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H_∞ CONTROL OF SINGULARLY PERTURBED TIME DELAY SYSTEMS: THE SEPARATION OF MOTIONS APPROACH

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Abstract. An infinite horizon H_∞ state-feedback control problem for singularly perturbed linear systems with multiple pointwise and distributed small state delays in the dynamics is considered. This problem is solved by the separation of motions method. Due to this method, the original H_∞ control problem is decomposed asymptotically into two much simpler parameter-free H_∞ control problems, the slow and fast ones. A solution (controller) of the original problem is obtained as a proper composition of solutions of the slow and fast problems. Gain matrices of this controller are parameter-free, and it solves the original problem for all sufficiently small values of the parameter of singular perturbation. An illustrative example is presented.

Keywords. H_∞ control problem; Time-delay system; Singular perturbation; Separation of motions; Slow H_∞ control problem; Fast H_∞ control problem; Composite controller.

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1. Introduction

Dynamics of many real-life systems has a multi-time-scale (mainly, two-time-scale) structure. Such systems are modeled by singularly perturbed equations (see, e.g., [4, 10, 13, 15, 23] and references therein). The infinite horizon H_∞ control problem for singularly perturbed systems without delays has been analyzed extensively in the literature. The following two main methods were used to solve this problem: (I) the separation of motions (the separation of time scales) method (see, e.g., [1, 14, 17, 18, 21, 22, 24, 25] and references therein); (II) the method based on the asymptotic analysis of the game theoretic matrix Riccati equation associated with the original problem by the sufficient conditions for the existence of its solution (see, e.g., [1, 2, 12, 17, 18, 22] and references therein).

It should be noted that in contrast with a large number of works, devoted to studying the infinite horizon H_∞ control problem for singularly perturbed undelayed systems, there exist (to the best of our knowledge) only few works devoted to investigation of such a problem for singularly perturbed time delay systems (see [3, 7, 8, 9, 11]). Thus, in [3, 9, 11], the case where the dynamic system has only a single point-wise small state delay of order of a small parameter

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$\varepsilon > 0$, multiplying a part of its derivatives, was considered. It should be noted that the case of the small delay arises in various applications (see, e.g., [10, 16, 19, 20] and references therein).

In [3], the LMI method was applied to solve the considered problem. In [11], the solution of the original H_∞ problem was derived by the reduction of this problem to the singularly perturbed set of three Riccati-type matrix equation (algebraic and ordinary and partial differential ones). The zero-order asymptotic solution of this set of equations was formally constructed and justified. Using this asymptotic solution, ε-free sufficient conditions for the existence of solution to the original H_∞ control problem, valid for all sufficiently small values of ε, were established. The simplified controller with ε-free gain matrices, solving the original H_∞ problem for all sufficiently small values of ε, was designed. In the short conference paper [9], the infinite horizon H_∞ control problem for singularly perturbed linear systems with multiple point-wise and distributed small delays in the state variable was considered. This problem was solved by its asymptotic slow-fast decomposition (the separation of motions) approach. This approach yields two much simpler parameter-free H_∞ control problems, the slow and fast ones. To solve each of these two problems, the Riccati Matrix Inequality/Equation method was applied. The controller, solving the original problem, was design as the composition of the controllers solving the slow and fast problems. Gain matrices of the composite controller are parameter-free, and it solves the original problem for all sufficiently small values of the parameter of singular perturbations. Since the paper [9] is a short conference one, it presents the results in a brief form and without proofs of the assertions. Another type of singular perturbation in an infinite horizon H_∞ control problem with nonsmall state delays in the dynamics, which is generated by the cheap control in the cost functional, was studied in [7, 8].

In the present paper, we consider (similarly to [9]) the infinite horizon H_∞ control problem for singularly perturbed linear systems with multiple point-wise and distributed small delays in the state variable. However, in contrast with [9], here we present a complete analysis and solution of the considered problem by the separation of motions approach with rigorous proofs of all the assertions. A more general illustrative example also is presented.

The paper is organised as follows. In the next (second) section, the original H_∞ control problem is rigorously formulated. Its asymptotic decomposition into the slow and fast H_∞ control problems is carried out. The objectives of the paper are presented. In Sections 3 and 4, the solutions of the fast and slow H_∞ control problems, respectively, are derived. Composite controller, solving the original H_∞ control problem, is designed in Section 5. In Section 6, the illustrative example is solved. Conclusions are placed in Section 7.

The following main notations are applied in the paper:

(1) E^n denotes the n-dimensional real Euclidean space.
(2) $L_2[a,b;E^n]$ denotes the space of n-dimensional real vector-valued functions quadratically integrable in the interval (a,b).
(3) $\|\cdot\|_{L_2(a,b)}$ denotes the norm in $L_2[a,b;E^n]$.
(4) $\mathrm{col}(x,y)$, where $x \in E^n$ and $y \in E^m$, denotes the column block-vector with the upper block x and the lower block y.
(5) The upper index "T" denotes the transposition either of a vector x (x^T) or of a matrix A (A^T).
(6) I_n denotes the n-dimensional identity matrix.

(7) The inequality $A \geq (>) B$, where A and B are symmetric matrices of the same dimension, implies that the matrix $A - B$ is positive semidefinite (positive definite).

2. Problem Statement

2.1. Original H_∞ Problem. Consider the system

$$E_\varepsilon \frac{dz(t)}{dt} = \sum_{j=0}^{N} A_j z(t - \varepsilon h_j) + \int_{-h}^{0} G(\eta) z(t + \varepsilon\eta) d\eta + Bu(t) + Fw(t), \quad t > 0, \tag{2.1}$$

$$v(t) = \mathrm{col}\{Cz(t), u(t)\}, \quad t > 0, \tag{2.2}$$

where for any $t \in [-h, +\infty)$, $z(t) \in E^{n+m}$; for any $t \in [0, +\infty)$, $u(t) \in E^r$, $(r \leq n+m)$, (u is a control); for any $t \in [0, +\infty)$, $w(t) \in E^q$, (w is a disturbance); for any $t \in [0, +\infty)$, $v(t) \in E^{p+r}$, (v is an output); $\varepsilon > 0$ is a small parameter ($\varepsilon << 1$); $0 = h_0 < h_1 < ... < h_N = h$ are given constants independent of ε; $N \geq 1$ is an integer; the vector z and the matrices E_ε, A_j, $(j = 0, 1, ..., N)$, $G(\eta)$, B, F, C have the block-form

$$z = \mathrm{col}(x, y), \quad x \in E^n, \quad y \in E^m, \tag{2.3}$$

and

$$E_\varepsilon = \begin{pmatrix} I_n & 0 \\ 0 & \varepsilon I_m \end{pmatrix}, \quad A_j = \begin{pmatrix} A_{j1} & A_{j2} \\ A_{j3} & A_{j4} \end{pmatrix}, \quad G(\eta) = \begin{pmatrix} G_1(\eta) & G_2(\eta) \\ G_3(\eta) & G_4(\eta) \end{pmatrix},$$
$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad C = (C_1, C_2). \tag{2.4}$$

Here, the blocks A_{j1} and $G_1(\eta)$ are of the dimension $n \times n$; the blocks A_{j4} and $G_4(\eta)$ are of the dimension $m \times m$; the blocks B_1 and B_2 are of the dimensions $n \times r$ and $m \times r$, respectively; the blocks F_1 and F_2 are of the dimensions $n \times q$ and $m \times q$, respectively; the blocks C_1 and C_2 are of the dimensions $p \times n$ and $p \times m$, respectively.

The matrix-valued function $G(\eta)$ is piece-wise continuous for $\eta \in [-h, 0]$.

Assuming that $u(t) \in L_2[0, +\infty; E^r]$ and $w(t) \in L_2[0, +\infty; E^q]$, we consider the cost functional

$$J(u, w) = \|v(t)\|^2_{L_2(0,+\infty)} - \gamma^2 \|w(t)\|^2_{L_2(0,+\infty)}, \tag{2.5}$$

where $\gamma > 0$ is a given constant.

Definition 2.1. The infinite horizon H_∞ control problem for a performance level γ is to design a controller $u^*[z(\cdot)](t)$ such that, for any $w(t) \in L_2[0, +\infty; E^q]$: (i) there exists the unique locally absolutely continuous solution $z^*(t; w(t))$ of the equation (2.1) with $u(t) = u^*[z(\cdot)](t)$ and the initial condition $z(t) = 0$, $t \leq 0$ in the interval $(0, +\infty)$; (ii) the following inequality is valid

$$J\Big(u^*[z^*(\cdot; w(t))](t), w(t)\Big) \leq 0. \tag{2.6}$$

In what follows, we call this problem the Original H_∞ Control Problem (OHCP).

