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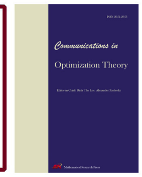

REAL OPTION VALUATION UNDER INCOMPLETE MARKETS VIA DYNAMIC PROGRAMMING

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Dedicated to Professor Alfredo N. Iusem on the occasion of his 75th birthday

Abstract. A real option value is obtained as a bid price, the maximum price an agent is willing to pay given underlying investment opportunities, such as financial or real investments. For evaluations, agent's multi-period utility function is employed in a multi-stage portfolio model. The bid price provides a unique real option value consistent with arbitrage pricing theory. Properties of exponential utility allow use of dynamic programming. Resulting real option values are independent of utility discounting factors. Under partially complete markets PCM, such results are well known. We generalize the results relaxing all market completeness assumptions. For further computational improvements, we also propose a locally complete market assumption LCM for which PCM is a special case. Such relaxation is important, if realizations of real option cash flow convey information on subsequent price processes of underlying assets; then PCM may be violated but LCM not. The approach is illustrated with examples in flexible manufacturing systems under PCM, LCM and without any completeness assumption. The results demonstrate vast computational savings in comparison with stochastic programming. Bid price principle is easy to explain to managers and spread sheet calculations suit for valuation.

Keywords. Asset pricing; Dynamic programming; Incomplete markets; Real options.

2020 Mathematics Subject Classification. 90C39, 91G10.

1. Introduction

Consider a firm evaluating a given real investment proposal. For example, the project proposal may concern expansion of the company's existing businesses in alternative ways concerning timing and choices of location; or the company may face a choice and timing of investments in a given set of potential technologies with uncertain prices of variable inputs and final outputs. As described by Trigeorgis [27] such cases of real options concern decisions on whether to defer, expand, contract, abandon, switch use, or otherwise alter a capital investment; see also Dixit and Pindyck [6] and Copeland and Antikarov [4]. In arbitrage pricing theory, valuation of

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a contingent claim is often based on a replicating portfolio[1] , and the option value is the largest value taking into account all contingent claims obtained from alternative ways of exercising. For such conventional approaches of real option valuation; see, e.g., Dixit and Pindyck [6] and Trigeorgis [27].

The literature on applications of real options is extensive and here we just mention some studies. Smith and Nau [25] propose an approach for real options valuation assuming both market and non-market uncertainties for the project under evaluation; Smith and McCardle [26] employ this dynamic programming approach for oil field valuation. Kamrad and Ernst [12] also consider market and private uncertainty for mining and manufacturing investments; the valuing is based on optimization of production rate over a binomial lattice of the market price of output. Meier, Christofides and Salkin [18] propose capital budgeting schemes for optimizing a project portfolio with deferable projects. The project valuation is based on replication with computations employing stochastic integer programming. Hilli, Kallio and Kallio [9] study optimal capital budgeting for a forest industry production technology portfolio. De Neufville and Scholtes [20] evaluate contingency plans for construction projects using option pricing theory. They consider both technical and market uncertainties as well as the time it takes to execute the project. Valuation is based on a replicating portfolio in the market. Van den Boomen et al. [3] found that in the presence of market price uncertainty, real option analysis is the first choice to evaluate flexible engineering options in public infrastructure projects. Tseng and Barz [28] study optimal short-term operation of an electricity generation asset based on expected profits. Given price processes for electricity and fuel, expectations can be calculated with risk neutral probabilities and the approach optimizes the real option value of the generating asset. Martinez-Ceseña and Mutale [17] demonstrate how the real option valuation support the decision marking process on renewable power generation projects. Their valuation proceeds in two steps: first, timing and design of power plant investments is optimized, and second, the expected value calculation is based on Monte Carlo simulation. For the decision making and capital budgeting process in renewable electricity generation projects, Loncar et al. [15] find the *market asset disclaimer (MAD)* approach by Copeland and Antikarov [4] valuable; the underlying asset *MAD* is the project itself, without flexibility. In valuation of leasing contracts by Kenyon and Tompaidis [13] the central concern is the idle time of an asset under lease between consecutive contracts. The underlying asset is a market resource, an asset which is leased continuously without interruption. Valuation is based on the rate of return for the market resource and on a market utilization rate, the share of assets under lease. An ask price is considered for valuation, which is based on expected returns (risk neutral preferences) and real probabilities. Kallio, Kuula and Oinonen [10] employ real option valuation on forest plantation investments in Brazil. He et al. [8] employ an actuarial real-option approach to mitigate disruption risk in the supply chain. Using prospect theory, Khan et al. [14] find empirically biases in real option exercise decisions in the context of a single IT project and in a portfolio setting. Regan et al. [23] propose a new simulation method for real options in land use investment decisions and indicate this approach not only provide a more realistic assessment of landholder investment decision making but also provide insights for improved policy performance.

In order to evaluate investment projects as real options we consider simultaneously investments in competing assets. For example, these underlying assets may concern investments in

[1]A replicating portfolio produces a stochastic cash flow stream which is equivalent to the one under valuation.

bonds, stocks, commodities, etc. We determine the real option value as the bid price; i.e., the maximum price which a firm is willing to pay for the real option, given competing investment alternatives. Such real option value is consistent with arbitrage pricing theory for valuing financial assets, such as derivatives. Based on indifference, the ask price is defined similarly as the minimum price at which an agent is willing to sell an option. The indifference principle of bid/ask price valuation was proposed by Pratt [22] and it has been used extensively thereafter.

For bid price valuation, stochastic programming (SP) and dynamic programming (DP) are possible computational approaches; however, the former often suffers from curse of dimensionality leading to intractability. This motivates our study to explore the use of DP whose possibilities have not been fully studied.

Since the development of the binomial lattice approach for option valuation (Cox et al. [5]), uncertainty on market prices often is described by complete market models. However, in portfolio optimization it is common to employ incomplete market models, where the number of successor nodes in a scenario tree exceeds the number of assets. Indeed, relatively few valuation problems afford the luxury of market completeness resulting in the simple risk neutral valuation approach commonly encountered in the analysis of contingent claims. The general concept of valuing real options in an incomplete (or partially complete) market is an important research avenue that is far from being fully explored and we aim to address this research gap.

We characterize a real option by a set of alternative actions for exercising the real option, and by resulting outcomes. An outcome is a stochastic cash flow stream, a contingent claim. Hence, a real option may be defined by feasible choices of contingent claims in a set of stochastic cash flow streams resulting from all possible actions of exercising the real option. Additionally, there may be an exogenous cash flow stream resulting, for instance, from the existing business of the firm. If this plays an important role, then we simply suggest finding the incremental value of the real option cash flow in the firm's total cash flow.

Our presentation is in a discrete time, discrete probability space framework. Given a single cash flow stream resulting from an exercising strategy for the real option, a multi-stage portfolio model is employed to determine an optimal portfolio strategy for underlying assets, and subsequently, to determine an optimal exercising strategy for the real option. Under additive exponential utility, we show how dynamic programming (Bellman [1]) is applied to such portfolio problems, and subsequently, to real option valuation.

For our article, a central reference is the seminal paper by Smith and Nau [25]. Under the assumption of partially complete markets (PCM), they consider two types of uncertainties, market uncertainty associated with underlying assets and private (non-market) uncertainty associated with the particular project under evaluation. For valuing a real option, Smith and Nau devise an integrated roll back procedure. Employing this procedure, Smith and McCardle [26] propose an approach for valuing oil fields. Also Kamrad and Ernst [12] are concerned with market and private uncertainty in evaluation of mining and manufacturing ventures. Their valuation is based on replication. However, it can also be interpreted as an integrated roll back procedure with risk neutral preferences.

