatorname{conv}}(F_0(M \cap \Omega) \cup V) \qquad (M \subseteq X)$$
satisfies $F(X) \subseteq Y$, then $F\colon 2^X \to 2^Y$ is of countable type (because F_0 is a composition of $\overline{id}$ with a map of finite type).

Theorem 6.12. *Let $F\colon 2^X \to 2^X$ be monotone and $Y \subseteq X$.*

(1) If F is strictly monotonically Γ-condensing where $\Gamma \succeq_F \mathscr{H}_Y$ and if for each $\gamma \in \Gamma$ there is some n_γ with $\gamma(F^{n_\gamma}(X)) < \infty$, then F is condensifying with respect to Y.

(2) If $\mathscr{U}$ is a uniformity, F is of countable type and strictly monotonically countably Γ-condensing where $\Gamma \succeq_F^c \mathscr{H}_Y$, and if for each $\gamma \in \Gamma$ there is some n_γ with $\gamma(C) < \infty$ for each countable $C \subseteq F^{n_\gamma(X)}$, then F is condensifying with respect to Y.

The proof of Theorem 6.12 is based on the following observation.

Lemma 6.13. *Let $F\colon 2^X \to 2^X$ be monotone and of countable type, and let $C_n \subseteq F^n(X)$ countable. Then there are countable sets $D_n \subseteq F^n(X)$ with $C_n \subseteq D_n \subseteq F^n(X)$ and $D_{n+1} \subseteq D_n \cap F(D_n)$ for all $n = 0, 1, 2, \ldots$*

Proof. Define by induction on $k = 0, 1, 2, \ldots$ a countable family of countable sets $\{C_{k,0}, C_{k,1}, \ldots\}$ with $C_{k,n} \subseteq F^n(X)$ as follows. For $k = 0$, put $C_{k,n} := C_n$, and if $\{C_{k,0}, C_{k,1}, \ldots\}$ are already defined, observe that for each $n = 0, 1, 2, \ldots$ the set $C_{k,n+1}$ is countable and contained in $F^{n+1}(X) \subseteq F^n(X)$ and that $F^{n+1}(X) = F(F^n(X))$. Since F is monotone and of countable type, there is a countable set $C_{k+1,n} \subseteq F^n(X)$ containing $C_{k,n+1}$ such that $C_{k,n+1} \subseteq F^n(C_{k+1,n})$. This defines the family $\{C_{k+1,0}, C_{k+1,1}, \ldots\}$, completing the inductive definition.

Now the sets $D_n := \bigcup_k C_{k,n}$ have the required properties. Indeed, $C_n = C_{n,0} \subseteq D_n$, and since $C_{n,k} \subseteq F^n(X)$ for all k, we have $D_n \subseteq F^n(X)$. Moreover, if $x \in D_{n+1}$, say $x \in C_{k,n+1}$, we have $x \in C_{k+1,n} \cap F^n(C_{k+1,n})$ by construction, and so $x \in D_n \cap F^n(D_n)$ since F^n is monotone. $\square$

Proof of Theorem 6.12. For the proof of the first claim, we put $Y_n := F^n(X)$.

For the proof of the second claim, it suffices in view of Lemma 5.7 to show that for each countable $C_n \subseteq F^n(X)$ the sequence $(C_n)_n$ is condensifying with respect to Y; in this case, we put $Y_n := D_n$ with D_n as in Lemma 6.13 and use $C_n \subseteq Y_n$ and Proposition 5.5.

In both cases, we thus have to show that $(Y_n)_n$ is condensifying with respect to Y, and we have

$$Y_{n+1} \subseteq Y_n \quad \text{and} \quad Y_{n+1} \subseteq F(Y_n)$$

for all n. Let $\gamma \in \Gamma$. Since F is strictly monotonically (countably) Γ-condensing, there is some $\varphi_\gamma\colon [0,\infty] \to [0,\infty]$ which is continuous from the right and satisfies $\varphi_\gamma(r) < r$ on $(0,\infty)$ and such that $\gamma(Y_{n+1}) \le \varphi_\gamma(\gamma(Y_n))$ for all n. By hypothesis, we find some n_γ such that $r_\gamma := \gamma(Y_{n_\gamma}) < \infty$. An induction by k implies

$$\gamma(Y_{n_\gamma+k}) \le \varphi_\gamma^k(r_\gamma) \qquad (k = 0,1,2,\dots). \tag{6.8}$$

Note that an induction implies that the sequence $r_k := \varphi_\gamma^k(r_\gamma)$ is strictly decreasing and thus convergent from the right to some number $\ell \in [0,\infty)$. Since φ_γ is continuous from the right, we obtain, passing to the limit $k \to \infty$ in $r_{k+1} = \varphi_\gamma(r_k)$ that $\ell = \varphi_\gamma(\ell)$, and so $\ell = 0$. Hence, $\varphi_\gamma^k(r_\gamma) \to 0$ as $k \to \infty$, and so (6.8) implies $\lim_{n\to\infty} \gamma(Y_n) = 0$. Since $\gamma \in \Gamma$ was arbitrary and $\Gamma \succeq_F \mathscr{H}_Y$ (or $\Gamma \succeq_F^c \mathscr{H}_Y$, respectively) we obtain $\bigcap_n \mathscr{H}_Y(Y_n) = \emptyset$. Since $Y_{n+1} \subseteq Y_n$ for all n this means that $(Y_n)_n$ is condensifying. ☐

6.2. **Condensing Maps.** Throughout this section, we assume that $(X,\mathscr{U})$ is a quasi-uniform space, $Y \subseteq X$, and that $F\colon 2^X \to 2^X$.

Definition 6.14. Let Γ_Y denote the class of all functions γ such that there is a set R equipped with a partial order $\le$ such that $\gamma\colon 2^Y \to R$ has the following properties.

1. (1) $(R,\le)$ is super Dedekind complete, i.e. each nonempty order bounded from above set $M \subseteq R$ has a supremum and there is a countable set $C \subseteq M$ with $\sup M = \sup C$.
2. (2) γ is monotone, i.e. $A \subseteq B \subseteq Y$ implies $\gamma(A) \le \gamma(B)$.
3. (3) $\gamma(C \cup \{y\}) = \gamma(C)$ for each countable $C \subseteq Y$ and each $y \in Y$.

We denote by Γ_Y^* the subclass of all $\gamma \in \Gamma_Y$ which satisfy

$$\gamma(\{x_1,x_2,\dots\}) = \gamma(\{y_1,y_2,\dots\})$$

whenever $x_n, y_n \in Y$ are such that for each $U \in \mathscr{U}$ there is some n_U with $(x_n,y_n) \in U$ for all $n \ge n_U$.

The most important example of some $\gamma \in \Gamma_Y^*$ is the Hausdorff (or Kuratowski) measure of noncompactness.

Note that typically one will have $R = [0,\infty]^I$ with some index set I (which may depend on γ) and the coordinatewise order $\le$. Obviously, $(R,\le)$ is super Dedekind complete in this example.

Our crucial assumption will involve $\operatorname{Fin}_Y F := \operatorname{Fin}_Y (F^n(X))_n$, where the right-hand side is defined by (5.7).

Definition 6.15. We call a subclass $\Gamma \subseteq \Gamma_Y$ *admissible for* F if, for each $C \in \operatorname{Fin}_Y F$, there exists some $A \in \operatorname{Fin}_Y F$ with $\gamma(F(A)) \ge \gamma(C)$ $(\gamma \in \Gamma)$.

For multivalued maps $F\colon X \multimap X$ this is no additional requirement for Γ. More general, this is the case for all monotone maps of finite type (Definition 6.10) as the following result shows. Even for the most important examples of maps of countable type this is no severe restriction on Γ. In fact, we will show that e.g. the maps (3.1) and (3.2) are contained in our considerations. Nevertheless, the hypothesis is slightly more restrictive than the hypothesis we made in the previous section.

Proposition 6.16. Suppose that $F\colon 2^X \to 2^X$ is monotone and of finite type.

1. If there is some k with $F^k(X) \subseteq Y \subseteq X$ then each subclass $\Gamma \subseteq \Gamma_Y$ is admissible for F.
2. Define $\overline{F}\colon 2^X \to 2^X$ by
$$\overline{F}(M) := \overline{F(M)} \qquad (M \subseteq X).$$
If there are k_1, k_2 such that $(\overline{F})^{k_1}(X) \subseteq Y$ and the restriction of $\mathscr{U}$ to $(\overline{F})^{k_2}(X)$ has a countable filter base (i.e. is induced by a q-quasi-pseudometric) then each subclass $\Gamma \subseteq \Gamma_Y^*$ is admissible for F.