Remark 2.2. Using (2.3) and (2.4), we can represent the OHCP in the following equivalent form:

$$\frac{dx(t)}{dt} = \sum_{j=0}^{N} \left[A_{j1}x(t-\varepsilon h_j) + A_{j2}y(t-\varepsilon h_j)\right] + \int_{-h}^{0} \left[G_1(\eta)x(t+\varepsilon\eta) + G_2(\eta)y(t+\varepsilon\eta)\right]d\eta$$
$$+B_1u(t) + F_1w(t), \quad t \geq 0, \tag{2.7}$$

$$\varepsilon\frac{dy(t)}{dt} = \sum_{j=0}^{N} \left[A_{j3}x(t-\varepsilon h_j) + A_{j4}y(t-\varepsilon h_j)\right] + \int_{-h}^{0} \left[G_3(\eta)x(t+\varepsilon\eta) + G_4(\eta)y(t+\varepsilon\eta)\right]d\eta$$
$$+B_2u(t) + F_2w(t), \quad t \geq 0, \tag{2.8}$$

$$J(u,w) = \left\|\mathrm{col}\big(C_1x(t) + C_2y(t), u(t)\big)\right\|^2_{L_2(0,+\infty)} - \gamma^2\|w(t)\|^2_{L_2(0,+\infty)}. \tag{2.9}$$

2.2. **Asymptotic Decomposition of the OHCP.** Consider the following two much simpler ε-free H_∞ control problems associated with the OHCP. Equation of dynamics and cost functional of the first problem are obtained from the OHCP by formal setting there $\varepsilon = 0$ and re-denoting z, u, w, v, J with z_s, u_s, w_s, v_s, J_s, respectively . Thus, we have

$$\frac{d[E_0z_s(t)]}{dt} = \bar{A}z_s(t) + Bu_s(t) + Fw_s(t), \quad E_0z_s(0) = 0, \tag{2.10}$$

$$J_s(u_s,w_s) = \|v_s(t)\|^2_{L_2(0,+\infty)} - \gamma^2\|w_s(t)\|^2_{L_2(0,+\infty)}, \quad v_s(t) = \mathrm{col}\{Cz_s(t), u_s(t)\}, \tag{2.11}$$

where for all $t \in [0,+\infty)$, $z_s(t) \in E^{n+m}$, $u_s(t) \in E^r$, $w_s(t) \in E^q$ and $v_s(t) \in E^{p+r}$; u_s is a control, w_s is a disturbance. v_s is an output;

$$E_0 = E_\varepsilon|_{\varepsilon=0} = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{A} = \sum_{j=0}^{N} A_j + \int_{-h}^{0} G(\eta)d\eta. \tag{2.12}$$

Note that, due to the block form of the matrix E_0, the system (2.10) is a descriptor (differential-algebraic) system.

Definition 2.3. The infinite horizon H_∞ control problem with the dynamics system (2.10) and the cost functional (2.11) for the performance level γ is to design a controller $u_s^*[z_s(\cdot)](t)$ such that, for any $w_s(t) \in L_2[0,+\infty;E^q]$: (i$_s$) there exists the unique solution $z_s^*(t;w_s(t))$ of the system (2.10) with $u_s(t) = u_s^*[z_s(\cdot)](t)$ and the initial condition $z_s(0) = 0$ in the interval $(0,+\infty)$; (ii$_s$) $E_0z_s^*(t;w_s(t))$ is a locally absolutely continuous function in the interval $(0,+\infty)$; (iii$_s$) the following inequality is valid

$$J_s\Big(u_s^*[z_s^*(\cdot;w_s(t))](t), w_s(t)\Big) \leq 0. \tag{2.13}$$

In what follows, we call this problem the Slow H_∞ Control Problem (SHCP).

The second problem, associated with the OHCP, is obtained from the equations (2.8) and (2.9) in the following two stages. First, the variable $x(\cdot)$ is removed from these equations. Second, the following transformation of variables is made in the resulting differential equation and cost functional:

$$t = \varepsilon\xi, \quad y(\varepsilon\xi) = y_f(\xi), \quad u(\varepsilon\xi) = u_f(\xi), \quad w(\varepsilon\xi) = w_f(\xi), \quad J\big(u(\varepsilon\xi), w(\varepsilon\xi)\big) = J_f(u_f, w_f),$$

yielding the dynamics equation

$$\frac{dy_f(\xi)}{d\xi} = \sum_{j=0}^{N} A_{j4} y_f(\xi - h_j) + \int_{-h}^{0} G_4(\eta) y_f(\xi + \eta) d\eta + B_2 u_f(\xi) + F_2 w_f(\xi), \quad \xi > 0,$$
$$y_f(\xi) = 0, \quad \xi \le 0, \tag{2.14}$$

and the cost functional

$$J_f(u_f, w_f) = \|v_f(\xi)\|^2_{L_2(0,+\infty)} - \gamma^2 \|w_f(\xi)\|^2_{L_2(0,+\infty)}, \quad v_f(\xi) = \mathrm{col}\{C_2 y_f(\xi), u_f(\xi)\}, \ \xi > 0, \tag{2.15}$$

where $y_f(\xi) \in E^m$, $u_f(\xi) \in E^r$, (u_f is a control), $w_f(\xi) \in E^q$, (w_f is a disturbance), $v_f(\zeta) \in E^{p+r}$, (v_f is an output).

Definition 2.4. The infinite horizon H_∞ control problem with the dynamics equation (2.14) and the cost functional (2.15) for the performance level γ is to design a controller $u_f^*[y_f(\cdot)](\xi)$ such that, for any $w_f(\xi) \in L_2[0,+\infty; E^q]$: (i$_f$) there exists the unique locally absolutely continuous solution $y_f^*(\xi; w_f(\xi))$ of the equation (2.14) with $u_f(\xi) = u_f^*[y_f(\cdot)](\xi)$ and the initial condition $y_f(\xi) = 0$, $\xi \le 0$ in the interval $(0,+\infty)$; (ii$_f$) the following inequality is valid

$$J_f\Big(u_f^*[y_f^*(\cdot; w_f(\xi))](\xi), w_f(\xi)\Big) \le 0. \tag{2.16}$$

In what follows, we call this problem the Fast H_∞ Control Problem (FHCP).

Comparing the SHCP and the FHCP with problem OHCP, we can conclude that the SHCP and the FHCP are much simpler than the OHCP. The SHCP is without delays. The FHCP is of a lower dimension. Moreover, these problems do not depend on ε.

2.3. **Objectives of the Paper.** The objectives of the present paper are the following: (a) to design the controllers $u_f^*[z_f(\cdot)](\xi)$ and $u_s^*[z_s(\cdot)](t)$ solving the FHCP and the SHCP; (b) based on these controllers, to construct a composite controller solving the OHCP for all sufficiently small $\varepsilon > 0$.

3. Solution of the Fast H_∞ Control Problem

Consider the following hybrid set of Riccati type matrix inequality and ordinary and partial differential equations for $m \times m$-matrices P, $Q(\eta)$ and $R(\eta,\chi)$ in the domain $\Omega = \{(\eta,\chi) : -h \le \eta \le 0, \ -h \le \chi \le 0\}$:

$$PA_{04} + A_{04}^T P + PS_3 P + Q(0) + Q^T(0) + D_3 \le 0, \tag{3.1}$$

$$\frac{dQ(\eta)}{d\eta} = (A_{04}^T + PS_3)Q(\eta) + \sum_{j=1}^{N-1} PA_{j4}\delta(\eta + h_j) + PG_4(\eta) + R(0,\eta), \tag{3.2}$$

$$\left(\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\chi}\right)R(\eta,\chi)=G_4^T(\eta)Q(\chi)+Q^T(\eta)G_4(\chi)$$
$$+\sum_{j=1}^{N-1}A_{j4}^TQ(\chi)\delta(\eta+h_j)+\sum_{k=1}^{N-1}Q^T(\eta)A_{k4}\delta(\chi+h_k)+Q^T(\eta)S_3Q(\chi), \tag{3.3}$$

where $\delta(\cdot)$ is the delta-function of Dirac,

$$S_3=\gamma^{-2}F_2F_2^T-B_2B_2^T,\quad D_3=C_2^TC_2. \tag{3.4}$$

The set of the equations (3.1)-(3.3) is subject to the boundary conditions

$$Q(-h)=PA_{N4},\;\; R(-h,\eta)=A_{N4}^TQ(\eta),\;\; R(\eta,-h)=Q^T(\eta)A_{N4}. \tag{3.5}$$

Let us assume:

A1. The problem (3.1)-(3.3), (3.5) has a solution $\{P,Q(\eta),R(\eta,\chi)\}$ in Ω such that: (I) $P=P^T$; (II) $Q(\eta)$ is a piecewise absolutely continuous matrix-valued function with breaks at $\eta=-h_j$, $(j=1,...,N-1)$, and $Q(-h_j+0)-Q(-h_j-0)=PA_{j4}$; (III) $R(\eta,\chi)=R^T(\chi,\eta)$; (IV) $R(\eta,\chi)$ is piecewise absolutely continuous in each argument in the domain Ω with breaks at $\eta=-h_j$, $(j=1,...,N-1)$, $(R(-h_j+0,\chi)-R(-h_j-0,\chi)=A_{j4}^TQ(\chi))$ and at $\chi=-h_k$, $(k=1,...,N-1)$, $(R(\eta,-h_k+0)-R(\eta,-h_k-0)=Q^T(\eta)A_{k4})$.