PCM assumes that the price processes of the competing assets are independent of the realizations of private uncertainties. Smith and Nau [25]: *Intuitively, market uncertainties are those that can be perfectly hedged by trading securities and private uncertainties are project-specific uncertainties that cannot be hedged. For example, the level of demand is a market uncertainty* *and the plant's efficiency is a private uncertainty.* However, such important (private) uncertainties may concern as it will affect the decision of whether the real option project is undertaken or not. Therefore, in this study, we also consider a locally complete market (LCM), where realizations of private uncertainty can impact, with a possible delay, the subsequent price processes of the competing assets. The resulting DP approach includes the integrated roll back procedure by Smith and Nau [25]as a special case. Also Hilli et al. [9] consider correlation among the real option cash flow and the return on an industry index used as a competing asset. In this study, a firm considers investments, within its investment budget over a given period, in a forest products technology portfolio. Exponential utility of the terminal value is used for bid price valuation to find the present value of each feasible portfolio. To capture the inter-dependencies of forest product prices, a vector error correction model is used. This high-dimensional time series model is the underlying price process in the Kalman filter used to determine means, variances and the correlation of the firm's profits and the return on the index. Then, a double binary tree is created for profits and index returns, and contingent claim valuation of each portfolio cash flow is carried out by dynamic programming.

An interesting question is, when is it justified to relax the market efficiency condition of PCM and obtain LCM? While the efficient market hypothesis (EMH) for capital markets (in weak and semi-strong form) is widely accepted, it is essential to keep in mind that the scenarios depicting both market and private uncertainties reflect the views of the user, the management responsible for the real option project and its valuation. Example 3 in the paper is a minor technical illustration, where the future probability distribution of the price of a competing asset depends on the realization of private uncertainty. For a better fitting example, consider a real option concerning new product developmento in a major pharmaceutical company investing in the development of emerging and cutting-edge technologies. The private uncertainty on the progress in such a long lasting research process and accumulation of information is held as company secret at least until completion of the project, and therefore, EMH would not apply. Yet, afterwards a successful major outcome is reflected in the stock market.

The contributions of the paper are as follows. First, we show that the bid price provides a unique real option value consistent with arbitrage pricing theory. Second, using properties of exponential utility, we extend the real option valuation approach by Smith and Nau [25], based on PCM, in two ways: (i) we develop a DP based approach for real option valuation under incomplete markets - thus relaxing all completeness assumptions, such as the conditions for PCM, and (ii) we employ the notion of a locally complete market (LCM), for which PCM is a special case. We show how the DP approach simplifies under LCM and under PCM becomes equivalent to the approach of Smith and Nau. Third, in the context of this paper an interesting issue is whether DP provides computational advantage over alternative approaches, such as multi-stage stochastic programming (SP). Indeed, our numerical test runs indicate a huge improvement by DP over SP. Furthermore, the run time under LCM increases quite modestly in comparison with the PCM, while the case without completeness assumptions is more demanding; nevertheless, the DP run time even in this case is vastly smaller than for SP.

The rest of the paper is organized as follows. In Section 2, we present the bid price valuation with a relevant framework and results. In Section 3, we discuss properties of exponential utility functions which allow a dynamic programming framework for solving the portfolio problems.

While no completeness assumption is necessary for applying dynamic programming, computational savings can be achieved if certain completeness assumptions hold. In Section 4 we propose the notion of a locally complete market (LCM), which is less restrictive than PCM. In Section 5, we develop dynamic programming approaches for valuation of a contingent claim. These approaches are extended in Section 6 to real option valuation. In Section 7, numerical illustrations are given with examples in flexible manufacturing systems. Section 8 concludes. All proofs, excluding well known results, are in the Appendix.

2. Bid Price Valuation

In this section we discuss the bid price valuation of real options based on stochastic optimization. For this aim, we begin by formulating suitable multi-stage portfolio models. Thereafter we state properties of such models. An outcome is that valuation based on indifference is consistent with risk neutral valuation of arbitrage pricing theory.

We employ a discrete time framework, where periods are defined by stages $t = 0, 1, \ldots, T$. Hence, the time horizon is subdivided into T periods. An index $t > 0$ also refers to a period between stage $t-1$ and t. The duration in years between t and $t-1$ may depend on t. All vectors are column vectors, except price vectors, return vectors, vectors of dual multipliers and sum vectors $e = (1, 1, \ldots, 1)$ are row vectors.

Uncertainty in our analysis may concern, for example, market prices of inputs and outputs of the real option project in consideration as well as market prices of underlying assets. Realizations of uncertainties over time are depicted by a scenario tree denoted by Γ. Let k denote a node of the scenario tree with $k = 0$ referring to the root. Let k_- denote the predecessor of node k, for $k \neq 0$, and let K be the set of terminal nodes. We let Γ refer to the set of nodes in the tree as well. Node k appears at stage $t_k \in \{0, 1, 2, \ldots, T\}$. For the root, $t_0 = 0$, and for terminal nodes $k \in K$, $t_k = T$. For $k \neq 0$, we assume $t_{k_-} = t_k - 1$ for the predecessor node k_-. For $k \notin K$, let J_k denote the set of immediate successor nodes of k. Hence, for all $j \in J_k$, we have $j_- = k$. The probability of attaining node k is π_k with $\pi_k > 0$, for all k.

A real option specifies a set of possible actions of exercising the real option. Such actions may specify choices of technological investments and production strategy, for instance. A choice of actions yields a cash flow f_k, for all k. We denote the stochastic cash flow stream by $f = (f_k)$ and define a real option by a set F of feasible choices f.

The following technical assumption is used to guarantee the existence of an optimal solution for portfolio problems used for bid price valuation.

Assumption A0: Real option. The choice set F of stochastic cash flow streams $f = (f_k)$, for $k \in \Gamma$, is nonempty, closed and bounded. A short position in the real option is not allowed.

Consider finitely many competing assets including a *risk free asset*. For all such assets, let $\tilde{S}_t$ denote the vector of gross returns from stage $t-1$ to stage t. A component of $\tilde{S}_t$ is a risk free return, denoted by R_t. While R_t is assumed strictly positive and deterministic, other components of $\tilde{S}_t$ may be random. The risk free return from stage $t = 0$ to stage t is $R_{0,t} = \Pi_{t' \leq t} R_{t'}$. For stages t' and t with $t' \leq t$, $R_{t',t} = R_{0,t}/R_{0,t'}$ is the risk free return from stage t' to t. For node $k \neq 0$, the vector of a single period gross returns is denoted by S_k, the realization of $\tilde{S}_t$ observed at node k.

Let vector s_k denote monetary amounts invested in the competing assets at node k so that the investment expenditure at node k is es_k, where $e = (1, 1, \ldots, 1)$. Let $s = (s_k)$ denote the investment strategy with a sub-vector s_k for all k. Initially, the positions are zero so that $s_{k_-} = 0$, for $k = 0$. At stage T, all positions are closed and we denote $s_k = 0$, for terminal nodes $k \in K$. Transaction costs and other market frictions are excluded from consideration.

For each node k, we define an endogenous cash flow d_k, to be paid at node k. We call d_k the *yield* for which a number of interpretations are possible. For instance, yield d_k may be a free cash flow, profit or dividend. If d_k is negative, it may be interpreted as an increase in equity capital. At the end of the horizon, the yield may be interpreted as terminal wealth.

To unify cash balance equations, define $v = (v_k)$ with $v_0 = V$ and $v_k = 0$, for all $k \neq 0$. The portfolio dynamics with initial and terminal conditions is given as follows

$$es_k + d_k + v_k = S_k s_{k_-} + f_k \;\forall\, k, \tag{2.1}$$

$$s_{k_-} = 0 \text{ for } k = 0, \tag{2.2}$$

$$s_k = 0 \;\forall\, k \in K. \tag{2.3}$$

For all nodes k, the left side in (2.1) sums up the cash use in investments (in competing assets), the yield (e.g., dividends), and at the root node $k = 0$, the real option charge V. The right side of (2.1) shows the sources of cash from investments and the real option cash flow.