Proof. Let $C \in \operatorname{Fin}_Y F$. Then $C = \bigcup_n C_n$ with finite sets $C_n \subseteq F^n(X)$. Since an induction shows $F^{n+1}(X) \subseteq F^n(X)$, we have for $n > k$ that the set $I_n := F^{n-1}(X)$ is contained in $F^k(X) \subseteq Y$. Since $C_n \subseteq F(I_n)$ and F is of finite type, we thus find a finite set $A_n \subseteq I_n \subseteq Y$ with $C_n \subseteq F(A_n)$. Then $A := \bigcup_{n>k}^{\infty} A_n$ belongs to $\operatorname{Fin}_Y F$, and with the remaining set $C_* := \bigcup_{n \le k} C_n$ we have $F(A) \supseteq C \setminus C_*$. Since $C_* \subseteq C$ is finite, this implies for each $\gamma \in \Gamma_Y$ that
$$\gamma(F(A)) \ge \gamma(C \setminus C_*) = \gamma((C \setminus C_*) \cup C_*) = \gamma(C),$$
and so each $\Gamma \subseteq \Gamma_Y$ is admissible for F.

For the proof of the second claim, put $X_n := (\overline{F})^n(X)$, and let $U_1, U_2, \ldots \in \mathscr{U}$ be such that $U_j \cap (X_{k_2} \times X_{k_2})$ is a base for the inherited uniformity on X_{k_2}. Let $C \in \operatorname{Fin}_Y \overline{F}$, i.e. $C = \bigcup_n C_n$ with finite sets $C_n \subseteq (\overline{F})^n(X)$. For all $n > k := \max\{k_1, k_2\}$, we see similarly as above that $X_{n-1} \subseteq X_{k_1} \cap X_{k_2} \subseteq Y \cap X_{k_2}$ satisfies $C_n \subseteq \overline{F}(X_{n-1})$. For each $x \in C_n$ choose some $y_{n,x} \in F(X_n)$ with $(x, y_{n,x}) \in U_1 \cap \cdots \cap U_n$. Since F is of finite type, we find a finite set $A_{n,x} \subseteq X_{n-1} \subseteq Y$ such that $y_{n,x} \in F(A_{n,x})$. Then $A := \bigcup_{n>k} \bigcup_{x \in C_n} A_{n,x}$ belongs to $\operatorname{Fin}_Y F$, and
$$F(A) \supseteq A_0 := \{y_{n,x} : n > k,\ x \in C_n\}.$$
By construction, we can write $A_0 = \{y_1, y_2, \ldots\}$ and $C = \{x_1, x_2, \ldots\}$ such that for each j, we find some n_j with $x_n, y_n \in X_{k_2}$ and $(x_n, y_n) \in U_k$ for all $n \ge k$. Since $U_j \cap (X_{k_2} \times X_{k_2})$ is a base for the inherited uniform structure on X_{k_2}, we thus find for each $U \in \mathscr{U}$ some n_u with $(x_n, y_n) \in U$ for all $n \ge n_U$. Consequently, we have for each $\gamma \in \Gamma_Y^*$ that
$$\gamma(C) = \gamma(A_0) \le \gamma(F(A)) \le \gamma(\overline{F}(A)),$$
and so each $\Gamma \subseteq \Gamma_Y^*$ is admissible for F. □

Example 6.17. Let X be a metrizable subset of a topological vector space and $F_0\colon X \multimap X$ and $\Omega, V \subseteq X$ be such that
$$F(M) := \overline{\operatorname{conv}}(F_0(M \cap \Omega) \cup V) \qquad (M \subseteq X)$$
satisfies $F(X) \subseteq X$. Then $F =: G_0$ for a map $G_0\colon X \multimap X$ of finite type (Example 6.11), and so each subclass $\Gamma \subseteq \Gamma_Y^*$ is admissible for F, provided that $F^n(X) \subseteq Y \subseteq X$ for some n.

Definition 6.18. We call F *countably condensing* with respect to $\Gamma \subseteq \Gamma_Y$ if for each countable set $C \subseteq Y$ which is not precompact and satisfies $F(C) \subseteq Y$ there is some $\gamma \in \Gamma$ such that

$$\gamma(F(C)) \not\geq \gamma(C).$$

Note that $\gamma \in \Gamma$ may even depend on C.

Now we are in a position to prove that countably condensing maps are condensifying.

Theorem 6.19. *Let $F\colon 2^X \to 2^X$ be monotone and countably condensing with respect to a countable family $\Gamma \subseteq \Gamma_Y$ which is admissible for F. Then each set in $\mathrm{Fin}_Y F$ is precompact. If $\mathscr{U}$ is a uniformity, we have in addition:*

(1) $Y \cap \underline{\lim}_n F^n(X)$ is precompact.
(2) If $F^n(X) \subseteq Y$ for some n then F is condensifying with respect to Y.

The proof of Theorem 6.19 makes use of the following generalization of [2, Lemma (I.5.30)]. It is not surprising that the proof makes use of such type of lemma, because Theorem 6.19 will allow us to prove Theorem 7.20 which is a generalization of [2, Theorem (I.5.29)] (whose proof also depends on [2, Lemma (I.5.30)]). Actually, the idea to use a similar lemma for the proof of the compactness of $\bigcap_n F^n(X)$ with $F(M) := \overline{\mathrm{conv}}(F_0(M) \cup \{x_0\})$ (which in view of Example 6.17 is contained in the second claim of Theorem 6.19, even for a multivalued map F_0) goes back to [15] and [27].

Lemma 6.20. *Let $\Gamma \subseteq \Gamma_Y$ be a countable family, and let $X_n, Y \subseteq X$. Then there is a set $C_\Gamma \in \mathrm{Fin}_Y\,(X_n)_n$ such that $\gamma(C_\Gamma) = \max\{\gamma(C) : C \in \mathrm{Fin}_Y\,(X_n)_n\}$ $(\gamma \in \Gamma)$, i.e., these maxima actually exist and are attained for the same set C_Γ.*

Proof. We use the shortcurt $\mathrm{Fin} := \mathrm{Fin}_Y\,(X_n)_n$. Let $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$ with $\gamma_k\colon 2^X \to R_k$, and let $M_k := \{\gamma_k(C) : C \in \mathrm{Fin}\}$. Since $\gamma_k(C) \leq \gamma_k(X)$ for all $C \subseteq X$, the set $M_k \subseteq R_k$ is order bounded from above. Hence, $s_k := \sup M_k$ exists, and there are countably many $C_{k,1}, C_{k,2}, \ldots \in \mathrm{Fin}$ with $s_k = \sup \bigcup_j \gamma(C_{k,j})$. Put $C_m := \bigcup_{k,j \leq m} C_{k,j}$. Since a finite union of sets from Fin belongs to Fin, we have $C_m \in \mathrm{Fin}$. Thus, we find finite sets $A_{m,n} \subseteq X_n$ with $C_m = \bigcup_n A_{m,n}$. The set $A_n := \bigcup_{m \leq n} A_{m,n}$ is finite and satisfies $A_n \subseteq X_n$. Hence, $C_\Gamma := \bigcup_n A_n = \bigcup_m \bigcup_{n \geq m} A_{m,n}$ belongs to Fin.

To see that C_Γ has the required property, let k and j be given. Put $m := \max\{k, j\}$. Since $\bigcup_{n<m} A_{m,n}$ is finite, we have

$$\gamma_k(C_\Gamma) \geq \gamma_n\Bigl(\bigcup_{n \geq m} A_{m,n}\Bigr) = \gamma_k\Bigl(\bigcup_n A_{m,n}\Bigr) = \gamma_k(C_m) \geq \gamma_k(C_{k,j}).$$

Since this holds for all j, we conclude

$$s_k = \sup M_k \geq \gamma_k(C_\Gamma) \geq \sup \gamma_k(C_{k,j}) = s_k,$$

and so $\gamma_k(C_\Gamma) = s_k$ for all k, as required. □

Proof of Theorem 6.19. Let C_Γ be the set of Lemma 6.20. Since Γ is admissible for F, we find some set $A \in \mathrm{Fin}_Y F$ with $\gamma(F(A)) \geq \gamma(C_\Gamma)$ $(\gamma \in \Gamma)$. Let $C \in \mathrm{Fin}_Y F$ be arbitrary. Since finite unions of sets from $\mathrm{Fin}_Y F$ belong to $\mathrm{Fin}_Y F$, we have $C_C := A \cup C_\Gamma \cup C \in \mathrm{Fin}_Y F$. Hence, the choice of C_Γ implies, for each $\gamma \in \Gamma$,

$$\gamma(C_C) \leq \gamma(C_\gamma) \leq \gamma(F(A)) \leq \gamma(F(C_C)).$$

Since F is countably condensing with respect to Γ, this implies that C_C is precompact, and so also $C \subseteq C_C$ is precompact. The remaining claims follow now from Proposition 5.9. □

7. EXISTENCE OF COMPACT UNIFORM ATTRACTORS

For multi-valued maps, it makes sense to distinguish between orbitwise attractors, uniform attractors, and attractors. While the former means that each orbit is attracted, the other mean that the attraction is uniform or uniform with respect to the initial value of the orbit. In case of single-valued maps, orbitwise attractors and attractors coincide, of course. If one considers more general monotone set-maps or even sequences of sets, only the notion of a uniform attractor makes sense. Although it might appear redundant to use the word "uniform" in this connection (since no other type of attractor can be defined in this general setting), we keep this word anyway to avoid confusion when we will pass later to the setting of single- or multivalued maps.