A2. All roots λ of the quasi-polynomial equation
$\det\left[\lambda I_m-A_{04}+B_2B_2^TP-\sum_{j=1}^{N}A_{j4}\exp(-\lambda h_j)-\int_{-h}^{0}[G_4(\eta)-B_2B_2^TQ(\eta)]\exp(\lambda\eta)d\eta\right]=0$
lie strictly inside the left-hand half-plane.

Lemma 3.1. *Let the assumptions A1 and A2 be valid. Then, the controller*

$$u_f^*[y_f(\cdot)](\xi)=-B_2^T\left[Py_f(\xi)+\int_{-h}^{0}Q(\eta)y_f(\xi+\eta)d\eta\right] \tag{3.6}$$

solves the FHCP.

Proof. For any $\xi\ge 0$, let us consider the Lyapunov-Krasovskii-like functional

$$V[y_{f,\xi},\xi]=y_f^T(\xi)Py_f(\xi)+2y_f^T(\xi)\int_{\xi-h}^{\xi}Q(\tau-\xi)y_f(\tau)d\tau$$
$$+\int_{\xi-h}^{\xi}\int_{\xi-h}^{\xi}y_f^T(\tau)R(\tau-\xi,\rho-\xi)y_f(\rho)d\tau d\rho, \tag{3.7}$$

where $y_{f,\xi}\overset{\triangle}{=}y_f(\theta)$, $\theta\in[\xi-h,\xi]$. Let $V^*(\xi;w_f)\overset{\triangle}{=}V\left[y_f^*(\xi;w_f(\xi)),\xi\right]$, where, for any given $w_f(\xi)\in L_2[0,+\infty;E^q]$, $y_f^*(\xi;w_f(\xi),\xi\in[0,+\infty)$ is the solution of the equation (2.14) generated by the controller $u_f(\xi)=u_f^*[y_f(\cdot)](\xi)$ (see the equation (3.6)) and subject to the initial condition $y_f(\xi)=0$, $\xi\le 0$. Due to the assumption A1 and the form of $u_f^*[y_f(\cdot)](\xi)$, the aforementioned solution of the equation (2.14) exists, is unique and locally absolutely continuous in the interval $[0,+\infty)$. Differentiating $V^*(\xi;w_f)$ with respect to ξ and using the assumption A1, we obtain after a routine algebra the following expression (in this expression for the sake of simplicity we omit the designation of the dependence of $y_f^*\big(\xi;w_f(\xi)\big)$ on $w_f(\xi)$):

$$\frac{dV^*(\xi;w_f)}{d\xi} = 2\left(\frac{dy_f^*(\xi)}{d\xi}\right)^T\left(Py_f^*(\xi)+\int_{\xi-h}^{\xi}Q(\tau-\xi)y_f^*(\tau)d\tau\right)$$
$$+2\big(y_f^*(\xi)\big)^T\Bigg(Q(0)y_f^*(\xi)-Q(-h)y_f^*(\xi-h)$$
$$-\int_{\xi-h}^{\xi}\frac{dQ(\eta)}{d\eta}\Big|_{\eta=\tau-\xi}y_f^*(\tau)d\tau\Bigg)+2\big(y_f^*(\xi)\big)^T\int_{\xi-h}^{\xi}R(0,\rho-\xi)y_f^*(\rho)d\rho$$
$$-2\big(y_f^*(\xi-h)\big)^T\int_{\xi-h}^{\xi}R(-h,\rho-\xi)y_f^*(\rho)d\rho$$
$$-\int_{\xi-h}^{\xi}\int_{\xi-h}^{\xi}\big(y_f^*(\tau)^T\big)\left(\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\chi}\right)R(\eta,\chi)\Big|_{\eta=\tau-\xi,\chi=\rho-\xi}y_f^*(\rho)d\tau d\rho,\quad \xi\geq 0. \tag{3.8}$$

Changing the variables $\eta=\tau-\xi$, $\chi=\rho-\xi$ in the integrals of the equation (3.8) and using the equation (2.14) with $u_f(\xi)=u_f^*[y_f(\cdot)](\xi)$, as well as the problem (3.1)-(3.3), (3.5), we can convert (3.8) to the following inequality:

$$\frac{dV^*(\xi;w_f)}{d\xi}\leq -\big(y_f^*(\xi)\big)^T\big(D_3+\gamma^{-2}PF_2F_2^TP+PB_2B_2^TP\big)y_f^*(\xi)$$
$$+2w_f^T(\xi)F_2^T\left(Py_f^*(\xi)+\int_{-h}^{0}Q(\eta)y_f^*(\xi+\eta)d\eta\right)$$
$$-2\big(y_f^*(\xi)\big)^T\big(\gamma^{-2}PF_2F_2^T+PB_2B_2^T\big)\int_{-h}^{0}Q(\eta)y_f^*(\xi+\eta)d\eta$$
$$-\int_{-h}^{0}\big(y_f^*(\xi+\eta)\big)^TQ^T(\eta)d\eta\big(\gamma^{-2}F_2F_2^T+B_2B_2^T\big)\int_{-h}^{0}Q(\chi)y_f^*(\xi+\chi)d\chi,\quad \xi\geq 0. \tag{3.9}$$

Finally, using (3.6) and the notation

$$w_f^*(\xi)\triangleq\gamma^{-2}F_2^T\left[Py_f^*(\xi)+\int_{-h}^{0}Q(\eta)y_f^*(\xi+\eta)d\eta\right],$$

we see that (3.9) can be rewritten as:

$$\frac{dV^*(\xi;w_f)}{d\xi}\leq -\big(y_f^*(\xi)\big)^TD_3y_f^*(\xi)-\gamma^2\big(w_f(\xi)-w_f^*(\xi)\big)^T\big(w_f(\xi)-w_f^*(\xi)\big)$$
$$+\gamma^2w_f^T(\xi)w_f(\xi)-\big(u_f^*[y_f^*(\cdot)](\xi)\big)^Tu_f^*[y_f^*(\cdot)](\xi),\quad \xi\geq 0. \tag{3.10}$$

Let, for any given $\xi_0\geq 0$, $y_{f,in}(\tau)$ be a given m-dimensional vector-valued continuous function in the interval $[\xi_0-h,\xi_0]$. Let $y_{f,0}(\xi)$, $\xi\geq\xi_0$ be the solution of the equation (2.14) generated by the controller $u_f(\xi)=u_f^*[y_f(\cdot)](\xi)$ (see the equation (3.6)), the disturbance $w_f(\xi)\equiv 0$ and the initial condition $y_f(\tau)=y_{f,in}(\tau)$, $\tau\in[\xi_0-h,\xi_0]$. Due to the assumption A2,

$$\lim_{\xi\to+\infty} y_{f,0}(\xi)=0. \tag{3.11}$$

Let $V_0(\xi)\stackrel{\triangle}{=}V\left[y_{f,0}(\xi);\xi\right]$, $\xi\ge\xi_0$. Then, quite similarly to the inequality (3.10), we obtain

$$\frac{dV_0(\xi)}{d\xi}\le -y_{f,0}^T(\xi)D_3y_{f,0}(\xi)-\gamma^2w_{f,0}^T(\xi)w_{f,0}(\xi)-\left(u_{f,0}^*(\xi)\right)^Tu_{f,0}^*(\xi),\quad \xi\ge\xi_0, \tag{3.12}$$

where

$$w_{f,0}(\xi)\stackrel{\triangle}{=}\gamma^{-2}F_2\left[Py_{f,0}(\xi)+\int_{-h}^{0}Q(\eta)y_{f,0}(\xi+\eta)d\eta\right],\quad u_{f,0}^*(\xi)\stackrel{\triangle}{=}u_f^*[y_{f,0}(\cdot)](\xi),\quad \xi\ge\xi_0.$$