The agent is a risk-averse expected utility maximizer (von Neumann and Morgenstern [21]), and we assume a separable utility function as follows:

Assumption A1: Utility function. The utility function $u(d_0, d_1, \ldots, d_T) = \sum_t u_t(d_t)$ is separable. Let $\Phi \subset \{0, 1, \ldots, T\}$ such that $T \in \Phi$. For all $t \in \Phi$, assume $u_t(d_t)$ is differentiable, increasing and strictly concave, for all d_t, with $\lim_{d_t \to -\infty} u_t'(d_t) = \infty$. For all $t \notin \Phi$, define $u_t(d_t) = 0$, for $d_t \geq 0$, and $u_t(d_t) = -\infty$, for $d_t < 0$.

Hence, in some valuation cases we may account for d_t in some periods only, but the terminal yield always counts. For example, the utility may only depend on the terminal wealth d_T. Furthermore, if $t \in \Phi$, then u_t is increasing and strictly concave such that marginal utility increases without limit as d_t decreases. On the other hand, if $t \notin \Phi$, then we require the yield d_t to be non-negative. Given an option charge V, in order to find the optimal expected utility for the portfolio problem of simultaneous investments in the real option and the competing assets, we begin by considering separately each single real option cash flow $f \in F$. Thereafter, the optimal value is the maximum value over $f \in F$; see (2.5) below.

The notation is summarized as follows:

f_k	real option cash flow
F	set of cash flow streams
V	real option charge
d_k	yield
s_k	level of competing investment
S_k	return of competing investment
π_k	node probability

For the evaluation of a cash flow stream f, we define the portfolio problem as follows. Given any V and $f = (f_k) \in F$, consider the problem of finding an investment strategy $s = (s_k)$ and yield $d = (d_k)$ to

$$\max \sum_k \pi_k u_{t_k}(d_k) \ s.t. \ (2.1) - (2.3) \tag{2.4}$$

Let $u(V, f)$ denote the optimal objective function value of (2.4). Then, the exercising problem of finding the best cash flow stream of the real option is

$$\max_{f \in F} u(V, f). \tag{2.5}$$

If $\hat{f}(V) \in F$ denotes an optimal choice in (2.5), then the optimal expected utility $\hat{u}(V)$ in (2.5) is $\hat{u}(V) = u(V, \hat{f}(V))$.

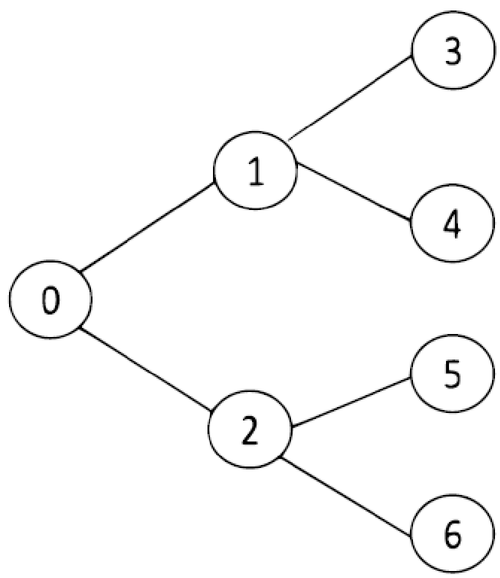

FIGURE 1. A binary tree with $T = 2$ time periods, seven nodes $k = 0, 1, \ldots, 6$ and two edges branching from each non-terminal node.

To illustrate the problem (2.4) using a scenario tree, Figure 2 shows an example of a binary tree with $T = 2$ and the horizon of 2 years subdivided into 2 time steps. The tree is composed of nodes $k \in \{0, 1, \ldots, 6\}$ at time stages $t = 0, 1, 2$, and edges joining the nodes. The root node $k = 0$ is at time $t = 0$, and the set of terminal nodes is $K = \{3, 4, 5, 6\}$ are at stage $t = 2$. From each node k at time $t < T$, there are 2 equally likely edges branching from node k. For the node probabilities π_k, we have $\pi_0 = 1$, $\pi_1 = \pi_2 = 0.5$, and $\pi_3 = \pi_4 = \pi_5 = \pi_6 = 0.25$, and for all $k > 0$, the immediate predecessor node of k_- is given by $1_- = 2_- = 0$, $3_- = 4_- = 1$ and $5_- = 6_- = 2$. Associated with nodes k, there is an exogenous return vector S_k, for $k > 0$, and an endogenous vector s_k of asset values in the portfolio, real option cash flow f_k, and yield d_k. Additionally, at the root node, there is a real option charge V.

The structure of constraint matrix and the right hand side (*rhs*) in (2.1) - (2.3) for a two period binary tree case in Figure 2 is shown in the following table:

s_0	s_1	s_2	d_0	d_1	d_2	d_3	d_4	d_5	d_6	*rhs*
e			1							$f_0 - V$
$-S_1$	e			1						f_1
$-S_2$		e			1					f_2
	$-S_3$					1				f_3
	$-S_4$						1			f_4
		$-S_5$						1		f_5
		$-S_6$							1	f_6

We assume no arbitrage opportunities exist in the scenario tree. Formally, this is stated as follows.

Assumption A2: No arbitrage. There is no homogenous solution $s = (s_k)$ and $d = (d_k)$, of (2.1) - (2.3) such that $d \geq 0$ and $d \neq 0$.

Note that a homogenous solution $s = (s_k)$ and $d = (d_k)$ of (2.1) - (2.3) such that $d \geq 0$ and $d \neq 0$ defines an arbitrage opportunity where the cash flow stream $d = (d_k)$ resulting from investment strategy s is non-negative for all nodes k and strictly positive for some k. Assumption A2 implies the well known result of arbitrage pricing theory: for all k, there exist node prices $y_k > 0$ such that

$$y_k e = \sum_{j \in J_k} y_j S_j \tag{2.6}$$

for all $k \notin K$; see, e.g., Luenberger [16], or Kallio and Ziemba [11]. We may scale the vector y such that $y_0 = 1$ to obtain a state price vector. Given a risk free asset exists, let R_t denote the total risk free return over the period t immediately succeeding node k. Then equivalently with (2.6), there are risk neutral probabilities $q_j = R_t y_j / y_k > 0$, for $j \in J_k$, with $\sum_{j \in J_k} q_j = 1$, such that

$$R_t e = \sum_{j \in J_k} q_j S_j. \tag{2.7}$$

The following auxiliary results under assumptions A0 - A2 rely on standard optimization theory. They provide some insight on the optimization problems (2.4)-(2.5) and prove useful to show the subsequent main results in Theorem 2.2.

Lemma 2.1. *For problem (2.4), assume that A0 - A2,*

- **(i):** *an optimal solution exists, and an optimal dual vector $y > 0$ for (2.1) exist such that (2.6) holds;*
- **(ii):** *the optimal objective function value $u(V, f)$ is concave in f and V, strictly increasing in f and strictly decreasing in V; furthermore, $(-y_0, y)$ is a subgradient of $u(V, f)$ at (V, f);*
- **(iii):** *for any V, an optimal solution $\hat{f} = \hat{f}(V) \in F$ exists for the exercising problem (2.5) and the optimal value $\hat{u}(V) = u(V, \hat{f}(V))$ is strictly decreasing and continuous in V, such that, for some $V_1 \geq V_2$,*

$$\hat{u}(V_1) \leq u(0,0) \leq \hat{u}(V_2). \tag{2.8}$$

In item (i) we first worry about the existence of an optimal solution; not all optimization problems have an optimal solution. The rest of (i) relates to Karush-Kuhn-Tucker (KKT) optimality conditions for (2.4). We show that $y = (y_k) > 0$ holds for the dual multiplier vector; for any terminal node $k \in K$ and the yield d_k, KKT implies $y_k = \pi_k u'_T > 0$, and for all other nodes k, $y_k > 0$ follows by induction. In item (i), (2.6) follows from KKT applied to the vector s_k of investments at node $k \notin K$; to see this, it helps to look at the table above showing the constraint matrix of (2.4). Item (ii) states properties of the optimal objective function value $u(V, f)$ for (2.4) as a function of the charge V and the cash flow f of the real option. Item (iii) states properties of the optimal objective function value $\hat{u}(V)$ for (2.5) as a function of V given the best choice $f \in F$. For bid price valuation, we also consider the problem (2.4) with $f = 0$ and $V = 0$, in other words, without a cash flow stream from the real option and without an option charge. Hence, in this case the problem represents optimal investment in competing assets and we denote the optimal objective function value by $u^* = u(0,0)$. Indifference holds given an option charge $V = \hat{V}$ is such that $\hat{u}(\hat{V}) = u^*$; based on this fundamental equation, $\hat{V}$ is the bid price of the real option.