7.1. **Uniform Attractors for Sequences of Sets.** Throughout this section, let $(X,\mathscr{U})$ be a quasi-uniform space, and $X_n \subseteq X$.

Definition 7.1. A set $M \subseteq X$ is an *(almost/weak) uniform attractor* for the sequence $(X_n)_n$ if for each open set $O \supseteq M$ we have $X_n \subseteq O$ for almost all n (infinitely many n/some n).

$M \subseteq X$ is an *(almost/weak) uniform $\mathscr{U}$-attractor* for the sequence $(X_n)_n$ if for each $U \in \mathscr{U}$ we have $X_n \subseteq U(M)$ for almost all n (infinitely many n/some n).

Proposition 7.2. Each (almost/weak) uniform attractor M for $(X_n)_n$ is an (almost/weak) uniform $\mathscr{U}$-attractor for $(X_n)_n$. The converse holds if M is compact.

Proposition 7.2 is an immediate consequence of the following well-known observation. For completeness, we provide a proof anyway.

Lemma 7.3. *Let $M \subseteq X$.*

1. *For each $U \in \mathscr{U}$ there is an open set $O \subseteq X$ with $M \subseteq O \subseteq U(M)$.*
2. *Conversely, if M is compact then for each open $O \supseteq M$ there is some $U \in \mathscr{U}$ with $U(M) \subseteq O$.*

Proof. The first statement follows from the fact that $U(M)$ is a neighborhood of M (i.e. of each point $x \in M$). For the second statement, assume that $M \subseteq X$ is compact, and let $O \supseteq M$ be open. For each $x \in M$ we find some $U \in \mathscr{U}$ with $U(x) \subseteq O$ and some $V \in X$ with $V^2 \subseteq U$. Then $V^2(x) \subseteq U(x) \subseteq O$. Since $V(x)$ is a neighborhood of x, we thus have proved that the family

$$\mathscr{M} := \{N : N \text{ is open and there are } x \in M,\, V \in \mathscr{U} \text{ with } x \in N \subseteq V(x) \text{ and } V^2(x) \subseteq O\}$$

is an open cover of M. By the compactness of M, we thus find finitely many $N_1,\dots,N_n \in \mathscr{M}$ and corresponding $x_1,\dots,x_n \in M$ and $V_1,\dots,V_n \in \mathscr{U}$ such that

$$M \subseteq N_1 \cup \cdots \cup N_n \subseteq V_1(x_1) \cup \cdots \cup V_n(x_n)$$

and $V_k^2(x_k) \subseteq O$ $(k = 1,\dots,n)$. Put $U := V_1 \cap \ldots \cap V_n$. For each $x \in M$, we find some k with $x \in V_k(x_k)$, and so $x_k \in V_k(x)$ implies $U(x) \subseteq V_k(x) \subseteq V_k(V_k(x_k)) = V_k^2(x_k) \subseteq O$. Hence, $U(M) \subseteq O$. □

It is the aim of this section to establish a connection between condensifying $(X_n)_n$ and those with a (pre)compact uniform $\mathscr{U}$-attractor. One of these implications is easily seen:

Proposition 7.4. If $(X_n)_n$ has a precompact (almost/weak) uniform $\mathscr{U}$-attractor M then $(X_n)_n$ is (almost/weakly) condensifying with respect to each set Y satisying $M \subseteq Y \subseteq X$.

Proof. Let $U \in \mathscr{U}$. Choose some $V \in \mathscr{U}$ with $V^2 \subseteq U$. By hypothesis, we have $X_n \subseteq V(M)$ for almost all n (infinitely many n/some n). Since M is precompact, there is a finite set $N \subseteq M \subseteq Y$ with $M \subseteq V(N)$. Hence, $X_n \subseteq V(V(N)) = V^2(N) \subseteq U(N)$ which implies $U \notin \mathscr{H}_Y^\circ(X_n)$ and thus $U \notin \mathscr{H}_Y(X_k)$ for almost all n (infinitely many n/some n). This means that $(X_n)_n$ is (almost/weakly) condensifying. □

However, even in $\mathbb{R}$, one cannot expect that condensifying sequences $(X_n)_n$ have some sort of compact uniform attractor if they are not decreasing as the following example shows:

Example 7.5. Let $x_n \in X$ be a sequence without a limit (or even without accumulation points). Then $X_n := \{x_n\}$ is condensifying with respect to X, but there is no precompact (almost) uniform $\mathscr{U}$-attractor.

Obviously, the problem in this example is that the sequence of sets X_n is not decreasing. For this reason, we will consider the converse implication essentially only for decreasing sequences. Actually, the following milder requirement will be sufficient.

Definition 7.6. $(X_n)_n$ is *$\mathscr{U}$-compactly decreasing* (with respect to $Y, Y_0 \subseteq X$) if for each $U \in \mathscr{U}$ there is some $V_U \in \mathscr{U}$ with the following property: For each n_0 there are j_0 and a set $K \subseteq X$ with $\mathscr{H}_{Y_0}(K) = \emptyset$ such that

$$V_U \notin \bigcup_{n=n_0}^{\infty} \mathscr{H}_Y^\circ(X_n) \implies \bigcup_{j=j_0}^{\infty} X_j \subseteq U(K).$$

Each decreasing sequence $X_{n+1} \subseteq X_n$ is $\mathscr{U}$-compactly decreasing. It suffices even that $X_{n+1} \subseteq \overline{X_n \cup K}$ or in locally convex spaces even, e.g., $X_{n+1} \subseteq \overline{\operatorname{conv}}(X_n \cup K)$ with a fixed set $K \subseteq X$ satisfying $\mathscr{H}_{Y_0}(K) = \emptyset$. More general, we have the following sufficient criteria.

Proposition 7.7. $(X_n)_n$ is $\mathscr{U}$-compactly decreasing with respect to $Y, Y_0 \subseteq X$ if one of the following holds.

(1) For each n there are $j_0 \geq m \geq n$ and a set $K \subseteq X$ satisfying $\mathscr{H}_{Y_0}(K) = \emptyset$ with

$$\bigcup_{j=j_0}^{\infty} X_j \subseteq \bigcup_{k=n}^{m} \overline{X}_k \cup K.$$

(2) X is a subset of a topological group Z, and there is a natural number c such that for each n there are $j_0 \geq m \geq n$, and a map $F\colon 2^Z \to 2^Z$ which is built from at most c operations of the form $A \mapsto \overline{A}$, $A \mapsto A^{-1}$, $(A,B) \mapsto A \cdot B$, $(A,B) \mapsto A \cup B$, composing either previous results or precompact sets such that $F(Y) \subseteq Y_0$ and

$$\bigcup_{j=j_0}^{\infty} X_j \subseteq F\Big(\bigcup_{k=n}^{m} X_k\Big).$$

(3) If Z in the previous part is a topological vector space (not necessarily Hausdorff), then in addition the operation $A \mapsto \Lambda A$ with a set Λ of scalars is allowed for F if all these scalars are bounded by c. Moreover, if Z is locally convex, then also the operation $A \mapsto \operatorname{conv} A$ is allowed for F.

Proof. We prove only the last claim, since the proof of the others is similar. Let $U \in \mathscr{U}$ be given and choose $V = V_U \in \mathscr{U}$ such that $V^{(2c)^c} \subseteq U$. Shrinking V if necessary, we may assume in addition that there is a balanced (convex) neighborhood $C \subseteq X$ of 0 with

$$V = \{(x,y) \in X : x - y \in C\}.$$

We verify the property of Definition 7.6, putting $n = m$ in our above hypotheses. Suppose that $V \notin \bigcup_{k=m}^{\infty} \mathscr{H}_Y^{\circ}(X_k)$. In particular, we find for each $k = n, \dots, m$ some finite set $N_k \subseteq Y$ with $X_k \subseteq V(N_k)$. Then

$$K := F\Big(\bigcup_{k=n}^{m} N_k\Big)$$

is a precompact subset of Y_0, hence $\mathscr{H}_{Y_0}^{\circ}(K) = \emptyset$, and

$$\bigcup_{j=j_0}^{\infty} X_j \subseteq F\Big(\bigcup_{k=n}^{m} X_k\Big) \subseteq V^{(2c)^c}(K) \subseteq U(K).$$

For the second inclusion, we used (by induction on the structure of F) that the operations are compatible with V in the sense that $\overline{M} \subseteq V(M)$, $-M \subseteq V(-M)$, $V^{k_1}(M_1) + V^{k_2}(M_2) \subseteq V^{k_1+k_2}(M_1 + M_2)$, $V^{k_1}(M_1) \cup V^{k_2}(M_2) \subseteq V^{k_1+k_2}(M_1 \cup M_2)$, $\Lambda V^n(A) \subseteq V^{cn}(\Lambda A)$, and (if C is convex) $\operatorname{conv} V^k(M) \subseteq V^k(\operatorname{conv} M)$. To see the latter, note that the convexity of kC and $\operatorname{conv} M$ implies that also $kC + \operatorname{conv} M$ is convex, and so $\operatorname{conv}(kC + M) \subseteq kC + \operatorname{conv} M$. □

Under the hypothesis of Definition 7.6, we can prove some powerful converse of Proposition 5.9.