The inequality (3.12) yields

$$-\frac{dV_0(\xi)}{d\xi}\ge 0,\quad \xi\ge\xi_0. \tag{3.13}$$

Integrating this inequality from $\xi=\xi_0$ to $+\infty$ and using the definition of $V_0(\xi)$ and the equations (3.7), (3.11), we have

$$\begin{aligned}&y_{f,0}^T(\xi_0)Py_{f,0}(\xi_0)+2y_f^T(\xi_0)\int_{\xi_0-h}^{\xi_0}Q(\tau-\xi_0)y_{f,0}(\tau)d\tau\\&+\int_{\xi_0-h}^{\xi_0}\int_{\xi_0-h}^{\xi_0}y_{f,0}^T(\tau)R(\tau-\xi_0,\rho-\xi_0)y_{f,0}(\rho)d\tau d\rho\ge 0\end{aligned}$$

implying the inequality

$$\begin{aligned}&y_{f,in}^T(\xi_0)Py_{f,in}(\xi_0)+2y_{f,in}^T(\xi_0)\int_{\xi_0-h}^{\xi_0}Q(\tau-\xi_0)y_{f,in}(\tau)d\tau\\&+\int_{\xi_0-h}^{\xi_0}\int_{\xi_0-h}^{\xi_0}y_{f,in}^T(\tau)R(\tau-\xi_0,\rho-\xi_0)y_{f,in}(\rho)d\tau d\rho\ge 0.\end{aligned} \tag{3.14}$$

Remember that in this inequality ξ_0 is any nonnegative number and $y_{f,in}(\tau)$ is any m-dimensional vector-valued continuous function in the interval $[\xi_0-h,\xi_0]$. Based on this observation, we continue to treat the inequality (3.10). Since $\gamma^2\left(w_f(\xi)-w_f^*(\xi)\right)^T\left(w_f(\xi)-w_f^*(\xi)\right)\ge 0$, the inequality (3.10) implies the inequality

$$\left(y_f^*(\xi)\right)^TD_3y_f^*(\xi)+\left(u_f^*[y_f^*(\cdot)](\xi)\right)^Tu_f^*[y_f^*(\cdot)](\xi)-\gamma^2w_f^T(\xi)w_f(\xi)\le-\frac{dV^*(\xi;w_f)}{d\xi},\quad \xi\ge 0.$$

Integrating the latter from $\xi = 0$ to any $\xi \geq 0$, using the equation (3.7) and taking into account that $y_f^*(\xi) = 0$, $\xi \leq 0$, we obtain

$$\int_0^\xi \Big[\big(y_f^*(\sigma)\big)^T D_3 y_f^*(\sigma) + \big(u_f^*[y_f^*(\cdot)](\sigma)\big)^T u_f^*[y_f^*(\cdot)](\sigma) - \gamma^2 w_f^T(\sigma) w_f(\sigma)\Big] d\sigma$$
$$\leq -\Big[\big(y_f^*(\xi)\big)^T P y_f^*(\xi) + 2\big(y_f^*(\xi)\big)^T \int_{\xi-h}^{\xi} Q(\tau-\xi) y_f^*(\tau) d\tau$$
$$+ \int_{\xi-h}^{\xi}\int_{\xi-h}^{\xi} \big(y_f^*(\tau)\big)^T R(\tau-\xi,\rho-\xi) y_f^*(\rho) d\tau d\rho\Big], \quad \xi \geq 0. \tag{3.15}$$

Now, taking into account that the function $y_f^*(\xi)$ is continuous for $\xi \geq -h$ and using the inequality (3.14), we have from inequality (3.15) that

$$\int_0^\xi \Big[\big(y_f^*(\sigma)\big)^T D_3 y_f^*(\sigma) + \big(u_f^*[y_f^*(\cdot)](\sigma)\big)^T u_f^*[y_f^*(\cdot)](\sigma) - \gamma^2 w_f^T(\sigma) w_f(\sigma)\Big] d\sigma \leq 0, \quad \xi \geq 0.$$

This inequality, along with the inclusion $w_f(\xi) \in L_2[0,+\infty; E^q]$, yields

$$\int_0^\xi \Big[\big(y_f^*(\sigma)\big)^T D_3 y_f^*(\sigma) + \big(u_f^*[y_f^*(\cdot)](\sigma)\big)^T u_f^*[y_f^*(\cdot)](\sigma)\Big] d\sigma \leq \|w_f(\xi)\|^2_{L_2(0,+\infty)}, \quad \xi \geq 0. \tag{3.16}$$

Since the matrix D_3 is symmetric and positive semidefinite, the integral in the left-hand side of the inequality (3.16) is a non-decreasing function of $\xi \geq 0$ and this function is bounded from above. Therefore, this integral has a finite limit for $\xi \to +\infty$. This observation, along with the equation (2.15) and the inequality (3.16), directly yields the inequality (2.16), where $u_f^*[y_f(\cdot)](\xi)$ is given by the equation (3.6). Thus, the lemma is proven. □

The following assertions present some useful properties of the solution to the problem (3.1)-(3.3), (3.5).

Proposition 3.2. *Let the assumption A1 be valid. Then the following equality holds:*

$$\int_{-h}^0 R(0,\eta) d\eta + \int_{-h}^0 R(\eta,0) d\eta = \left(\sum_{j=1}^N A_{j4}^T + \int_{-h}^0 G_4^T(\eta) d\eta\right)^T \int_{-h}^0 Q(\eta) d\eta$$
$$+ \int_{-h}^0 Q^T(\eta) d\eta \left(\sum_{j=1}^N A_{j4} + \int_{-h}^0 G_4(\eta) d\eta\right) + \int_{-h}^0 Q^T(\eta) d\eta S_3 \int_{-h}^0 Q(\eta) d\eta. \tag{3.17}$$

Proof. The proposition is proved by the consecutive integration of the equation (3.3) with respect to η (from $-h$ to 0) and with respect to χ (from $-h$ to 0), and using the boundary conditions for $R(\eta,\chi)$ given in (3.5). □

Consider the matrix

$$P_Q \triangleq P + \int_{-h}^0 Q(\eta) d\eta. \tag{3.18}$$

Lemma 3.3. *Let the assumption A1 be valid. Then, the matrix P_Q satisfies the Riccati matrix inequality*

$$P_Q^T\bar{A}_4+\bar{A}_4^TP_Q+P_Q^TS_3P_Q+D_3\le 0, \tag{3.19}$$

where $\bar{A}_4$ is the lower right-hand block of the matrix $\bar{A}$ of the dimension $m\times m$.

Proof. Integrating the equation (3.2) from $\eta=-h$ to $\eta=0$ and using the initial condition for $Q(\eta)$ given in (3.5), we have

$$Q(0)=P\left(\sum_{j=1}^{N}A_{j4}+\int_{-h}^{0}G_4(\eta)d\eta\right)+(A_{04}^T+PS_3)\int_{-h}^{0}Q(\eta)d\eta+\int_{-h}^{0}R(0,\eta)d\eta. \tag{3.20}$$

Substitution of (3.20) into (3.1) and use of the expressions for A_j, $(j=0,1,...,N)$, $G(\eta)$, $\bar{A}$ (see the equations (2.4),(2.12)) and the definition of $\bar{A}_4$ yield after a routine matrix algebra

$$\begin{aligned}P\bar{A}_4+\bar{A}_4^TP+PS_3P+(A_{04}^T+PS_3)\int_{-h}^{0}Q(\eta)d\eta+\int_{-h}^{0}Q^T(\eta)d\eta(A_{04}+S_3P)\\+\int_{-h}^{0}R(0,\eta)d\eta+\int_{-h}^{0}R(\eta,0)d\eta+D_3\le 0.\end{aligned} \tag{3.21}$$

Now, substituting (3.17) into (3.21) and using the expression for P_Q (see the equation (3.18)), we obtain after a routine matrix rearrangement in the left-hand side of the resulting inequality the statement of the lemma. □

The following assertion is a direct consequence of the assumptions A1 and A2.

Proposition 3.4. *Let the assumptions A1 and A2 be valid. Then, the matrix $\widetilde{M}\stackrel{\triangle}{=}\bar{A}_4-B_2B_2^TP_Q$ is nonsingular.*

4. Solution of the Slow H_∞ Control Problem

Consider the generalized Riccati matrix inequality with respect to the $n\times n$-matrix K

$$\Theta(K)\stackrel{\triangle}{=}K^T\bar{A}+\bar{A}^TK+K^TSK+D\le 0, \tag{4.1}$$

where

$$S=\gamma^{-2}FF^T-BB^T,\quad D=C^TC. \tag{4.2}$$

The inequality (4.1) is subject to the condition

$$E_0K=K^TE_0. \tag{4.3}$$

Based on a proper solution of (4.1),(4.3), we are going to construct the controller solving the SHCP. We seek the solution of (4.1),(4.3) in the block form

$$K=\begin{pmatrix}K_1 & 0\\ K_2 & K_3\end{pmatrix},\quad K_1^T=K_1, \tag{4.4}$$

where the blocks K_1 and K_3 are of the dimensions $n\times n$ and $m\times m$, respectively. Such a form of the matrix K satisfies the condition (4.3).