The following valuation result implies a unique real option value $\hat{V}$ with interpretations in terms of arbitrage pricing theory.

Theorem 2.2. *Assume that A0 - A2,*

- **(i):** *there is a unique real option value $\hat{V}$ with an optimal choice $\hat{f} = \hat{f}(\hat{V}) \in F$ for (2.5), such that the indifference equation $\hat{u}(\hat{V}) = u^*$ holds,*
- **(ii):** *there exist a state price vector $y > 0$ with $y_0 = 1$ satisfying (2.6) such that*

$$\hat{V} = \sum_k y_k \hat{f}_k, \tag{2.9}$$

- **(iii):** *if $V(f)$ denotes the indifference value of $f \in F$ such that $u(V(f), f) = u^*$, then*

$$\hat{V} = \max_{f \in F} V(f). \tag{2.10}$$

The valuation results (2.9) and (2.10) resemble those provided by standard arbitrage pricing theory extended to incomplete markets. However, under incomplete markets the state price vector is not unique so that a unique option value is not obtained based on no-arbitrage only. Besides, because short position in the real option is prohibited by A0, arbitrage pricing theory only yields a lower bound on the real option value $\hat{V}$; see e.g., Kenyon and Tompaidis [13]. Due to preference information, Theorem 2.2 yields a unique real option value $\hat{V}$.

In general, an approach for determining the real option value is to employ a stochastic model for uncertain parameters first, and to generate the scenario tree thereafter. Employing stochastic programming (see, e.g., Birge and Louveaux [2] or Wets and Ziemba [29]), valuation of the real option may follow from a search for $V = \hat{V}$, for instance, employing interpolation, to satisfy $\hat{u}(\hat{V}) = u^*$.

In summary, for computations, we deal with two cases as follows. First, we consider investments in the real option F and in the competing assets. In this case, both competing investments, the yield and the choice $f \in F$ are simultaneously optimized, while the cash flow at the root node is reduced by an option charge V. The resulting problem is given by (2.4) and (2.5), and the optimal expected utility is $\hat{u}(V)$. Second, we exclude the real option and the option charge so that the problem is given by (2.4) with $V = 0$ and $f = 0$, resulting in optimal expected utility u^*.

In case of indifference $\hat{u}(V) = u^*$; i.e. the optimal objective function values are equal in both cases. Furthermore, if F is a convex (polyhedral) set, then the joint problem of (2.4) and (2.5) satisfying the indifference equation $\hat{u}(\hat{V}) = u^*$ is a convex optimization problem. In this case, the real option value is obtained as an optimal value $V = \hat{V}$ for the following problem of finding V, $f = (f_k)$, $s = (s_k)$ and $d = (d_k)$ to

$$\max\{\, V \mid \sum_k \pi_k u_{t_k}(d_k) \geq u^*,\ f \in F,\ (2.1) - (2.3)\}.$$

Because the expected utility is a concave function of the yield d, the requirement for a lower bound u^* on the expected utility is a convex constraint. The market of competing assets is complete if a replicating investment strategy $s = (s_k)$ exists for any $d = (d_k)$; i.e., if for any vector d, there is s, such that s and d form a homogenous solution for (2.1) - (2.3). Given A2, if the market of competing assets is complete, then the state price vector y is unique (Harrison and Kreps [7]). Hence Theorem 2.2 implies the following well known result.

Lemma 2.3. *If A0 - A2 hold, and the market is complete, then*

$$\hat{V} = \max_{f \in F} \sum_k y_k f_k, \tag{2.11}$$

where $y = (y_k)$ is a unique vector of state prices with $y_k > 0$ and $y_0 = 1$ satisfying (2.6).

Hence, under complete market hypothesis, the valuation problem simplifies. We may then begin by solving for the set of prices y_k from (2.6), with $y_0 = 1$. Thereafter, we obtain the real option value $\hat{V}$ from (2.11), which is independent of the utility function.

Finally, we note that there may be an exogenous cash flow stream of the firm which can play an important role in real options valuation. Such exogenous stochastic cash flow may result, for instance, from the firm's business existing initially. In this case, the valuation of an investment project under incomplete market should not be done in isolation from other cash flow streams of the firm. Under complete markets such need does not arise, because the valuation result is independent of preferences and private cash flows. To keep our presentation simple, we just suggest the following. In the presence of an exogenous cash flow stream $c = (c_k)$, using the methods for real options valuation discussed in this article, we determine the value of c first and the value of $c + f$ thereafter. Then the value of f is the incremental improvement due to f in the value of $c + f$.

3. Properties of an Additive Exponential Utility

In the spirit of Smith and Nau [25] we now restrict preferences by exponential utility to allow dynamic programming approaches in Sections 5 and 6 for portfolio optimization problems (2.4). In addition to assumption A1 for the utility function, we consider exponential functions as follows.

Assumption A3: Exponential utility. In addition to A1, for all $t \in \Phi$, assume $u_t(d_t) = -\kappa_t \exp(-d_t/\gamma_t)$ with a utility discounting factor $\kappa_t > 0$ and a risk tolerance $\gamma_t > 0$.

Recalling the risk free return $R_{t,t'}$ from stage t to stage t' and defining parameters $\gamma_t = 0$ for $t \notin \Phi$, for subsequent use we define parameters $\hat{\gamma}_t$ as follows:

$$\hat{\gamma}_t = \sum_{t' \geq t} \gamma_{t'} / R_{t,t'} \tag{3.1}$$

for $t = 0, 1, \ldots, T$. From (3.1) we obtain

$$\hat{\gamma}_{t-1} = \gamma_{t-1} + \hat{\gamma}_t / R_{t-1,t} \tag{3.2}$$

Exponential utility functions accomodate favorable properties observed already by Wilson [30] and Merton [19]. The following proposition is a consequence of an observation, adopted from Merton [19], that under assumption A3 an optimal allocation in risky assets in the portfolio problem (2.4) is independent of the option charge V.

Lemma 3.1. *If A0 - A3 hold, then for the optimal objective function value in (2.5) and for a decrease Δ in the option charge V, we have*

$$\hat{u}(V - \Delta) = \hat{u}(V) \exp(-\Delta/\hat{\gamma}_0). \tag{3.3}$$

Furthermore, an optimal choice $\hat{f}$ in (2.5) is independent of the option charge V, and the real option value $\hat{V}$ is given by

$$\hat{V} \;=\; \hat{\gamma}_0 \, \log[\, u(0,0)/\hat{u}(0)\,]. \tag{3.4}$$

We obtain $\hat{u}(V-\Delta)\exp(-(V-\Delta)/\hat{\gamma}_0) = \hat{u}(V)\exp(-V/\hat{\gamma}_0)$ through multiplication of equation (3.3) by $\exp(-(V-\Delta)/\hat{\gamma}_0)$. Consequently, we conclude that the value of the expression $\hat{u}(V)\exp(-V/\hat{\gamma}_0)$ is independent of the option charge V. For $V' = V - \Delta$, (3.3) states $u(V', f) = \alpha u(V, f)$ where $\alpha = \exp(-\Delta/\hat{\gamma}_0)$ is a constant. Therefore, the optimal choice $\hat{f}$ is independent of V. Furthermore, after the evaluation of $u(V, f)$, for some V, the evaluation of $u(V', f)$, for any V', results simply from scaling by α. Subsequently, *leveling* refers to such an evaluation procedure and we use it extensively for DP valuation of contingent claims and real options.