Proposition 7.8. Let $(X_n)_n$ be $\mathscr{U}$-compactly decreasing with respect to $Y, Y_0 \subseteq X$ and condensifying with respect to Y. If a set $M \subseteq X$ satisfies

$$M \subseteq \overline{\lim_n} X_n \cup \overline{\bigcup_n K_n}$$

with some $K_n \subseteq X_n$ where $\mathscr{H}_{Y \cup Y_0}(K_n) = \emptyset$, then $\mathscr{H}_{Y \cup Y_0}^{\circ}(\overline{M}) = \emptyset$. In particular, each set

$$C \in \operatorname{Fin}_{Y \cup Y_0}(X_n)_n \cup \{\underline{\lim_n} X_n, \overline{\lim_n} X_n\}$$

satisfies $\mathscr{H}_{Y \cup Y_0}^{\circ}(\overline{C}) = \emptyset$ (hence, C is precompact if $\mathscr{U}$ is a uniformity).

Proof. For each $U_0 \in \mathscr{U}$ there is some $U \in \mathscr{U}$ with $U^3 \subseteq U_0$. Choose some V_U according to $U \in \mathscr{U}$ as in Definition 7.6. By Proposition 5.2, there is some n_0 such that $V_U \notin \bigcup_{n=n_0}^{\infty} \mathscr{H}_Y^{\circ}(X_n)$. With j_0 as in Definition 7.6, we thus find a set $K \subseteq X$ with $\mathscr{H}_{Y_0}(K) = \emptyset$ such that

$$\bigcup_{j=j_0}^{\infty} X_j \subseteq U(K).$$

Hence, putting $P_{j_0} := \bigcup_{j<j_0} K_j$, we have

$$M \subseteq P_{j_0} \cup \overline{\bigcup_{j=j_0}^{\infty} X_j} \subseteq \overline{P_{j_0} \cup U(K)} \subseteq \overline{U(P_{j_0} \cup K)} \subseteq U(U(P_{j_0} \cup K)).$$

Since Proposition 4.10 implies that $\mathscr{H}_{Y \cup Y_0}^{\circ}(P_{j_0} \cup K)$ is empty and thus does not contain $U \in \mathscr{U}$, there is a finite set $N \subseteq Y \cup Y_0$ such that $P_{j_0} \cup K \subseteq U(N)$. Hence, $M \subseteq U^3(N) \subseteq U_0(N)$, i.e.

$\mathscr{H}^{\circ}_{Y\cup Y_0}(M)=\emptyset$. Proposition 4.10 implies $\mathscr{H}^{\circ}_{Y\cup Y_0}(\overline{M})=\emptyset$. The second claim is an immediate consequence of the inclusion $\underline{\lim}_n X_n \subseteq \overline{\lim}_n X_n$ and of the fact that each finite set $K_n = N_n \subseteq Y\cup Y_0$ satisfies $\mathscr{H}_{Y\cup Y_0}(K_n)=\emptyset$. □

Nevertheless, we will have to require some additional properties on the quasi-uniform space X to prove the existence of a (pre)compact attractor: Either, we will have to require some sort of completeness or we assume that $\mathscr{U}$ is a uniformity and has a countable filter base.

Of course, the set $\overline{\lim}_n X_n$ is a good candidate for a precompact attractor. Nevertheless, even if this set is nonempty and compact, and if $X=\mathbb{R}$ and the set sequence $(X_n)_n$ is decreasing, this set need not necessarily be a uniform attractor.

Example 7.9. Let $X=\mathbb{R}$. Then $X_n := \{0\}\cup(n,\infty)$ is a decreasing sequence without any precompact uniform $\mathscr{U}$-attractor. Nevertheless, $\underline{\lim}_n X_n = \overline{\lim}_n X_n = \{0\}$.

We will see that it is not accidental that the sequence $(X_n)_n$ in this example fails to be condensifying. In fact, we will see soon that under some mild completeness hypothesis $\overline{\lim}_n X_n$ actually is a uniform attractor if $(X_n)_n$ is condensifying. Such a completeness hypothesis cannot be omitted, even if $X\subseteq\mathbb{R}$, $\operatorname{core}_n X_n$ is nonempty and compact and $(X_n)_n$ is decreasing.

Example 7.10. Let $X:=(0,1]$ and $X_n:=(0,1/n]\cup\{1\}$. Then $\underline{\lim}_n X_n = \overline{\lim}_n X_n = \{1\}$ is not a uniform attractor for the decreasing sequence $(X_n)_n$.

Nevertheless, in this example $(X_n)_n$ *has* a precompact uniform attractor (e.g. the whole space X is such an attractor). This is not accidental. In fact, the following result establishes the existence of such attractors without any completeness hypothesis, but under a certain metrizability assumption.

Theorem 7.11. *Let $(X_n)_n$ be $\mathscr{U}$-compactly decreasing with respect to $Y, Y_0\subseteq X$. Assume in addition that $\mathscr{U}$ is a uniformity with a countable filter base (i.e. it is induced by a q-pseudometric). Then the following three statements are equivalent and imply the fourth one.*

(1) $(X_n)_n$ is condensifying with respect to Y.
(2) $(X_n)_n$ has a precompact $\mathscr{U}$-attractor $M\subseteq Y$.
(3) $(X_n)_n$ has a precompact countable uniform $\mathscr{U}$-attractor $M\subseteq Y$.
(4) $(X_n)_n$ has a precompact countable uniform $\mathscr{U}$-attractor $M\in\operatorname{Fin}_X(X_n)_n$ (not necessarily $M\subseteq Y$).

Proof. In view of Proposition 7.4, we only have to show that if $(X_n)_n$ is condensifying with respect to Y, then also the other statements hold. Put $H_n := \bigcup_{k=n}^{\infty}\mathscr{H}^{\circ}_{Y}(X_k)$. We have (Proposition 5.2)

$$\bigcap_{n=1}^{\infty} H_n=\emptyset, \qquad H_{n+1}\subseteq H_n. \tag{7.1}$$

Choose some countable filter base $U_1,U_2,\ldots\in\mathscr{U}$ for $\mathscr{U}$. Replacing U_k by $\widetilde{U}_k := U_1\cap\cdots\cap U_k$ if necessary, we may assume without loss of generality that $U_{k+1}\subseteq U_k$ $(k=1,2,\ldots)$. For each $k=1,2,\ldots$ we find by (7.1) some index n_k with $U_k\notin H_n$ $(n\geq n_k)$. Choosing n_k inductively, we can assume that $n_{k+1}>n_k$ $(k=1,2,\ldots)$. For each $n\geq n_1$, let k_n denote the unique index satisfying $n_{k_n}\leq n<n_{k_n+1}$ and observe that $U_{k_n}\in\mathscr{U}\setminus H_n\subseteq\mathscr{U}\setminus\mathscr{H}^{\circ}_{Y}(X_n)$. Hence, there is a finite set $N_n\subseteq Y$ with $X_n\subseteq U_{k_n}(N_n)$. Then the set $M_n := N_n\cap U^{+}_{k_n}(X_n)$ is finite and satisfies

$$X_n\subseteq U_{k_n}(N_n)\cap X_n\subseteq U_{k_n}(M_n) \qquad (n\geq n_1). \tag{7.2}$$

Since $M_n \subseteq U_{k_n}^{+}(X_n) = U_{k_n}^{-1}(X_n)$ is finite, there is a finite set $A_n \subseteq X_n$ (of at most the same cardinality than M_n) such that

$$M_n \subseteq U_{k_n}^{-1}(A_n) \qquad (n \geq n_1). \tag{7.3}$$

We claim that the countable sets

$$M := \bigcup_{n=n_1}^{\infty} M_n \subseteq Y \quad \text{and} \quad A := \bigcup_{n=n_1}^{\infty} A_n \in \operatorname{Fin}_X (X_n)_n$$

are precompact uniform $\mathscr{U}$-attractors for $(X_n)_n$.