Substitution of (4.4) into (4.1) yields after a routine matrix algebra

$$\Theta(K)=\begin{pmatrix}\Theta_1 & \Theta_2\\ \Theta_2^T & \Theta_3\end{pmatrix}\le 0, \tag{4.5}$$

where

$$\begin{aligned}\Theta_1=\Theta_1\big(K_1,K_2\big)=K_1\bar{A}_1+\bar{A}_1^TK_1+K_2^T\bar{A}_3+\bar{A}_3^TK_2+K_1S_1K_1\\ +K_1S_2K_2+K_2^TS_2^TK_1+K_2^TS_3K_2+D_1,\end{aligned} \tag{4.6}$$

$$\Theta_2=\Theta_2\big(K_1,K_2,K_3\big)=K_1\bar{A}_2+K_2^T\bar{A}_4+\bar{A}_3^TK_3+K_1S_2K_3+K_2^TS_3K_3+D_2, \tag{4.7}$$

$$\Theta_3=\Theta_3(K_3)=K_3^T\bar{A}_4+\bar{A}_4^TK_3+K_3^TS_3K_3+D_3, \tag{4.8}$$

$\bar{A}_1,\bar{A}_2$ and $\bar{A}_3$ are the upper left-hand, upper right-hand and lower left-hand blocks of the matrix $\bar{A}$ of the dimensions $n\times n$, $n\times m$ and $m\times n$, respectively; $S_i=\gamma^{-2}F_1F_i^T-B_1B_i^T$, $D_i=C_1^TC_i$, $(i=1,2)$; the matrices S_3 and D_3 are given in the equation (3.4).

Remark 4.1. Note, that the right-hand side of the equation (4.8) with $K_3=P_Q$ coincides with the left-hand side of the inequality (3.19), meaning that $\Theta_3(P_Q)\le 0$.

Based on Remark 4.1, let us assume
A3. Matrix inequality (4.5)-(4.8) has a solution $\{K_1,K_2,K_3\}$ such that $K_3=P_Q$.
Consider the following matrices:

$$\widetilde{A}=\bar{A}_1-L_1\bar{A}_3+B_1B_2^TL_2^T-L_1B_2B_2^TL_2^T,\quad \widetilde{S}=\widetilde{B}\widetilde{B}^T, \tag{4.9}$$

where

$$L_1=(\bar{A}_2-B_1B_2^TP_Q)\widetilde{M}^{-1},\quad L_2=-(K_1L_1+K_2^T),\quad \widetilde{B}=B_1-L_1B_2. \tag{4.10}$$

Let us assume
A4. The matrix $(\widetilde{A}-\widetilde{S}K_1)$ is a Hurwitz one.

Lemma 4.2. *Let the assumptions A1-A3 be valid. Then the system*

$$\frac{d[E_0\bar{z}(t)]}{dt}=(\bar{A}-BB^TK)\bar{z}(t),\quad t>0 \tag{4.11}$$

is asymptotically stable if and only if the assumption A4 is satisfied.

Proof. Let us represent the vector $\bar{z}(t)$ in the block form as $\bar{z}(t)=\mathrm{col}\big(\bar{x}(t),\bar{y}(t)\big)$, where $\bar{x}(t)\in E^n$, $\bar{y}(t)\in E^m$. Since $K_3=P_Q$ and the matrix $\widetilde{M}$ is invertible (see Proposition 3.4), we can rewrite the system (4.11) in the equivalent form

$$\frac{d\bar{x}(t)}{dt}=(\bar{A}_1-B_1B_1^TK_1-B_1B_2^TK_2)\bar{x}(t)+(\bar{A}_2-B_1B_2^TP_Q)\bar{y}(t),\quad t\ge 0, \tag{4.12}$$

$$\bar{y}(t)=-\widetilde{M}^{-1}(\bar{A}_3-B_2B_1^TK_1-B_2B_2^TK_2)\bar{x}(t),\quad t\ge 0. \tag{4.13}$$

Substitution of (4.13) into (4.12) and use of the equations (4.9)-(4.10) transform the equation (4.12) after a routine matrix algebra to the equation

$$\frac{d\bar{x}(t)}{dt}=(\widetilde{A}-\widetilde{S}K_1)\bar{x}(t),\quad t\ge 0. \tag{4.14}$$

This equation is asymptotically stable if and only if the assumption A4 is satisfied. Hence, the system (4.12)-(4.13) (and, therefore, the system (4.11)) is asymptotically stable if and only if the assumption A4 is satisfied. This completes the proof of the lemma. □

Lemma 4.3. *Let the assumptions A1-A4 be valid. Then, the controller*

$$u_s^*[z_s(t)] = -B^T K z_s(t) \tag{4.15}$$

solves the SHCP.

Proof. Based on Lemma 4.2, the lemma is proven similarly to Lemma 3.1 using the functional $V[z_s] = z_s^T E_0 K z_s$ and that $E_0 K$ is a symmetric matrix. □

5. Composite Controller for the Original H_∞ Control Problem

The composite controller design consists of two stages. At the first stage, for any $t \geq 0$, the following auxiliary vector-valued functional is constructed

$$u_{\rm a}^*[z(\cdot)](t) = u_s^*[Z(t)] + u_f^*[\widehat{y}(\cdot)](t/\varepsilon), \quad z = \mathrm{col}(x, y), \quad x \in E^n, \ \ y \in E^m, \tag{5.1}$$

where $u_s^*[\cdot]$ and $u_f^*[\cdot]$ are given by (4.15) and (3.6), respectively;

$$Z = \mathrm{col}(x, y_s), \quad \widehat{y}(t/\varepsilon) = y(t) - y_s(t), \quad y_s(t) = 0 \ \forall t < 0, \tag{5.2}$$

and $y_s(t)$ is the lower block of the state vector $z_s(t)$ (see the equation (2.10)) of the dimension m. Substituting (5.2) into (5.1) and using the equation (4.4), the assumption A3 and the equation (3.18), we obtain after a routine rearrangement

$$u_{\rm a}^*[z(\cdot)](t) = -[B_1^T K_1 + B_2^T K_2]x(t) - B_2^T \Big[P y(t) + \int_{-h}^{0} Q(\eta) y(t + \varepsilon\eta) d\eta \Big] \\ -B_2^T \int_{-h}^{0} Q(\eta)[y_s(t) - y_s(t + \varepsilon\eta)] d\eta. \tag{5.3}$$

Equation (5.3) means that $u_{\rm a}^*[z(\cdot)](t)$ cannot be used immediately as a controller for the OHCP, because it contains the state variable $y_s(\cdot)$ of the SHCP. At the second stage, we eliminate $y_s(\cdot)$ from (5.3) by the formal setting $\varepsilon = 0$ in the term depending on $y_s(\cdot)$. Such an elimination yields the composite controller for the OHCP

$$u_{\rm c}^*[z(\cdot)](t) = -[B_1^T K_1 + B_2^T K_2]x(t) - B_2^T \Big[P y(t) + \int_{-h}^{0} Q(\eta) y(t + \varepsilon\eta) d\eta \Big]. \tag{5.4}$$

To show that this controller solves the OHCP, we need two more assumptions and two more auxiliary assertion. Namely, let us assume

A5. All roots λ of the quasi-polynomial equation
$\det \Big[\lambda I_m - A_{04} - S_3 P - \sum_{j=1}^{N} A_{j4} \exp(-\lambda h_j) - \int_{-h}^{0} [G_4(\eta) + S_3 Q(\eta)] \exp(\lambda\eta) d\eta \Big] = 0$
lie strictly inside the left-hand half-plane.

The following assertion is a direct consequence of the assumptions A1 and A5.