The following result shows, that estimation of utility discounting factors is unnecessary, only risk tolerance parameters are needed for real option valuation.

Lemma 3.2. *If A0 - A3 hold, then the same real option value $\hat{V}$ is obtained using any utility discounting factors $\kappa_t > 0$, for $t \in \Phi$.*

4. Market Completeness Assumptions

For valuation with dynamic programming in Sections 5 and 6, assumptions A0-A3 are sufficient. However, if additional assumptions concerning the market of competing assets hold, then some computational advantage is achieved. For this aim, we first look at the scenario tree in more detail. Thereafter, we discuss the conventional assumption of a complete market (CM), a partially complete market (PCM) similar to the one by Smith and Nau [25], and a locally complete market (LCM), a concept proposed in this article.

Consider node $k \notin K$ and the set J_k of its successors. The collection of distinct return vectors S_j among all nodes $j \in J_k$ is $\{S_i\}$ with indices i in a set I_k. We partition the set J_k into subsets J_{ik} so that $j \in J_{ik}$ if and only if $S_j = S_i$. Let $p_j = \pi_j/\pi_k$ denote the conditional probability of node $j \in J_k$ given k. Then for all $i \in I_k$, $p_i = \sum_{j \in J_{ik}} p_j$ is the conditional probability of return S_i, given k, and $r_{ij} = p_j/p_i$ is the conditional probability of node $j \in J_{ik}$, given return S_i. Hence, For $j \in J_{ik} \subseteq J_k$, we have $p_j = p_i r_{ij}$. In terms of Smith and Nau [25], events $i \in I_k$ refer to *market uncertainty*, and events $j \in J_{ik}$, given i, refer to *private uncertainty*.

Under no-arbitrage assumption A2, there are risk neutral probabilities $q_j > 0$ for $j \in J_k$, such that $Re = \sum_{j \in J_k} q_j S_j = \sum_{i \in I_k} \sum_{j \in J_{ik}} q_j S_i$. Hence, we have

$$Re = \sum_{i \in I_k} q_i S_i, \tag{4.1}$$

where $q_i = \sum_{j \in J_{ik}} q_j > 0$ is the risk neutral probability of return S_i.

In Section 2, the standard definition of a complete market was used to confirm the well known valuation result in Lemma 2.3.

Assumption CM: Complete market. A replicating investment strategy $s = (s_k)$ exists for any stochastic cash flow stream $d = (d_k)$.

Under CM and A2, for all $k \notin K$, the risk neutral probabilities $q_j > 0$ satisfying $Re = \sum_{j \in J_k} q_j S_j$ are unique. Hence, the return vectors S_j, $j \in J_k$, are linearly independent and distinct. Therefore, there is a single node in each set J_{ik}. Although completeness is commonly assumed for real option valuation, the following assumption, which is much less restrictive than CM, appears powerful.

Assumption LCM: Locally complete market. Consider a stochastic cash flow stream $d = (d_k)$, and assume that for all $k \notin K$, $d_j = d_i$ and $S_j = S_i$, for all $i \in I_k$ and $j \in J_{ik}$. Then a replicating investment strategy $s = (s_k)$ exists for the stochastic cash flow stream d.

Under LCM and no-arbitrage A2, for all $k \notin K$, the risk neutral probabilities $q_i > 0$ satisfying (4.1) are unique, and the vectors S_i, $i \in I_k$, are linearly independent.

The concept of partially complete market (PCM) proposed by Smith and Nau [25] is defined as follows:

Assumption PCM: Partially complete market. In addition to the conditions for LCM, asume for all $k \notin K$, that the realizations of private uncertainties convey no information about the future price processes of competing assets.

Thus, PCM implies LCM. If A2 and PCM hold, then there exist unique risk neutral probabilities $q_i > 0$, $i \in I_k$, satisfying (4.1), for all $k \notin K$. Unlike PCM, LCM allows realizations of private events to provide information about the future price processes of the competing assets. Thereby, valuation based on LCM generalizes the valuation result based on PCM. For an illustration, see Example 3 in Section 7 where the future probability distribution of the price of a competing asset depends on the realization of private uncertainty.

As mentioned already, the widely accepted efficient market hypothesis (EMH) for capital markets is not equivalent to the efficiency in PCM, and in many instances, it is justified to relax such condition of PCM and obtain LCM. Different from PCM, the price processes depicting the market involve individual choices in setting up the scenario tree considered in LCM. If the management finds it justified to deviate from the theoretical EMH, the interpretation may be that realization of an important non-market uncertainty impact market prices with delay. This can be the case even if the uncertainty is classified as private, but its realizations in fact refer to private beliefs. For example, the management of a real estate development company may hold strong beliefs concerning possible impacts of climate change. Such expectations may be based on beliefs which support or deny the phenomenon, and these beliefs may not obey EMH. Other contexts where such private views on uncertainties can lead to LCM are as follows: (a) for a firm competing in international markets: exchange rate and tariff policies of major governments; (b) capital investments in Russia: legislation protecting or endangering foreign investments; (c) impact of internet, social media and AI in advertisement; (d) barriers in the maritime supply chains; (e) emergence of new technologies concerns several sectors; (f) political acceptance of small modular reactors for nuclear power and environmental control measures.

5. Recursive Valuation of Contingent Claims

For valuation of contingent claims via DP backward recursion, for each node k of the scenario tree Γ, define a sub-tree Γ_k composed by the root node k, and by all nodes and arcs from stage t_k to T succeeding node k. We also let Γ_k denote the set of nodes in such sub-tree. In the sequel, Lemma 3.1 is employed for optimization over all sub-trees Γ_k of the entire scenario tree Γ. Because node k is at stage t_k, in place of $\hat{\gamma}_0$ we now have $\hat{\gamma}_{t_k}$ in (3.3) and (3.4). Note also that a decrease in option charge in Lemma 3.1 is interpreted as an increase in the cash flow at root node. Lemma 3.1 allows us to develop a dynamic programming recursion for valuation of a stochastic cash flow stream $f \in F$, and by Theorem 2.2 (iii), the real option value is obtained by optimizing over $f \in F$. We begin with a dynamic programming formulation, which employs relatively simple optimization problems. Analytical expressions are obtained for the optimal expected utility over sub-trees. Such expressions need relatively simple numerical evaluation of risk neutral probabilities q_i satisfying (4.1) at the root node k associated with the agents preferences and the cash flow stream f. Subsequently, under a locally complete market LCM or under a partially complete market PCM, such risk neutral probabilities for each node k are unique and their evaluation can be based on (4.1).

Consider a node $k \notin K$ and the sub-tree Γ_k of the scenario tree. Let $t_k = t-1$ so that all successor nodes of k appear at stage t. Denote the single period risk free (total) return over period t succeeding node k by $R = R_t$, suppressing t. Recall that J_k is the set of successor nodes j of node k, S_j is the single period return vector at node $j \in J_k$, and $p_j = \pi_j/\pi_k$ is the conditional probability for node $j \in J_k$ given k. Applying the formulation (2.4) without any option charge to a sub-tree Γ_k with root node k and nodes $l \in \Gamma_k$, consider the problems $\hat{P}_k = \hat{P}_k(f)$ with cash flow stream f and $P_k^* = \hat{P}(0)$, where $\hat{P}_k(f)$ is given by

$$\max \sum_{l \in \Gamma_k} (\pi_l/\pi_k)\, u_{t_l}(d_l) \tag{5.1}$$

s.t.