Let $U \in \mathscr{U}$. Choose some $V \in \mathscr{U}$ with $V^2 \in \mathscr{U}$ and note that $W := V \cap V^{-1} \in \mathscr{U}$. Hence, there is some $n_U \geq n_1$ such that $U_k \subseteq W$ whenever $k \geq k_n$ for some $n \geq n_U$. By (7.2) and (7.3), we have for all $n \geq n_U$ that

$$X_n \subseteq U_{k_n}(M_n) \subseteq U_{k_n}(U_{k_n}^{-1}(A_n)) \subseteq W(W^{-1}(A_n)) \subseteq V^2(A_n)) \subseteq U(A_n) \subseteq U(A),$$

and

$$X_n \subseteq U_k(M_n) \subseteq U(M_n) \subseteq U(M).$$

Hence, A and M are uniform $\mathscr{U}$-attractors for $(X_n)_n$. Moreover, A is precompact by Proposition 7.8, and we have for all $n \geq n_U$ by (7.3) that

$$M_n \subseteq U_{k_n}^{-1}(A_n) \subseteq W^{-1}(A_n) \subseteq U(A).$$

Hence, putting $K := \bigcup_{n<n_U} M_n$, we have

$$M \subseteq K \cup U(A) \subseteq U(K \cup A).$$

Since $K \cup A$ is precompact, this shows $U \notin \mathscr{H}_X^{\infty}(M)$. Since $U \in \mathscr{U}$ was arbitrary, we thus have shown that $\mathscr{H}_X^{\infty}(M) = \emptyset$, i.e. M is precompact in view of Proposition 4.2/4.9. ☐

Since we do not want to assume completeness of the whole space, we introduce a weaker form of sequential completeness with respect to the sequence of sets $X_n \subseteq X$.

Definition 7.12. X is core-complete with respect to $(X_n)_n$ if each infinite set from $\operatorname{Fin}_X (X_n)_n$ with a precompact closure actually has a compact closure.

Recall that these sets are precisely those whose elements consist of sequences $x_k \in X$ such that there are $n_1 \leq n_2 \leq \dots$ with $n_k \to \infty$ and $x_k \in X_{n_k}$ and such that $\overline{\{x_1, x_2, \dots\}}$ is precompact. The set of all accumulation points of such sequences is called the *completeness core* $\operatorname{core}_n X_n$ of the sequence $(X_n)_n$.

Using the notation (5.5), we have clearly

$$\bigcup_k \bigcap_{n=k}^{\infty} X_n \subseteq \operatorname*{core}_n X_n \subseteq \operatorname*{\overline{lim}}_n X_n. \tag{7.4}$$

Since $\overline{\lim}_n X_n$ is closed, we conclude, using also the notation (5.6), that

$$\operatorname*{\underline{lim}}_n X_n \subseteq \overline{\operatorname*{core}_n X_n} \subseteq \operatorname*{\overline{lim}}_n X_n. \tag{7.5}$$

In particular, if $X_{n+1} \subseteq \overline{X}_n$ for all n, we have

$$\bigcap_n \overline{X}_n = \operatorname*{\underline{lim}}_n \overline{X}_n = \operatorname*{core}_n \overline{X}_n = \overline{\operatorname*{core}_n \overline{X}_n} = \operatorname*{\overline{lim}}_n \overline{X}_n, \tag{7.6}$$

and so $\operatorname{core}_n \overline{X}_n$ is closed in this case. In general, it is usually not true that $\operatorname{core}_n X_n$ is closed, even if the sets X_n are closed, because the definition involves only sequences and not arbitrary nets/filters. However, if the uniformity is quasi-pseudometrizable, we have a much stronger statement even if the sets X_n are not necessarily closed.

Lemma 7.13. *Let $\mathscr{U}$ be a uniformity with a countable filter base. If X is core-complete with respect to $(X_n)_n$, then $\operatorname{core}_n X_n$ is closed and complete.*

The converse is of course not true, as can be seen in the space $X = (0,\infty)$ with $X_n := (0,1/n)$.

Proof. Since $\mathscr{U}$ has a countable filter base, it suffices to show that each Cauchy sequence in $\operatorname{core}_n X_n$ converges to an element in $\operatorname{core}_n X_n$. Let $(x_n)_n$ be a Cauchy sequence with $x_n \in \operatorname{core}_n X_n$. By definition, we find $n_{k,1} \le n_{k,2} \le \cdots$ with $n_{k,n} \to \infty$ as $n \to \infty$ and $x_{k,n} \in X_{n_{k,n}}$ such that x_k is an accumulation point of the sequence $(x_{k,n})_n$. Let $U_1, U_2, \ldots \in \mathscr{U}$ generate a filter base for $\mathscr{U}$. Passing to subsequences of $(x_{k,n})_n$ and $(n_{k,n})_n$ if necessary, we may assume without loss of generality that $n_{k,n} \ge n$. Moreover, using a diagonal argument to pass to a subsequence, we can also assume without loss of generality that $x_{k,n} \in U_j(x_k)$ for all $n \ge j$. Finally, since $(x_n)_n$ is a Cauchy sequence we can assume, passing to subsequences of $(x_k)_k$, $(x_{k,n})_k$ and $(n_{k,n})_k$ if necessary, that $(x_n, x_m) \in U_j$ for all $n,m \ge j$.

Let $N = (N_1, N_2)\colon \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ be one-to-one and onto, and put $y_n := x_{N_1(n),N_2(n)}$ and $k_n := n_{N_1(n),N_2(n)}$. Choose m_j such that $N_1(n), N_2(n) \ge j$ for all $n \ge m_j$. Then $k_n \ge j$ for all $n \ge m_j$, and so $k_n \to \infty$ as $n \to \infty$. Moreover, we have for all $n,m \ge m_j$ that

$$y_n \in U_j(x_{N_1(n)}) \subseteq U_j(U_j(x_{N_1(m)})) \subseteq U_j^2(U_j^{-1}(y_m)),$$

Since we find for each $U \in \mathscr{U}$ some j with $U_j^2 \circ U_j^{-1} \subseteq U$, the sequence $(y_n)_n$ is a Cauchy sequence; in particular, $\{y_1, y_2, \ldots\}$ is precompact. Since $y_n \in X_{k_n}$ and $k_n \to \infty$ as $n \to \infty$, the assumption implies that $(y_n)_n$ has some accumulation point $x \in \operatorname{core}_n X_n$. For $V \in \mathscr{U}$, choose some j with $U_j^{-1} \circ U_j \subseteq V$. For infinitely many n, we have

$$x_{N_1(n)} \in U_j^{-1}(y_n) \subseteq U_j^{-1}(U_j(x)) \subseteq V(x).$$

We thus have shown that x is also an accumulation point of (a subsequence of our original sequence) $(x_n)_n$. Since the original sequence was a Cauchy sequence, we have $x_n \to x$. ☐

Recall that a *quasi-component* Q of a subset $M \subseteq X$ is a maximal (with respect to set-inclusion) set $Q \subseteq M$ with the property that the relations

$$M = A \cup B, \qquad \overline{A} \cap B = \emptyset, \qquad A \cap \overline{B} = \emptyset \tag{7.7}$$

imply that either $Q \subseteq A$ or $Q \subseteq B$. In particular, each connected subset of M has the latter property and thus is contained in some quasi-component, i.e. each connected component of M is contained in some quasi-component of M.

Theorem 7.14. *Let $\mathscr{U}$ be a uniformity. Let $(X_n)_n$ be $\mathscr{U}$-compactly decreasing with respect to $Y, Y_0 \subseteq X$, and let X be core-complete with respect to $(X_n)_n$. Assume in addition that for each $C \in \operatorname{Fin}_X (X_n)_n$ and each $C \subseteq \overline{\lim}_n X_n$ the implication*

$$\mathscr{H}^{\circ}_{Y\cup Y_0}(\overline{C}) = \emptyset \implies \overline{C} \text{ is precompact} \tag{7.8}$$

holds (if $\mathscr{U}$ is a uniformity, (7.8) holds automatically also without closures by Proposition 4.2).

Then the first of the following statements implies all others; in case $Y \supseteq \overline{\operatorname{core}_n X_n}$ *(if* (7.8) *holds without closures also in case* $Y \supseteq \operatorname{core}_n X_n$*) the statements are equivalent.*

(1) $(X_n)_n$ *is condensifying with respect to* Y*.*
(2) $(X_n)_n$ *has a precompact uniform* $\mathscr{U}$*-attractor.*
(3) $\overline{\operatorname{core}_n X_n}$ *is a uniform* $\mathscr{U}$*-attractor for* $(X_n)_n$ *with a precompact closure.*
(4) *(If* (7.8) *holds without closures).* $\operatorname{core}_n X_n$ *is a precompact uniform* $\mathscr{U}$*-attractor for* $(X_n)_n$*.*
(5) *(If* $\mathscr{U}$ *is a uniformity with a countable filter base.)* $\operatorname{core}_n X_n$ *is a compact uniform attractor for* $(X_n)_n$*.*
(6) $\overline{\lim}_n X_n$ *is a precompact uniform* $\mathscr{U}$*-attractor for* $(X_n)_n$*.*

Moreover, if one of these statements holds and $K \subseteq X$ *is closed and complete,* $\operatorname{core}_n X_n \subseteq K$*, and if*

$$K \text{ is contained in a quasi-component of } X_n \tag{7.9}$$

for infinitely many n*, then* K *is compact and connected.*

Proof. Let $(X_n)_n$ be condensifying with respect to Y. By Proposition 7.8, the set $B := \overline{\lim}_n X_n$ satisfies $\mathscr{H}^{\circ}_{Y \cup Y_0}(B) = \emptyset$. By (7.5), the set $C := \operatorname{core}_n X_n$ satisfies $\overline{C} \subseteq B$, and so also $\mathscr{H}^{\circ}_{Y \cup Y_0}(\overline{C}) = \emptyset$. Using (7.8), we obtain that $\overline{C} \subseteq B$ are both precompact (if (7.8) holds without closures, also C is precompact).