Proposition 5.1. *Let the assumptions A1 and A5 be valid. Then, the matrix* $\widehat{M} \stackrel{\triangle}{=} \bar{A}_4 + S_3 P_Q$ *is nonsingular.*

Consider the following matrices:

$$\widehat{A} \stackrel{\triangle}{=} \bar{A}_1 - N_1\bar{A}_3 - S_2N_2^T + N_1S_3N_2^T, \quad \widehat{S} \stackrel{\triangle}{=} \gamma^{-2}\widehat{F}\widehat{F}^T - \widehat{B}\widehat{B}^T, \tag{5.5}$$

where

$$N_1 \stackrel{\triangle}{=} (\bar{A}_2 + S_2P_Q)\widehat{M}^{-1}, \quad N_2 = -(K_1N_1 + K_2^T),$$
$$\widehat{F} \stackrel{\triangle}{=} F_1 - N_1F_2, \quad \widehat{B} \stackrel{\triangle}{=} B_1 - N_1B_2. \tag{5.6}$$

Let us assume

A6. The matrix $(\widehat{A} + \widehat{S}K_1)$ is a Hurwitz one.

Similarly to Lemma 4.2 (see also the results of [11] (Lemma 3.5)), we have the following assertion.

Proposition 5.2. *Let the assumptions A1, A3 and A5 be valid. Then, the system*

$$\frac{d[E_0\bar{z}(t)]}{dt} = (\bar{A} + SK)\bar{z}(t), \quad t > 0 \tag{5.7}$$

is asymptotically stable if and only if the assumption A6 is satisfied.

Theorem 5.3. *Let the assumptions A1-A6 be valid. Then, for all sufficiently small $\varepsilon > 0$, the controller (5.4) solves the OHCP.*

Proof. First of all, let us note the following. Since the composite control $u_c^*[z(\cdot)](t)$ is linear with respect to $x(t)$ and $y(t+\varepsilon\eta)$, $\eta \in [-h, 0]$, then, for any disturbance $w(t) \in L_2[0,+\infty; E^q]$, the system (2.1) with $u(t) = u_c^*[z(\cdot)](t)$ and zero initial condition has the unique locally absolutely continuous solution $z_c^*\big(t; w(t)\big)$, $t \in (0,+\infty)$. Thus, due to Definition 2.1, to prove the theorem, we should prove the inequality (2.6) with $u^* = u_c^*$ and $z^* = z_c^*$.

Consider the following $(n+m)\times r$-matrices:

$$\mathscr{K}_z \stackrel{\triangle}{=} (B_1^TK_1 + B_2^TK_2, B_2^TP), \quad \mathscr{V}_z(\eta) \stackrel{\triangle}{=} (0, B_2^TQ(\eta)). \tag{5.8}$$

Using these matrices, we can rewrite the composite controller, given by (5.4), in the form

$$u_c^*[z(\cdot)](t) = -\mathscr{K}_z z(t) - \int_{-h}^{0}\mathscr{V}_z(\eta)z(t+\varepsilon\eta)d\eta, \quad z = \mathrm{col}(x,y). \tag{5.9}$$

Substituting $u(t) = u_c^*[z(\cdot)](t)$ into system (2.1)-(2.2) and cost functional (2.5), we obtain the following differential system and cost functional:

$$E_\varepsilon\frac{dz(t)}{dt} = \mathscr{A}_0 z(t) + \sum_{j=1}^{N} A_j z(t-\varepsilon h_j) + \int_{-h}^{0}\mathscr{G}(\eta)z(t+\varepsilon\eta)d\eta + Fw(t), \tag{5.10}$$

$$\mathscr{J}(w) = J(u_c^*, w) = \int_0^{+\infty}\Big[z^T(t)(D+\mathscr{D}_P)z(t) + 2z^T(t)\int_{-h}^{0}\mathscr{D}_Q(\eta)z(t+\varepsilon\eta)d\eta$$
$$+ \int_{-h}^{0}\int_{-h}^{0} z^T(t+\varepsilon\eta)\mathscr{D}_R(\eta,\chi)z(t+\varepsilon\chi)d\eta d\chi - \gamma^2 w^T(t)w(t)\Big]dt, \tag{5.11}$$

where

$$\mathscr{A}_0 = A_0 - B\mathscr{K}_z, \quad \mathscr{G}(\eta) = G(\eta) - B\mathscr{V}_z(\eta), \quad \mathscr{D}_P = \mathscr{K}_z^T \mathscr{K}_z,$$
$$\mathscr{D}_Q(\eta) = \mathscr{K}_z^T \mathscr{V}_z(\eta), \quad \mathscr{D}_R(\eta,\chi) = \mathscr{V}_z^T(\eta)\mathscr{V}_z(\chi). \tag{5.12}$$

Thus, the fulfillment of the inequality (2.6) with $u^* = u_c^*$ and $z^* = z_c^*$ is a direct consequence of the fulfillment of the inequality

$$\mathscr{J}(w) \le 0 \quad \forall w(t) \in L_2[0,+\infty;E^q] \tag{5.13}$$

along trajectories of the equation (5.10) subject to the initial condition $z(t) = 0$, $t \le 0$.

To prove the inequality (5.13), first, we are going to show the asymptotic stability of the system (5.10) for $w(t) \equiv 0$ and all sufficiently small $\varepsilon > 0$. Decomposing asymptotically this singularly perturbed differential system, we directly obtain that its slow subsystem coincides with the descriptor (differential-algebraic) system (4.11), while its fast subsystem coincides with the differential equation (2.14) with $u_f(\xi) = u_f^*\big[y_f(\cdot)\big]$ (see the equation (3.6)) and $w_f(\xi) \equiv 0$. Due to Lemma 4.2 and the assumption A2, both subsystems are asymptotically stable. Therefore, by virtue of the results of [5], the system (5.10) with $w(t) \equiv 0$ is asymptotically stable for all sufficiently small $\varepsilon > 0$.

Now, let us consider the following hybrid set of Riccati type matrix inequality and ordinary and partial differential equations for $\mathscr{P}$, $\mathscr{Q}(\tau)$ and $\mathscr{R}(\tau,\rho)$ in the domain $\Omega_\varepsilon = \{(\tau,\rho) : -\varepsilon h \le \tau \le 0,\ -\varepsilon h \le \rho \le 0\}$

$$\mathscr{P}E_\varepsilon^{-1}\mathscr{A} + \mathscr{A}^T E_\varepsilon^{-1}\mathscr{P} + \mathscr{P}\mathscr{S}_\varepsilon\mathscr{P} + \mathscr{Q}(0) + \mathscr{Q}^T(0) + D + \mathscr{D}_P \le 0, \tag{5.14}$$

$$\frac{d\mathscr{Q}(\tau)}{d\tau} = (\mathscr{A}_0^T E_\varepsilon^{-1} + \mathscr{P}\mathscr{S}_\varepsilon)\mathscr{Q}(\tau) + \sum_{j=1}^{N-1} \mathscr{P}E_\varepsilon^{-1}A_j\delta(\tau+\varepsilon h_j) + (1/\varepsilon)\mathscr{P}E_\varepsilon^{-1}\mathscr{G}(\tau/\varepsilon)$$
$$+\mathscr{R}(0,\tau) + (1/\varepsilon)\mathscr{D}_Q(\tau/\varepsilon), \quad \mathscr{Q}(-\varepsilon h) = \mathscr{P}E_\varepsilon^{-1}A_N, \tag{5.15}$$

$$\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\rho}\right)\mathscr{R}(\tau,\rho) = (1/\varepsilon)\mathscr{G}^T(\tau/\varepsilon)E_\varepsilon^{-1}\mathscr{Q}(\rho) + (1/\varepsilon)\mathscr{Q}^T(\tau)E_\varepsilon^{-1}\mathscr{G}(\rho/\varepsilon)$$
$$+\sum_{j=1}^{N-1} A_j^T E_\varepsilon^{-1}\mathscr{Q}(\rho)\delta(\tau+\varepsilon h_j) + \sum_{j=1}^{N-1}\mathscr{Q}^T(\tau)E_\varepsilon^{-1}A_j\delta(\rho+\varepsilon h_j) + \mathscr{Q}^T(\tau)\mathscr{S}_\varepsilon\mathscr{Q}(\rho)$$
$$+(1/\varepsilon^2)\mathscr{D}_R(\tau/\varepsilon,\rho/\varepsilon), \quad \mathscr{R}(-\varepsilon h,\tau) = A_N^T E_\varepsilon^{-1}\mathscr{Q}(\tau), \quad \mathscr{R}(\tau,-\varepsilon h) = \mathscr{Q}^T(\tau)E_\varepsilon^{-1}A_N, \tag{5.16}$$

where $\mathscr{S}_\varepsilon = \gamma^{-2}\mathscr{F}_\varepsilon\mathscr{F}_\varepsilon'$, $\mathscr{F}_\varepsilon = E_\varepsilon^{-1}F$. Let, for a given $\varepsilon > 0$, $\{\mathscr{P}_\varepsilon, \mathscr{Q}_\varepsilon(\tau), \mathscr{R}_\varepsilon(\tau,\rho)\}$ be a solution of the problem (5.14)-(5.16) such that

$$\mathscr{P}_\varepsilon^T = \mathscr{P}_\varepsilon, \quad \mathscr{R}_\varepsilon^T(\tau,\rho) = \mathscr{R}_\varepsilon(\rho,\tau). \tag{5.17}$$