$$\begin{aligned} e s_l - S_l s_{l_-} + d_l &= f_l \quad \forall\, l \in \Gamma_k \\ s_{k_-} &= 0 \\ s_l &= 0 \quad \forall\, l \in K, \end{aligned}$$

and P_k^* is the problem (5.1) without the cash flow stream f. In the objective function, π_l/π_k is the conditional probability of node l, given root node k of the sub-tree. Let $\hat{u}_k = \hat{u}_k(f)$ and u_k^* denote the optimal objective function values of problems $\hat{P}_k$ and P_k^*, respectively. Then, at the root node $k = 0$, $\hat{u}_0 = u(0, f)$ and $u_0^* = u(0,0)$. Hence, the bid price $V(f)$ of cash flow stream f in Theorem 2.2 (iii) is obtained by (3.4) in Lemma 3.1 as

$$V(f) = \hat{\gamma}_0 \log(u_0^*/\hat{u}_0). \tag{5.2}$$

For terminal nodes of dynamic programming backward recursion, we have

$$\hat{u}_k = u_T(f_k) \text{ and } u_k^* = u_T(0)\ \forall\ k \in K. \tag{5.3}$$

For other nodes $k \notin K$ with $t_k = t-1 < T$, we employ leveling. Consider problem $\hat{P}_k$ first. Thereafter, analysis for P_k^* follows substituting $f = 0$. In DP, for solving the problem $\hat{P}_k$, we figure out the optimal expected utility $\hat{u}_k$ at the root node k of sub-tree Γ_k with cash balance equation $es_k + d_k = f_k$ excluding any extra cash flow at k from investments at the predecessor none k_- of k. It is assumed that in backward recursion, such optimal expected utilities $\hat{u}_j$ have already been evaluated for all successor nodes $j \in J_k$ of node k. Because the (total) return on investments s_k at node k yield a cash flow $S_j s_k$ at node $j \in J_k$, then by (3.3) of Lemma 3.1, the expected utility $\hat{u}_j$ is incremented by a factor $\exp(-S_j s_k/\hat{\gamma}_t)$. Recall that node j is at time stage t wherefore $\hat{\gamma}_0$ in (3.3) is here replaced by $\hat{\gamma}_t$. Consequently, application of (3.3) to each successor node $j \in J_k$ and optimization over d_k and s_k leads to the following problem:

$$\hat{u}_k = \max_{d_k,s_k} \{u_{t-1}(d_k) + \sum_{j\in J_k} p_j \hat{u}_j \exp(-S_j s_k/\hat{\gamma}_t) \mid es_k = f_k - d_k\}. \tag{5.4}$$

For all $k \notin K$, recall that $\{S_i\}$, $i \in I_k$, is the set of distinct return vectors S_j over all successors $j \in J_k$, and J_k is partitioned into subsets J_{ik} such that $j \in J_{ik}$ if and only if $S_j = S_i$. Furthermore, the conditional probability of j, given k, is $p_j = p_i r_{ij}$, where p_i is the conditional probability of S_i, given k, and r_{ij} is the conditional probability of $j \in J_{ik}$, given S_i. In this notation, problem (5.4) is rewritten as

$$\hat{u}_k = \max_{d_k}\{u_{t-1}(d_k) + \max_{s_k}\{\sum_{i\in I_k} p_i \hat{u}_i \exp(-S_i s_k/\hat{\gamma}_t) \mid es_k = f_k - d_k\}\}, \tag{5.5}$$

where

$$\hat{u}_i = \sum_{j\in J_{ik}} r_{ij}\hat{u}_j. \tag{5.6}$$

For further development of (5.5), we analyze the inner problem in more detail. Suppressing k, let the sub-vector x_k of s_k denote the vector of investments in risky assets and y_k the investment in risk free asset so that the cash balance equation becomes $y + ex = f_k - d_k$. Let R be the single period total risk free return of the period succeeding node k. For the risky assets, let E_i denote the vector of excess return over the risk free return R; i.e., $E_i = S_i - Re$ is obtained by subtracting R from the return of each risky asset. Substituting $y = f_k - d_k - ex$ from the cash balance equation, yields

$$S_i s_k = R(f_k - d_k) + E_i x. \tag{5.7}$$

Given A2, risk neutral probabilities $q_i > 0$ exist, although they may not be unique, and equation (4.1) becomes

$$\sum_{i\in I_k} q_i E_i = 0. \tag{5.8}$$

Assuming A2, an optimal solution exists for the inner problem in (5.5). Using (5.7), the inner problem is restated as

$$\max_x\ \exp[-R(f_k - d_k)/\hat{\gamma}_t] \sum_{i\in I_k} p_i \hat{u}_i \exp(-E_i x/\hat{\gamma}_t). \tag{5.9}$$

Let $x = \hat{x}$ denote optimal risky investments in (5.9). At the optimum, the gradient of the objective function w.r.t. x vanishes; hence, we have $\sum_{i\in I_k}[-p_i\hat{u}_i\exp(-E_i\hat{x}/\hat{\gamma}_t)]E_i = 0$. Here the expression in square brackets is positive, implying risk neutral probabilities satisfying (5.8) with

$$\hat{q}_i = -p_i\hat{u}_i\exp(-E_i\hat{x}/\hat{\gamma}_t)/\sigma > 0 \tag{5.10}$$

and the scaling factor $\sigma > 0$ determined such that $\sum_{i\in I_k}\hat{q}_i = 1$. Evaluation of risk neutral probabilities in (5.10) requires, in general, solving a small optimization problems (5.9); however, under LCM, PCM or CM, equation (4.1) can be used directly. Given risk neutral probabilities $\hat{q}_i$, we get the expected utility $\hat{u}_k$ from the following result.

Lemma 5.1. *For the problem $\hat{P}_k$, denote the risk neutral probabilities by $\hat{q}_i$, for $i \in I_k$. Then the optimal objective function value $\hat{u}_k$ is obtained from*

$$\hat{\gamma}_{t-1}\log(-\hat{u}_k) = \psi_{t-1} - f_k + (\hat{\gamma}_t/R)\sum_{i\in I_k}\hat{q}_i\log(-p_i\hat{u}_i/\hat{q}_i), \tag{5.11}$$

where

$$\psi_{t-1} = \gamma_{t-1}\log(\kappa_{t-1}/\gamma_{t-1}) - (\hat{\gamma}_t/R)\log(\hat{\gamma}_t/R) + \hat{\gamma}_{t-1}\log\hat{\gamma}_{t-1}, \tag{5.12}$$

for $t_k = t-1 \in \Phi$, and $\psi_{t-1} = 0$, for $t_k = t-1 \notin \Phi$.

Analysis of the problem P_k^* follows from the above by substituting $f = 0$ in $\hat{P}_k$. In this case, the definition (5.6) is replaced by

$$u_i^* = \sum_{j\in J_{ik}} r_{ij}u_j^* \tag{5.13}$$

and corresponding to $\hat{q}_i$ in (5.10), we have the risk neutral probabilities q_i^*, for $i \in I_k$, given by

$$q_i^* = -p_iu_i^*\exp(-E_ix^*/\hat{\gamma}_t)/\sigma' > 0 \tag{5.14}$$

with optimal risky investments x^* and scaling factor $\sigma' > 0$ chosen such that $\sum_{i\in I_k}q_i^* = 1$. Then, Lemma 5.1 implies that the optimal objective function value u_k^* is obtained from

$$\hat{\gamma}_{t-1}\log(-u_k^*) = \psi_{t-1} + (\hat{\gamma}_t/R)\sum_{i\in I_k}q_i^*\log(-p_iu_i^*/q_i^*). \tag{5.15}$$

Recursions (5.11) and (5.15) with (5.3) yield $\hat{u}_0$ and u_0^*. Thereafter the value $V(f)$ of the cash flow stream f is obtained from (5.2).

Alternatively, we may apply recursive valuation of cash flow stream f employing the bid price v_k at node k instead of optimal values $\hat{u}_k$ and u_k^*. Given a cash flow stream f, for the sub-tree with root node k, the bid price v_k is defined by indifference: similar to (5.2), $v_k = \hat{\gamma}_{t_k}\log(u_k^*/\hat{u}_k)$ with $t_k = t-1$. Hence, given $\hat{u}_k$ and u_k^* from recursions (5.11) and (5.15) with (5.3), and $\hat{q}_i$ and q_i^*, for $i \in I_k$, from (5.10) and (5.14), we obtain the following result.