We show that C (and thus also $B \supseteq C$) is a uniform $\mathscr{U}$-attractor. Assume by contradiction that this is not the case. Then there is some $U \in \mathscr{U}$ such that X_n is not contained in $U(C)$ for infinitely many n, i.e. we find a sequence $x_k \in X_{n_k} \setminus U(C)$ with $n_k \to \infty$. However, the set $A := \{x_1, x_2, \ldots\}$ belongs to $\operatorname{Fin}_X(X_n)_n$. By Proposition 7.8, we have $\mathscr{H}^{\circ}_{Y \cup Y_0}(A) = \emptyset$, and so $\overline{A}$ is precompact by (7.8). Since X is core-complete, the sequence $(x_k)_k$ has some accumulation point which by definition belongs to $\operatorname{core}_n X_n = C$. Hence, $x_k \in U(C)$ for infinitely many k, contradicting the choice of the sequence x_k.

If in addition $\mathscr{U}$ has a countable base, then Lemma 7.13 implies that the precompact set $\operatorname{core}_n X_n$ is actually compact and thus a uniform attractor by Proposition 7.2.

Conversely, if some of the other statement holds, then in particular $(X_n)_n$ has *some* precompact uniform $\mathscr{U}$-attractor (Proposition 7.2). Hence, Proposition 7.4 implies that $(X_n)_n$ is condensifying with respect to X. By what we have shown, then $\overline{\operatorname{core}_n X_n}$ (or even $\operatorname{core}_n X_n$) is a precompact uniform $\mathscr{U}$-attractor, and applying Proposition 7.4 once more, we obtain that $(X_n)_n$ is condensifying with respect to Y if $Y \supseteq \overline{\operatorname{core}_n X_n}$ (or $Y \supseteq \operatorname{core}_n X_n$, respectively).

To see the last statement of the theorem, note first that the previous results imply that if one the statement holds with some $Y \subseteq X$ then $(X_n)_n$ is condensifying with respect to $Y = X$. Note also that the hypothesis on K implies $K \subseteq \overline{\lim}_n X_n$, and so K is precompact by Proposition 7.8 and (7.8). Hence, K is actually compact. Assume by contradiction that $K = A_0 \cup A_1$ with disjoint nonempty closed (in K and thus in X) sets $A_0, A_1 \subseteq X$. Since A_0 is compact, we find by Lemma 7.3 some $U \in \mathscr{U}$ such that $U(A_0) \subseteq X \setminus A_1$. Choose some $V \in \mathscr{U}$ with $V^2 \subseteq U$, and let $O \subseteq X$ denote the interior of $V(A_0)$. Then

$$A_0 \subseteq O \subseteq \overline{O} \subseteq V(O) \subseteq V(V(A_0)) \subseteq U(A_0) \subseteq X \setminus A_1. \tag{7.10}$$

Let N denote the set of all indices n satisfying (7.9). For each $n \in N$, we have $X_n \cap \partial O \neq \emptyset$. Indeed, we would otherwise have $A := X_n \cap O = X_n \cap \overline{O}$. Hence, A is closed and open in X_n, and so also $B := X_n \setminus A$ is closed and open in X_n. This implies (7.7) (with $M := X_n$), and since K is contained in a quasi-component of X_n, we have either $K \subseteq A$ or $K \subseteq B$. But both is impossible, because (7.10) implies in view of $A_i \subseteq K \subseteq X_n$ that $A_0 \subseteq A$ and $A_1 \subseteq B$: In particular, $K = A_0 \cup A_1$ intersects both of the disjoint sets A and B (since $A_i \neq \emptyset$) and thus cannot be contained only in A or only in B.

The contradiction shows that for each $n \in N$ there is some $x_n \in X_n \cap \partial O$. The set $\{x_n : n \in N\}$ belongs to $\operatorname{Fin}_X (X_n)_n$ and thus has a precompact closure by Proposition 7.8 and (7.8). Since X is core-complete, the sequence $(x_n)_{n \in N}$ has some accumulation point x which by definition belongs to $\operatorname{core}_n X_n \subseteq K$. In particular, either $x \in A_0$ or $x \in A_1$. In view of (7.10) we thus have either $x \in O$ or $x \notin \overline{O}$. In both cases, $x \notin \partial O$. However, this is a contradiction because x is an accumulation point of a sequence x_n from the closed set ∂O. ☐

We obtain as a trivial special case the following generalization of Kuratowski's classical result [19] to quasi-uniform spaces. Recall in this connection that each quasi-component of M is contained in a quasi-component of $\overline{M}$.

Corollary 7.15. *Let $(X, \mathscr{U})$ be a complete quasi-uniform space. If $\mathscr{U}$ is not a uniformity, assume in addition*

$$\mathscr{H}_X^\circ(\overline{C}) = \emptyset \implies \overline{C} \text{ is precompact} \qquad (C \subseteq X). \tag{7.11}$$

Let $(X_n)_n$ be a condensifying sequence of nonempty sets in X satisfying $X_{n+1} \subseteq \overline{X}_n$. Then $\bigcap_n \overline{X}_n$ is nonempty and compact. Moreover, this set is connected if, for infinitely many n, it is contained in a quasi-component of $\overline{X}_n$.

Proof. Replacing X_n by $\overline{X}_n$ if necessary, we may assume without loss of generality that $X_n = \overline{X}_n$ for all n. By (7.6), we have then $K := \operatorname{core}_n X_n = \bigcap_n X_n$. Note that K is closed and thus complete. By Theorem 7.14, K is precompact (hence compact) and a uniform $\mathscr{U}$-attractor for $(X_n)_n$; moreover, K is connected if X_n is connected for infinitely many n. Now observe that $K = \emptyset$ cannot be a uniform $\mathscr{U}$-attractor for a sequence of nonempty sets. ☐

If X is compact, Corollary 7.15 boils down to the following two special cases which are well-known for a compact Hausdorff space X. However, our arguments did not need any separation properties, and it seems that even these special cases were unknown (for non-Hausdorff spaces).

Corollary 7.16. *Each quasi-component Q of a compact topological space X is connected, i.e. in X the notions "component" and "quasi-component" coincide.*

Proof. Recall that $X_n := Q$ is closed, and apply Corollary 7.15, using some quasi-uniformity $\mathscr{U}$ generating the topology of X, see [8, Section 2.1]. Note that the compactness of X automatically implies that $(X, \mathscr{U})$ is complete and that (7.11) holds, because closed subsets of compact sets are compact and thus precompact. ☐

Corollary 7.17. *Let X_n be closed subsets of a compact topological space X with $X_{n+1} \subseteq X_n$ for all n. Then $\bigcap_n X_n$ is nonempty, compact, and connected if each X_n is connected.*

Proof. Apply Corollary 7.15 using some quasi-uniformity $\mathscr{U}$ generating the topology of X. ☐

The previous corollaries were our main reason for considering quasi-uniformities instead of uniformities: Otherwise, the previous corollaries would apply only for those spaces whose topology stems from a uniform structure; this is precisely the class of T_{3a} spaces.

7.2. **Uniform Attractors for Monotone Maps of Power-Sets.** Let $(X,\mathscr{U})$ be a quasi-uniform space.

Definition 7.18. A set $M \subseteq X$ is a *uniform attractor* for a map $F\colon 2^X \to 2^X$ if for each open set $O \subseteq X$ with $M \subseteq O$ there is some index n_O with $F^n(X) \subseteq O$ for all $n \geq n_O$.

$M \subseteq X$ is a *uniform $\mathscr{U}$-attractor* for a map $F\colon 2^X \to 2^X$ if for each $U \in \mathscr{U}$ there is some index n_U with $F^n(X) \subseteq U(M)$ for all $n \geq n_U$.

Proposition 7.19. $M \subseteq X$ is a uniform attractor for F if and only if M is a uniform attractor for $(F^n(X))_n$.

$M \subseteq X$ is a uniform $\mathscr{U}$-attractor for F if and only if M is a uniform $\mathscr{U}$-attractor for $(F^n(X))_n$.

We call X F-core-complete with respect to $F\colon 2^X \to 2^X$ if it is core-complete with respect to $(F^n(X))_n$. Recall that this is in particular the case if X is complete. We use the shortcut $\operatorname{core} F := \operatorname{core}_n F^n(X)$ and note that, if F is monotone and thus $F^{n+1}(X) \subseteq F^n(X)$ for all n, (7.4) and (7.5) simplify to

$$\bigcap_n F^n(X) \subseteq \operatorname{core} F \subseteq \overline{\operatorname{core} F} \subseteq \bigcap_n \overline{F^n(X)}. \tag{7.12}$$

Now we can summerize some of our previous results as follows.