Then, for this ε, the inequality (5.13) is proven similarly to Lemma 3.1 using the asymptotic stability of the system (5.10) with $w(t) \equiv 0$ and the Lyapunov-Krasovskii-like functional

$$\mathscr{W}[z_t,t] = z^T(t)\mathscr{P}_\varepsilon z(t) + 2z^T(t)\int_{t-\varepsilon h}^{t} \mathscr{Q}_\varepsilon(\kappa - t)z(\kappa)d\kappa$$
$$+ \int_{t-\varepsilon h}^{t}\int_{t-\varepsilon h}^{t} z^T(\kappa)R(\kappa - t, \zeta - t)z(\zeta)d\kappa d\zeta, \tag{5.18}$$

where $z_t \overset{\triangle}{=} z(\kappa)$, $\kappa \in [t-\varepsilon h, t]$. As it is aforementioned, the system (5.10) with $w(t) \equiv 0$ is asymptotically stable for all sufficiently small $\varepsilon > 0$. Therefore, to complete the proof of the theorem, we should show the existence of the solution $\{\mathscr{P}_\varepsilon, \mathscr{Q}_\varepsilon(\tau), \mathscr{R}_\varepsilon(\tau,\rho)\}$ to the problem (5.14)-(5.16) for all sufficiently small $\varepsilon > 0$. By $-D_0$, let us denote the left-hand side of the inequality (4.1), i.e. $D_0 = -\Theta(K)$ where K is the solution of (4.1) mentioned in the assumption A3. The matrix D_0 is symmetric and positive semidefinite. Thus, the matrix K, satisfying the inequality (4.1), also is a solution of the equation

$$K^T\bar{A} + \bar{A}^T K + K^T SK + D + D_0 = 0. \tag{5.19}$$

Moreover, using assumption A3, as well as the equations (3.17), (3.18), (3.20), and (4.8), one can show that the matrices P, $Q(\eta), R(\eta,\chi)$, satisfying (3.1)-(3.3), (3.5), also satisfy the problem consisting of the equation

$$PA_4 + A_4^T P + PS_3P + Q(0) + Q^T(0) + D_3 + D_{03} = 0, \tag{5.20}$$

as well as equations (3.2), (3.3) and the boundary conditions (3.5), where D_{03} is the lower right-hand block of the matrix D_0 of the dimension $m \times m$.

Consider the equation

$$\mathscr{P}E_\varepsilon^{-1}\mathscr{A} + \mathscr{A}^T E_\varepsilon^{-1}\mathscr{P} + \mathscr{P}\mathscr{S}_\varepsilon\mathscr{P} + \mathscr{Q}(0) + \mathscr{Q}^T(0) + D + \mathscr{D}_P + D_0 = 0. \tag{5.21}$$

It is clear that if, for any $\varepsilon > 0$, matrices $\mathscr{P}, \mathscr{Q}(\tau), \mathscr{R}(\tau,\rho)$ satisfy (5.15), (5.16), and (5.21), they also satisfy problem (5.14)-(5.16).

Now, using the results of the papers [6, 11], as well as the aforementioned relations between the inequality (4.1) and the equation (5.19) and between the problem (3.1)-(3.3), (3.5) and the problem (5.20), (3.2), (3.3), (3.5), one can show that, for all sufficiently small $\varepsilon > 0$, there exists the solution to the problem (5.15), (5.16), (5.21) satisfying the symmetry properties (5.17). Hence, the problem (5.14)-(5.16) also has the solution for these ε satisfying the symmetry properties. Thus, the theorem is proven. □

6. Example

Consider the example of the OHCP (see the equations (2.1)-(2.2), (2.5)) with the following data:

$$n = m = r = q = p = 1, \quad N = 1,$$
$$A_{01} = -4,\ A_{02} = -2,\ A_{03} = 1,\ A_{04} = -2,\ A_{11} = -2,\ A_{12} = 2,\ A_{13} = -1,\ A_{14} = -1,$$
$$B_1 = 1,\ B_2 = 1,\ F_1 = 1,\ F_2 = 0.5,\ C_1 = \sqrt{2},\ C_2 = \sqrt{2},\ \gamma = 0.5,$$
$$G_1(\eta) \equiv 0,\ G_2(\eta) \equiv 0,\ G_3(\eta) \equiv 0,\ G_4(\eta) \equiv 1. \tag{6.1}$$

The example of the OHCP with the data (6.1) will allow us to clearly illustrate the theoretical results of the paper, while avoiding too complicated analytical/numerical calculations.

Let us start with the solution of the FHCP (see the equations (2.14)-(2.15)) subject to the data (6.1). Due to the results of Section 3, to design the controller of the FHCP, we should solve the problem (3.1)-(3.3), (3.5). Subject to the data (6.1), this problem becomes as:

$$-4P + 2Q(0) + 2 \leq 0, \tag{6.2}$$

$$\frac{dQ(\eta)}{d\eta} = -2Q(\eta) + P + R(0,\eta), \quad Q(-h) = -P, \tag{6.3}$$

$$\left(\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \chi}\right) R(\eta,\chi) = Q(\chi) + Q(\eta), \quad R(-h,\eta) = R(\eta,-h) = -Q(\eta), \tag{6.4}$$

where $-h \leq \eta \leq 0$, $-h \leq \chi \leq 0$. Solving (6.4), we obtain

$$R(\eta,\chi) = \Phi(\eta-\chi) + \int_{\max\{\eta-\chi-h,h\}}^{\eta} \big[Q(s-\eta+\chi) + Q(s)\big]ds, \quad (\eta,\chi) \in [-h,0] \times [-h,0], \tag{6.5}$$

where

$$\Phi(\sigma) = -\begin{cases} Q(-\sigma-h), & -h \leq \sigma \leq 0, \\ Q(\sigma-h), & 0 \leq \sigma \leq h. \end{cases} \tag{6.6}$$

Using (6.5)-(6.6) and calculating $R(0,\eta)$, we have after a routine rearrangement

$$R(0,\eta) = -Q(-\eta-h) + \int_{-h}^{\eta} Q(s_1)ds_1 + \int_{-\eta-h}^{0} Q(s_2)ds_2, \quad \eta \in [-h,0]. \tag{6.7}$$

Substitution of (6.7) into (6.3) yields the following initial-value problem for the functional-differential equation:

$$\begin{aligned} \frac{dQ(\eta)}{d\eta} = -2Q(\eta) + P - Q(-\eta-h) + \int_{-h}^{\eta} Q(s_1)ds_1 + \int_{-\eta-h}^{0} Q(s_2)ds_2, \\ \eta \in [-h,0], \quad Q(-h) = -P. \end{aligned} \tag{6.8}$$

Solving this initial-value problem, we obtain

$$Q(\eta) = P\omega(\eta,h), \quad \eta \in [-h,0], \tag{6.9}$$

where

$$\begin{aligned} \omega(\eta,h) = \frac{g_1(h)\exp\big(\sqrt{5}(\eta+h)\big) - g_2(h)\exp\big(-\sqrt{5}(\eta+h)\big)}{g_0(h)}, \\ g_1(h) = 6 - 2\sqrt{5} + 2\exp(-\sqrt{5}h), \quad g_2(h) = 6 + 2\sqrt{5} + 2\exp(\sqrt{5}h), \\ g_0(h) = 4\sqrt{5} + 2\exp(\sqrt{5}h) - 2\exp(-\sqrt{5}h). \end{aligned} \tag{6.10}$$

From (6.10), we have

$$\omega(0,h) = \frac{g_1(h)\exp\big(\sqrt{5}h\big) - g_2(h)\exp\big(-\sqrt{5}h\big)}{g_0(h)}. \tag{6.11}$$

This value is an increasing function of $h \in [0,+\infty)$ and

$$-1 \le \omega(0,h) < \sup_{h\in[0,+\infty)} \omega(0,h) = 3-\sqrt{5} < 2. \tag{6.12}$$

Substituting (6.9) into (6.2), solving the resulting inequality with respect to P and taking into account (6.12), we obtain

$$P \ge \frac{1}{2-\omega(0,h)}, \quad h \in [0,+\infty). \tag{6.13}$$

Due to the equation (6.12), any P, satisfying the inequality

$$P \ge \frac{1}{\sqrt{5}-1}, \tag{6.14}$$

satisfies the inequality (6.13) for all $h \in [0,+\infty)$.