Theorem 5.2. *If assumptions A1-A3 hold, and parameters $\hat{\gamma}_t$ are given by (3.1), then the bid price v_k of a cash flow stream f in a sub-tree with root node k is given recursively by*

$$v_k = f_k \;\forall\, k \in K \tag{5.16}$$

and for $k \notin K$,

$$v_k = f_k + (\hat{\gamma}_t/R)\sum_{i\in I_k}[q_i^*\log(-p_iu_i^*/q_i^*) - \hat{q}_i\log(-p_i\hat{u}_i/\hat{q}_i)]. \tag{5.17}$$

Under LCM, (5.17) becomes

$$v_k = f_k + (1/R)\sum_{i\in I_k} q_i^* C_i \tag{5.18}$$

where

$$C_i = -\hat{\gamma}_t \log[\sum_{j\in J_{ik}} r_{ij}^* \exp(-v_j/\hat{\gamma}_t)\,]. \tag{5.19}$$

with probabilities r_{ij}^* defined by

$$r_{ij}^* = r_{ij}(u_j^*/u_i^*). \tag{5.20}$$

Under PCM, (5.17) is replaced by (5.18) and $r_{ij}^* = r_{ij}$ in (5.19). Under CM, (5.17) is replaced by (5.18) with $C_i = v_i$, where $v_i = v_j$, for the single node $j \in J_{ik}$.

Under LCM, $\hat{q}_i = q_i^*$, so that we only need to evaluate risk neutral probabilities q_i^*, which are independent of the cash flow stream. Hence, they can be computed from (4.1) directly. In (5.19), C_i represents certainty equivalent for prices v_j obtained with probabilities r_{ij}^*, when an exponential utility function with risk tolerance $\hat{\gamma}_t$ is employed. In case assumption PCM holds, $r_{ij}^* = r_{ij}$, and C_i can be computed without evaluation of the values u_j^* for $j \in J_k$. Then the recursion in Theorem 5.2 becomes equivalent with the roll back procedure by Smith and Nau (2005). If the risk tolerance parameters γ_t increase indefinitely, the choice behavior is based on risk neutral preferences. In this case also $\hat{\gamma}_t$ approaches infinity, and C_i in (5.19) approaches the expected value $\sum_{j\in J_{ik}} r_{ij} v_j$. Such an approach is employed by Kamrad and Ernst [12] for valuing mining and manufacturing ventures. Under CM, the recursion simplifies to the well known formula $v_k = f_k + (1/R)\sum_i q_i^* v_i$.

6. Recursive Valuation of Real Options

Next, we apply DP valuation of the real option under the following extra assumption:

Assumption A4: F is finite. For each node $k \in \Gamma$, the real option permits a finite number of choices, and these contingent choices lead to stochastic cash flow streams $f = (f_k)$.

We use backward recursion for solving simultaneously problems (2.4) and (2.5) for optimal s, d and f, and the real option value $\hat{V}$. A cash flow stream $f \in F$ results from a multistage decision process as follows. For each node k in the scenario tree, assuming A4 there is a finite number of possible choices concerning the real option. For such a choice, the path from the root node up to node k as well as the choices preceding the node k are known. Possible choices at node k depend on k and possibly on the *relevant choice history* denoted by l. A choice preceding node k is relevant at k, if it carries some impact to the real option cash flow or to choice alternatives at node k or at nodes succeeding k. As mentioned, such relevant choices may concern decisions on whether to defer, expand, contract, abandon, switch use, or otherwise alter a capital investment (Trigeorgis [27]). For numerical illustration in Section 7, we consider flexible manufacturing systems, where an operating mode is chosen in each node k of Γ. A change in the mode causes a switching cost. Therefore the relevant choice history l at k is the mode chosen in the predecessor node k_-.

Let H_{kl} define the finite set of feasible choices at node k given a choice history l at k. Thus, if $h \in H_{kl}$ is chosen at k, then h enters the choice history at successor nodes $j \in J_k$. Because the choice h at k with l determines uniquely the choice history at $j \in J_k$, we let h refer to the relevant choice history at j as well. For all k, the real option cash flow $f_k = f_{klh}$ depends on k,

l, and h. We carry out DP backward recursion in the tree where node k with choice history l appear at stage t_k. Assuming A0-A4, recursion for solving (2.4)–(2.5) is developed employing results of Section 5. In particular, we take from Section 5 the analysis of the case without the real option cash flow. Thus for all k, the definitions of u_k^* in (5.15) u_i^* in (5.13) and q_i^* in (5.14), for $i \in I_k$, are as above. For the case with real option cash flows, we use leveling. Let $\hat{u}_{kl}$ denote the optimal expected utility over the sub-tree with root node k and choice history l excluding the cash flow from investments at the predecessor node k_-. For terminal nodes $k \in K$ with choice history l,

$$\hat{u}_{kl} = \max_{h \in H_{kl}} u_T(f_{klh}). \tag{6.1}$$

For nodes $k \notin K$ at stage $t_k = t-1 < T$, it follows from Lemma 3.1, that an optimal choice $h \in H_{kl}$ at node k with choice history l is independent of cash flow at node k. The optimal value $\hat{u}_{kl}$ is obtained in two steps. First, for each choice $h \in H_{kl}$, we optimize the yield d_{kl} and competing investments s_{kl} with cash balance equation $es_{kl} + d_{kl} = f_{klh}$. Given $h \in H_{kl}$, we obtain the optimal expected utility $\hat{u}_{klh}$, similarly as $\hat{u}_k$ in (5.5), as follows

$$\hat{u}_{klh} = \max_{d_{kl}} \{u_{t-1}(d_{kl}) + \max_{s_{kl}} \{\sum_{i \in I_k} p_i \hat{u}_{ih} \exp(-S_i s_{kl}/\hat{\gamma}_t) \mid es_{kl} + d_{kl} = f_{klh}\}\}, \tag{6.2}$$

where, for all $i \in I_k$,

$$\hat{u}_{ih} = \sum_{j \in J_k} r_{ij} \hat{u}_{jh}. \tag{6.3}$$

Again, under assumption A1, $d_{kl} = 0$ is optimal in (6.2), if $t_k \notin \Phi$. Second, optimization over choices $h \in H_{kl}$ yields

$$\hat{u}_{kl} = \max_{h \in H_{kl}} \hat{u}_{klh}. \tag{6.4}$$

For all nodes k and choice history l, the risk neutral probabilities $\hat{q}_{ih}$, for $i \in I_k$, are derived from the inner problem of (6.2) similarly as $\hat{q}_i$ in (5.10); however, probabilities $\hat{q}_{ih}$ may now depend on the choice h. In an adaptation to (5.11), we obtain

$$\hat{\gamma}_{t-1} \log(-\hat{u}_{klh}) = \psi_{t-1} - f_{klh} + (\hat{\gamma}_t/R) \sum_{i \in I_k} \hat{q}_{ih} \log(-p_i \hat{u}_{ih}/\hat{q}_{ih}), \tag{6.5}$$

where ψ_{t-1} is given by (5.12) if $t_k \in \Phi$, and $\psi_{t-1} = 0$ if $t_k \notin \Phi$. Given $\hat{u}_{kl}$ by (6.1)–(6.5), and given u_k^* from Section 5, we obtain the following real option value using (3.4) of Lemma 3.1 that $\hat{V} = \hat{\gamma}_0 \log(u_0^*/\hat{u}_{00})$, where $kl = 00$ refers to the root node with a given initial choice history.