Theorem 7.20. *Let $F\colon 2^X \to 2^X$ be monotone, and let $\mathscr{U}$ be a uniformity. Then for each $Y \subseteq X$ the first six of the following statements are equivalent and imply the others; if $Y = X$ (or at least if Y contains the respective attractors) all these statements are equivalent.*

(1) *F is condensifying with respect to Y.*
(2) *All sets in $\operatorname{Fin}_Y F$ are precompact.*
(3) *All sets in $\operatorname{Fin}_Y F$ and $\bigcap_n \overline{F^n(X)}$ are precompact.*
(4) *(If X is a q-pseudometric space.) $\lim\limits_{n\to\infty} \chi_Y(F^n(X)) = 0$.*
(5) *(If X is a q-pseudometric space.) F has a precompact uniform $\mathscr{U}$-attractor $M \subseteq Y$.*
(6) *(If X is a q-pseudometric space.) F has a countable precompact uniform $\mathscr{U}$-attractor $M \subseteq Y$.*
(7) *(If X is a q-pseudometric space.) F has a countable precompact uniform $\mathscr{U}$-attractor $M \in \operatorname{Fin}_X F$.*
(8) *(If X is F-core-complete.) F has a precompact uniform $\mathscr{U}$-attractor.*
(9) *(If X is F-core-complete.) $\operatorname{core} F$ is a precompact uniform $\mathscr{U}$-attractor for F.*
(10) *(If X is a q-pseudometric space and F-core-complete.) $\operatorname{core} F$ is a compact uniform attractor for F.*
(11) *(If X is complete). $\overline{\operatorname{core} F}$ is a compact uniform attractor for F.*
(12) *(If X is complete). $\bigcap_n \overline{F^n(X)}$ is a compact uniform attractor for F.*
(13) *(If X is complete and, for infinitely many n, $\overline{\operatorname{core} F}$ is contained in a quasi-component of $F^n(X)$.) $\overline{\operatorname{core} F}$ is a connected compact uniform attractors for F.*
(14) *(If X is complete and, for infinitely many k, $\bigcap_n \overline{F^n(X)}$ is contained in a quasi-component of $F^k(X)$.) $\bigcap_n \overline{F^n(X)}$ is a connected compact uniform attractors for F.*

These statements are satisfied if one of the following holds.

(1) *F is strictly monotonically (countably) Γ-condensing (and of countable type) where $\Gamma \succeq_F \mathscr{H}_Y$ (or $\Gamma \succeq_F^c \mathscr{H}_Y$) is such that for each $\gamma \in \Gamma$ there is some n_γ with $\gamma(F^{n_\gamma}(X)) < \infty$ (or $\gamma(C) < \infty$ for each countable $C \subseteq F^{n_\gamma}(X)$, respectively).*

(2) *F is countably condensing with respect to a countable family $\Gamma \subseteq \Gamma_Y$ which is admissible for F.*

We note that if $\mathscr{U}$ does not have a countable base (i.e. if it is not induced by a q-pseudometric), we assume the axiom of choice (or at least PI) for the last 4 of the above equivalent statements for the claim that the respective sets are actually *compact* uniform attractors; without this axiom, we claim only that they are precompact complete uniform $\mathscr{U}$-atttractors (recall Proposition 3.6).

Proof. In view of Proposition 7.19, each of the implications (implying or requiring that F is condensifying with respect to Y) is just a special case of one of the previous results, keeping (7.12) and Proposition 7.2 in mind. ☐

As remarked earlier, the last statement of Theorem 7.20 contains [2, Theorem I.5.29]. However, our result is more general in several respects.

(1) We allow a much wider class of measures of noncompactness, and we require only that F is countably condensing with respect to a (countable) *family* of such measures.
(2) We consider monotone set-maps instead of just multivalued maps.
(3) Even in case of multivalued maps, we obtain the existence of a *uniform* attractor, not only of an attractor.

In particular, the last statement will be important in the next section for multivalued maps. We point out that it would not have been possible to prove Theorem 7.20 directly along the lines of [2, Theorem I.5.29], because the argument given there only shows that $\bigcap_n \overline{F^n(X)}$ is compact for condensing multivalued F. Unfortunately, even if this set is nonempty and compact, it is in general not an attractor for F, even if $F\colon X \to X$ is a single-valued continuous map, as can be seen by the right-shift operator $F(\xi_1,\xi_2,\dots) = (0,\xi_1,\xi_2,\dots)$ in the space $X := \ell_\infty$. Although it was known that $\bigcap_n \overline{F^n(X)}$ is an attractor if F has precompact orbits (see e.g. [24, Lemma 6] or [2, Proposition I.5.9] for the single- or multivalued case, respectively), we could not directly use such kind of arguments, because we cannot speak about orbits in our situation and because we want a uniform attractor.

8. EXISTENCE OF COMPACT ATTRACTORS

Throughout this section, let $(X,\mathscr{U})$ be a quasi-uniform space, and $F\colon X \multimap X$ be a multivalued map. If X is e.g. an unbounded subset of a normed space, then F will typically not have a compact uniform attractor, but we will see that often F will have a compact attractor.

Definition 8.1. A set $M \subseteq X$ is an *attractor* for $F\colon X \multimap X$ if for each open set $O \subseteq X$ with $M \subseteq O$ and each $x \in X$ there is some index $n_{O,x}$ with $F^n(x) \subseteq O$ for all $n \geq n_{O,x}$.

$M \subseteq X$ is a *$\mathscr{U}$-attractor* for $F\colon X \multimap X$ if for each $U \in \mathscr{U}$ and each $x \in X$ there is some index $n_{U,x}$ with $F^n(x) \subseteq U(M)$ for all $n \geq n_{U,x}$.

Using Lemma 7.3, we obtain immediately:

Proposition 8.2. Each attractor M for F is an $\mathscr{U}$-attractor F. The converse holds if M is compact.

In order to show the existence of a compact attractor the following notions will be useful:

Definition 8.3. A set $B \subseteq X$ is a *collector* for $F\colon X \multimap X$ if for each $x \in X$ there is some index n_x with $F^n(x) \subseteq B$ for all $n \geq n_x$.

Definition 8.4. The multivalued map $F\colon X \multimap X$ is called *occasionally finite* if for each $x \in X$ the set $F^n(x)$ is finite (or empty) for infinitely many indices n.

Of course, each single-valued map $F\colon X \to X$ is occasionally finite. More general, each multivalued maps with finite images $F(x)$ is occasionally finite.

Nevertheless, most multivalued maps occurring in analytic problems fail to be occasionally finite. For this reason, it is crucial to observe that if the attractor M for the restriction of F is *uniform*, no such requirement is needed in the following result.

Theorem 8.5. *Suppose that $F\colon X \multimap X$ has a collector $B \subseteq X$. Define $F|_B \cap B\colon X \multimap X$ by*

$$(F|_B \cap B)(x) := \begin{cases} F(x) \cap B & \text{if } x \in B, \\ \emptyset & \text{if } x \in X \setminus B. \end{cases} \tag{8.1}$$

Then the following holds for $M \subseteq X$.

(1) If M is a uniform attractor for $F|_B \cap B$ then M is an attractor for F.
(2) If M is a uniform $\mathscr{U}$-attractor for $F|_B \cap B$ then M is an $\mathscr{U}$-attractor for F.
(3) If M is an attractor for $F|_B \cap B$ and if F is occasionally finite then M is an attractor for F.
(4) If M is an $\mathscr{U}$-attractor for $F|_B \cap B$ and if F is occasionally finite then M is an $\mathscr{U}$-attractor for F.

Proof. Let $M \subseteq X$ be a (uniform) ($\mathscr{U}$-)attractor for $F|_B$, and let $O := U(M)$ for some $U \in \mathscr{U}$ (or let $O \supseteq M$ be open, respectively). Let $x \in X$ be given. We are to show that there is some index n_0 with $F^n(x) \subseteq O$ for all $n \geq n_0$.

By assumption, we find some index n_1 with $F^n(x) \subseteq B$ for all $n \geq n_1$. We show now by induction on $k = 0, 1, 2, \ldots$ that

$$F^k(K) = (F|_B \cap B)^k(K) \subseteq B \qquad \text{for all } K \subseteq \bigcup_{n=n_1}^{\infty} F^n(x). \tag{8.2}$$

Indeed, since $B_0 := \bigcup_{n=n_1}^{\infty} F^n(x) \subseteq B$, we have for each $K \subseteq B_0$ that $id(K) = K \subseteq B_0 \subseteq B$, i.e. (8.2) holds with $k = 0$. Assume that (8.2) holds with some $k \geq 0$, and let $K \subseteq B_0$. By hypothesis, we have $F^k(K) = (F|_B \cap B)^k(K) \subseteq B$, and so

$$F^{k+1}(K) = F(F^k(K)) = F((F|_B \cap B)^k(K)) = (F|_B \cap B)((F|_B \cap B)^k(K)) = (F|_B \cap B)^{k+1}(K).$$

Hence, (8.2) holds with $k+1$ in place of k. This completes the proof of (8.2).