Now, let us show the validity of the assumptions A2 and A5. We start with the assumption A2. Subject to the data (6.1), the matrix appearing in this assumption becomes the following scalar function of λ:

$$\alpha_1(\lambda) = \lambda + 2 + P + \exp(-\lambda h) - \int_{-h}^{0} \big[1 - P\omega(\eta,h)\big] \exp(\lambda\eta)d\eta, \quad h \in [0,+\infty). \tag{6.15}$$

Using this equation, we have

$$\begin{aligned} \mathrm{Re}\big(\alpha_1(\lambda)\big) &= \mathrm{Re}(\lambda) + 2 + P + \mathrm{Re}\big(\exp(-\lambda h)\big) \\ &\quad - \int_{-h}^{0} \big[1 - P\omega(\eta,h)\big]\mathrm{Re}\big(\exp(\lambda\eta)\big)d\eta, \ h \in [0,+\infty). \end{aligned} \tag{6.16}$$

Let us observe that

$$\begin{aligned} &\big|\mathrm{Re}\big(\exp(-\lambda h)\big)\big| \le 1, \\ &\left|\int_{-h}^{0} \big[1 - P\omega(\eta,h)\big]\mathrm{Re}\big(\exp(\lambda\eta)\big)d\eta\right| \le \int_{-h}^{0} \Big(\max_{\eta\in[-h,0]} \big|1 - P\omega(\eta,h)\big|\Big)d\eta \le (1+P)h, \\ &\mathrm{Re}(\lambda) \ge 0, \quad h \in [0,+\infty). \end{aligned} \tag{6.17}$$

Using the equation (6.16) and the inequalities in (6.17), we obtain the inequality

$$\mathrm{Re}\big(\alpha_1(\lambda)\big) \ge \mathrm{Re}(\lambda) + 1 + P - (1+P)h, \quad \mathrm{Re}(\lambda) \ge 0, \quad h \in [0,+\infty). \tag{6.18}$$

This inequality directly shows that, for the validity of the assumption A2, it is sufficient that $1+P-(1+P)h > 0$, meaning that all values of h satisfying the inequality

$$0 \le h < 1 \tag{6.19}$$

provide the validity of the assumption A2. Proceed to the assumption A5. Subject to the data (6.1), the matrix appearing in this assumption becomes the following scalar function of λ:

$$\alpha_2(\lambda) = \lambda + 2 + \exp(-\lambda h) - \int_{-h}^{0} \exp(\lambda\eta)d\eta, \quad h \in [0,+\infty). \tag{6.20}$$

Due to this equation,

$$\mathrm{Re}\big(\alpha_2(\lambda)\big) = \mathrm{Re}(\lambda) + 2 + \mathrm{Re}\big(\exp(-\lambda h)\big) - \int_{-h}^{0} \mathrm{Re}\big(\exp(\lambda\eta)\big)d\eta, \ h \in [0,+\infty). \tag{6.21}$$

Using equation (6.21), the first inequality in (6.17), and $\left|\mathrm{Re}\big(\exp(\lambda\eta)\big)\right| \le 1$ for all $\mathrm{Re}(\lambda) \ge 0$, $\eta \in [-h,0]$, $h \in [0,+\infty)$, we obtain

$$\mathrm{Re}\big(\alpha_2(\lambda)\big) \ge \mathrm{Re}(\lambda) + 1 - h, \quad \mathrm{Re}(\lambda) \ge 0, \quad h \in [0,+\infty), \tag{6.22}$$

meaning that, for the validity of the assumption A5, it is sufficient that $1-h > 0$. Thus, the inequality (6.19), providing the validity of the assumption A2, provides also the validity of the assumption A5. Using Lemma 3.1 and equations (3.6), (6.1), and (6.9), we obtain the controller solving the FHCP for all $h \in [0,1)$

$$u_f^*[y_f(\cdot)](\xi) = -P\left[y_f(\xi) + \int_{-h}^{0} \omega(\eta,h) y_f(\xi+\eta)d\eta\right], \tag{6.23}$$

where P satisfies the inequality (6.13).

Now, proceed to solution of the SHCP (see the equations (2.10)-(2.11), (2.12)) subject to the data (6.1). Due to the results of Section 4, to design the controller of the SHCP, we should solve the matrix inequality (4.5), (4.6)-(4.8). Subject to the data (6.1), this matrix inequality is with respect of 2×2-matrix K of the form (4.4). The matrix $\Theta(K)$ in the left-hand side of (4.5) also is of the dimension 2×2 with the following entries:

$$\begin{aligned} \Theta_1 = \Theta_1\big(K_1,K_2\big) &= -12K_1 + 3K_1^2 + 2K_1K_2 + 2, \\ \Theta_2 = \Theta_2\big(K_1,K_2,K_3\big) &= (h-3)K_2 + K_1K_3 + 2, \\ \Theta_3 = \Theta_3(K_3) &= 2(h-3)K_3 + 2, \\ h &\in [0,1). \end{aligned} \tag{6.24}$$

The matrix inequality (4.5) with the entries of the matrix $\Theta(K)$, given in (6.24), has multiple solutions. Let us chose the following solution of this inequality:

$$K_1 = 1, \quad K_2 = \frac{3}{3-h}, \quad K_3 = 1, \quad h \in [0,1). \tag{6.25}$$

For this solution, $\Theta_1\big(K_1,K_2\big) < 0$, $\Theta_2\big(K_1,K_2,K_3\big) = 0$, $\Theta_3(K_3) < 0$, $h \in [0,1)$, i.e., for this solution, the matrix $\Theta(K)$ becomes negative definite. Once the solution of (4.5), (6.24) is obtained, we can equate K_3 and P_Q in order to satisfy the assumption A3. Thus, using equations (3.18), (6.9), and (6.25), we have

$$1 = \left(1 + \int_{-h}^{0} \omega(\eta,h)d\eta\right)P, \quad h \in [0,1), \tag{6.26}$$

yielding

$$P = \bar{P}(h) \stackrel{\triangle}{=} \frac{1}{1 + \int_{-h}^{0} \omega(\eta,h)d\eta}, \quad h \in [0,1). \tag{6.27}$$

By direct calculation, we have that $\min_{h\in[0,1]} \bar{P}(h) = 1$, meaning that $P = \bar{P}(h)$ satisfies the inequality (6.14). Therefore, $P = \bar{P}(h)$ is the component of the solution to the problem (6.2)-(6.4) for all $h \in [0,1)$. Now, let us show the validity of the assumptions A4 and A6. Using Propositions 3.4 and 5.1, as well as equations (4.9), (4.10), (5.5), (5.6), (6.1), (6.25), and $P_Q =$

$K_3 = 1$, we obtain

$$\widetilde{A} - \widetilde{S}K_1 = -6 + \frac{4h-15}{(4-h)^2} - \left(\frac{3-h}{4-h}\right)^2, \quad h \in [0,1),$$
$$\widehat{A} - \widehat{S}K_1 = \frac{3h-5}{3-h}, \quad h \in [0,1), \tag{6.28}$$

meaning that $\widetilde{A} - \widetilde{S}K_1 < 0$ and $\widehat{A} - \widehat{S}K_1 < 0$ for all $h \in [0,1)$. Thus, the asuumptions A4 and A6 are valid for these values of h. Using Lemma 4.3 and equations (4.15), (6.1), and (6.25), we obtain the controller solving the SHCP for all $h \in [0,1)$

$$u_s^*[z_s(t)] = -\left(\frac{6-h}{3-h}\ ,\ 1\right) z_s(t). \tag{6.29}$$

Finally, using Theorem 5.3, as well as equations (5.4), (6.1), (6.23), and (6.29), we obtain the composite controller solving the OHCP for all sufficiently small $\varepsilon > 0$

$$u_{\mathrm{c}}^*[z(\cdot)](t) = -\frac{6-h}{3-h}x(t) - P\Big[y(t) + \int_{-h}^{0} \omega(\eta,h) y(t+\varepsilon\eta) d\eta\Big].$$

7. Conclusions

In this paper, the infinite horizon state-feedback H_∞ control problem for singularly perturbed linear systems with state delays (multiple point-wise and distributed) was studied. The delays are small of order of the small parameter $\varepsilon > 0$ multiplying a part of the derivatives in the system. It was shown the applicability of the separation of motions method to solution of this problem. Namely, the original H_∞ problem was decomposed asymptotically into two much simpler ε-free H_∞ control problems, the slow and fast ones. Based on the solutions (controllers) of these problems, the composite controller, having ε-free gain matrices and solving the original problem for all sufficiently small $\varepsilon > 0$, was designed. This result is valid for both, standard and nonstandard, forms of the singularly perturbed system in the original H_∞ control problem.

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