We consider first the case that M is a uniform ($\mathscr{U}$-)attractor for $F|_B \cap B$. In this case, we find some index n_2 with $(F|_B \cap B)^k(X) \subseteq O$ for all $k \geq n_2$. Using the shortcut $K := F^{n_1}(x)$, we obtain in view of (8.2) that

$$F^{n_1+k}(x) = F^k(K) = (F|_B \cap B)^k(K) \subseteq (F|_B \cap B)^k(X) \subseteq O \qquad (n \geq n_2),$$

i.e. $F^n(x) \subseteq O$ for all $n \geq n_0 := n_1 + n_2$.

Now we assume that F is eventually finite. In this case, we find some index $n_1' \geq n_1$ such that $K := F^{n_1'}(x)$ is finite or empty. Since M is a ($\mathscr{U}$-)attractor for $F|_B \cap B$ and since K is finite, there is some index n_2 with $(F|_B \cap B)^k(y) \subseteq O$ for each $k \geq n_2$ and each $y \in K$. Hence, (8.2) implies

$$F^{n_1'+k}(x) = F^k(K) = (F|_B \cap B)^k(K) \subseteq O \qquad (k \geq n_2),$$

i.e. $F^n(x) \subseteq O$ for all $n \geq n_0' := n_1' + n_2$. □

Combining Theorems 7.20 and 8.5, we obtain:

Theorem 8.6. *Let* $(X,\mathscr{U})$ *be a complete uniform space, and let* $F\colon X \multimap X$*. Suppose that there is a collector* $B \subseteq X$ *for* F *such that the restriction* $F|_B \cap B\colon X \multimap X$ *defined by* (8.1) *satisfies at least one of the following properties for some* $Y \subseteq X$*.*

1. *(1)* $F|_B \cap B$ *is condensifying with respect to* Y*.*
2. *(2)* $\lim_{n\to\infty} \chi_Y((F|_B \cap B)^n(B)) = 0$.
3. *(3)* $F|_B \cap B$ *is strictly monotonically (countably)* Γ*-condensing (and of countable type) where* $\Gamma \succeq_{F|_B\cap B} \mathscr{H}_Y$ *(or* $\Gamma \succeq^c_{F|_B\cap B} \mathscr{H}_Y$*) is such that for each* $\gamma \in \Gamma$ *there is some* n_γ *with* $\gamma((F|_B \cap B)^{n_\gamma}(X)) < \infty$ *(or* $\gamma(C) < \infty$ *for each countable* $C \subseteq (F|_B \cap B)^{n_\gamma}(X)$*, respectively).*
4. *(4)* $F|_B \cap B$ *is countably condensing with respect to a countable family* $\Gamma \subseteq \Gamma_Y$ *which is admissible for* $F|_B \cap B$*.*

If $\overline{B}$ *is complete, then* F *has a compact attractor in* $\overline{B}$*. If the restriction of* $\mathscr{U}$ *to* B *is induced by a q-pseudometric, then* F *has a countable precompact* $\mathscr{U}$*-attractor in* Y*.*

If $\mathscr{U}$ does not have a countable base (i.e. if it is not induced by a q-pseudometric), we assume the axiom of choice (or at least PI) for the first claim in Theorem 8.6. Without this axiom, we claim only that F has a precompact complete $\mathscr{U}$-attractor in $\overline{B}$ (recall Proposition 3.6).

Proof. Theorem 7.20 implies that $F|_B \cap B\colon \overline{B} \multimap \overline{B}$ has a compact uniform attractor $M \subseteq \overline{B}$ or that $F|_B \cap B\colon X \multimap X$ has a countable precompact uniform $\mathscr{U}$-attractor in Y, respectively. In both cases, M is a uniform ($\mathscr{U}$-)attractor for $F|_B \cap B\colon X \multimap X$, and so the claim follows from Theorem 8.5. ☐

We point out once more that for multivalued maps we were only able to apply Theorem 8.5 in the previous proof because we proved in Theorem 7.20 that the attractor is uniform.

REFERENCES

[1] J. Andres, G. Gabor, L. Górniewicz, Acyclicity of solution sets to functional inclusions, Nonlinear Anal. 49 (2002), 671-688.

[2] J. Andres, L. Górniewicz, Topological fixed point principles for boundary value problems, Kluwer, Dordrecht, 2003.

[3] J. Andres, L. Górniewicz, Periodic solutions of dissipative systems revisited, Fixed Point Theory and Applications 2006 (2006), Article ID 65195, 12 pages.

[4] J. Appell, M. Väth, A. Vignoli, Compactness and existence results for ordinary differential equations in Banach spaces, Z. Anal. Anwendungen 18 (1999), no. 3, 569-584.

[5] N. Arronszajn, Le correspondant topologique de l'unicité dans la théorie des Équations differentielles, Ann. Math. 43 (1942), no. 4, 326-350.

[6] G. Darbo, Punti uniti in trasformazioni a codominio non compatto, Rend. Sem. Mat. Univ. Padova 24 (1955), 84-92.

[7] Z. Dzedzej, Fixed point index theory for a class of nonacyclic multivalued mappings, Dissertationes Math. (Rozprawy Mat.), no. 253, Polish Scientific Publ., Warszawa, 1985.

[8] P. Fletcher, W.F. Lindgren, Quasi-uniform spaces, Lecture Notes in Pure and Applied Mathematics, vol. 77, Marcel Dekker, New York, Basel, 1982.

[9] G. Fournier, D. Violette, A fixed point index for compositions of acyclic multivalued maps in Banach spaces, Operator Equations and Fixed Point Theorems (Singh, S. P., Sehgal, V. M., and Burry, J. H. W., eds.), vol. 1, The MSRI-Korea Publications, 1986, 203-224.

[10] L. Górniewicz, Homological methods in fixed-point theory of multi-valued maps, Dissertationes Math. (Rozprawy Mat.), no. 129, Polish Scientific Publ., Warszawa, 1976.

[11] L. Górniewicz, Topological fixed point theory of multivalued mappings, Kluwer, Dordrecht, 1999.

[12] G. Haddad, Topological properties of the set of solutions for functional differential inclusions, Nonlinear Anal. 5 (1981), no. 12, 1349-1366.

[13] J.D. Halpern, The independence of the axiom of choice from the Boolean prime ideal theorem, Fund. Math. 55 (1964), 57-66.

[14] H.-P. Heinz, On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions, Nonlinear Anal. 7 (1983), no. 12, 1351-1371.

[15] L. Janos, M. Martelli, Sequentially condensing maps, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 16 (1986), no. 1, 85-94.

[16] M.I. Kamenskiĭ, V.V. Obukhovskiĭ, P. Zecca, Condensing multivalued maps and semilinear differential inclusions in Banach spaces, de Gruyter, Berlin, 2001.

[17] M.A. Krasnoselskiĭ, P.P. Zabreĭko, P. P., Geometrical methods of nonlinear analysis (in Russian), Nauka, Moscow, 1975, Engl. transl.: Springer, New York 1984.

[18] M. Kunze, G. Schlüchtermann, Strongly generated Banach spaces and measures of noncompactness, Math. Nachr. 191 (1998), 197-214.

[19] C. Kuratowski, Sur les espaces complètes, Fund. Math. 15 (1930), 301-309.

[20] Lasry, J. H. and Robert, R., Acycliceté de l'ensemble des solutions de certaines Équations fonctionelles, C. R. Acad. Sci. 282 (1976), no. 22, A1283-A1286.

[21] Loś, J. and Ryll-Nardzewski, C., *Effectiveness of the representation theory for Boolean algebras*, Fund. Math. 41 (1954), 49-56.

[22] W.A.J. Luxemburg, Reduced powers of the real number system and equivalents of the Hahn-Banach extension theorem, (Intern. Symp. on the) Applications of Model Theory to Algebra, Analysis, and Probability (1967) (Toronto) (Luxemburg, W. A. J., ed.), Holt, Rinehart and Winston, 1969, 123-137.

[23] H. Mönch, H. and G.-F. von Harten, On the Cauchy problem for ordinary differential equations in Banach spaces, Arch. Math. (Basel) 39 (1982), 153-160.

[24] R.D. Nussbaum, Some asymptotic fixed point theorems, Trans. Amer. Math. Soc. 171 (1972), 349-375.

[25] R.D. Nussbaum, Generalizing the fixed point index, Math. Ann. 228 (1977), 259-278.

[26] Siegberg, H. W. and Skordev, G., Fixed point index and chain approximations, Pacific J. Math. 102 (1982), 455-486.

[27] M. Väth, Fixed point theorems and fixed point index for countably condensing maps, Topol. Methods Nonlinear Anal. 13 (1999), no. 2, 341-363.

[28] D. Violette, G. Fournier, Un indice de point fixe pour les composées de fonctions multivoques acycliques dans des espaces de Banach, Ann. Sci. Math. Québec 22 (1998), no. 2, 225-244.

[29] T. Yoshizawa, Stability theory and the existence of periodic solutions and almost periodic solutions, Springer, Berlin, 1975.