Alternatively, let v_{kl} denote the optimal bid price at node k with choice history l. By indifference and Lemma 3.1, $\hat{u}_{kl} \exp(v_{kl}/\hat{\gamma}_{t-1}) = u_k^*$ so that $v_{kl} = \hat{\gamma}_{t-1} \log(u_k^*/\hat{u}_{kl})$. Using the same argumentation as in the case of Theorem 5.2, given $\hat{u}_{ih}$ in (6.3), $\hat{q}_{ih}$ derived from the inner problem of (6.2) and u_k^* from Section 5, we state the following result (without proof):

Theorem 6.1. *If assumptions A0-A4 hold, and parameters $\hat{\gamma}_t$ are given by (3.1), then for the bid price v_{kl} of the real option, in a sub-tree with root node k and choice history l,*

$$v_{kl} = \max_{h \in H_{kl}} f_{klh} \; \forall \; k \in K \tag{6.6}$$

and for nodes $k \notin K$,

$$v_{kl} = \max_{h \in H_{kl}} \{f_{klh} + (\hat{\gamma}_t/R) \sum_{i \in I_k} [q_i^* \log(-p_i u_i^*/q_i^*) - \hat{q}_{ih} \log(-p_i \hat{u}_{ih}/\hat{q}_{ih})]\}. \tag{6.7}$$

Under LCM, (6.7) becomes

$$v_{kl} = \max_{h \in H_{kl}} \{ f_{klh} + (1/R) \sum_{i \in I_k} q_i^* C_{ih} \} \tag{6.8}$$

where

$$C_{ih} = -\hat{\gamma}_t \log[\sum_{j \in J_{ik}} r_{ij}^* \exp(-v_{jh}/\hat{\gamma}_t)]. \tag{6.9}$$

with probabilities r_{ij}^* defined by (5.20). Under PCM, (6.7) is replaced by (6.8) with $r_{ij}^* = r_{ij}$ in (6.9). Under CM, (6.7) is replaced by (6.8) with $C_{ih} = v_{ih}$, where $v_{ih} = v_{jh}$, for the single node $j \in J_{ik}$.

Similarly as in Section 5, under LCM, $\hat{q}_{ih} = q_i^*$ in Theorem 6.1. Hence, we only need to evaluate risk neutral probabilities q_i^*, which are independent of the real option cash flow streams and obtained from (4.1) directly. Under assumption PCM, $r_{ij}^* = r_{ij}$ and Theorem 6.1 results in the integrated roll back procedure by Smith and Nau (1995). Again, under CM, we obtain the familiar recursion $v_{kl} = \max_h \{ f_{klh} + (1/R) \sum_i q_i^* v_{ih} \}$.

7. EXAMPLES

As numerical illustrations, we consider valuation of investments in flexible manufacturing systems (FMS). The first example, the basic version of these two period valuation models, is a complete market case adopted from Trigeorgis [27]. We present this case in a unified manner with three other examples. The second example is an incomplete market case satisfying PCM and demonstrating the integrated roll back procedure of Smith and Nau [25]. The third example satisfies LCM, but PCM does not hold. Yet, we show how minor additional computational effort allows valuation employing dynamic programming. The fourth example does not meet any of the completeness assumptions. Nevertheless, the dynamic programming approach proves computationally efficient. The examples are computed with a wide range of risk tolerance parameters. All examples demonstrate a vast computational improvement by dynamic programming over valuations based on stochastic programming.

The FMS allows two modes h of operation, $h = A, B$. The mode determines profits of the FMS and it can be chosen separately at each stage and at each state of nature, including initial and terminal stages. There is a switching cost at the moment the mode is changed.

Consider two competing investment opportunities for an FMS project. One is a risk free asset with a single period total returns $R = 1.08$. The other is a risky asset with two possible outcomes in each period: the total return is $R_1 = 1.80$ (high) with probability p_1 or $R_2 = 0.60$ (low) with probability p_2. At the initial stage, the profits of each mode h are given. In each time period, profits go up by a factor u_h or down by a factor d_h, where $u_A = 1.8$, $d_A = 0.6$, $u_B = 1.5$ and $d_B = 0.8$. In a given period, if the risky asset yields high return R_1, then the profits of each mode in the same period go up with probability r_{11} and down with probability r_{12}. Similarly, if the risky asset yields low return R_2, then profits go up and down with probabilities r_{21} and r_{22}, respectively.

There are two time steps with stages $t = 0, 1, 2$. Here $t = 0$ is the initial stage, where the valuation is done, and $t = 2$ is the terminal stage. Uncertainty is depicted by a scenario tree. The root node $k = 0$ is at stage $t = 0$. For each period, there are two branches (high and low) for return R_i, each branch followed by two branches (up and down) for changes in profits. There are four successor nodes for each node at stages $t = 0, 1$; i.e., succeeding the root node, there are four nodes at stage $t = 1$, each one having four successor nodes at the terminal stage $t = 2$. One

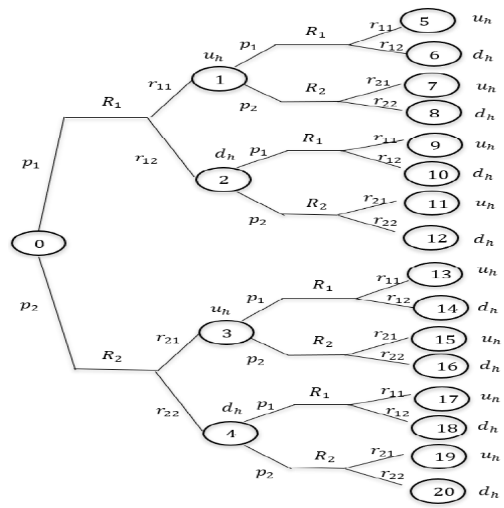

FIGURE 2. An illustration of the scenario tree with nodes $k \in \{0,1,\dots,20\}$ in Examples 1–3; the root node $k=0$ is at $t=0$, $J_0 = \{1,\dots,4\}$, $J_{10} = \{1,2\}$, $J_{20} = \{3,4\}$, the terminal nodes $K = \{5,\dots,20\}$ are at $t=2$, and nodes $\{1,\dots,4\}$ are at $t=1$.

can check that assumption A2 holds. The set J_k of successor nodes of k is subdivided into two subsets J_{ik}, $i=1,2$, referring high and low return. In the two nodes of J_{1k}, the return is high, while in the other two nodes of J_{2k} the return is low. If mode h is chosen at node k, then profits c_{kh} result at k. The initial profits c_{0h} at the root node are given by $c_{0A} = 100$ and $c_{0B} = 85$. For all other nodes k, c_{kh} is defined by the profits of the preceding node with mode h multiplied by the up or down factor u_h or d_h. If the mode is l while entering node k and mode h is chosen at k, then the switching cost at node k is s_{lh}, where $s_{AB} = 8$, $s_{BA} = 2$ and $s_{AA} = s_{BB} = 0$. Initially, the mode is A, while at node $k=0$ it may be altered. If the mode is l while entering node k and mode h is chosen at k, then the FMS cash flow at node k is $f_{klh} = c_{kh} - s_{lh}$. Because the number of nodes is finite and in each node there are two possible choices, the number of alternative real option cash flow streams in the choice set F is finite, and hence, assumption A0 holds. The exogenous cash flow stream form the firm's existing business is assumed zero.

Figure 7 illustrates the scenario tree and Table 1 shows the transition probabilities for Examples 1–3. At the root node $k=0$, the mode is initially A and the profit is c_{0h}, given mode $h \in \{A,B\}$ is chosen. For each node $k>0$, the return realization (R_1 or R_2) is given and so are the factors u_h or d_h for updating the profits for each mode h; consequently, the profits c_{kh} are given for all k and h. In each node $k \geq 0$, the operating mode $h \in \{A,B\}$ is chosen. For node $k>0$, the choice history of relevance (while optimizing over the sub-tree with node k as the root) is the choice of mode l in the predecessor node k_- of k. Then the switching cost at k is s_{lh} and the real option cash flow f_k at node k is $f_{klh} = c_{kh} - s_{lh}$, given the choices l and h. The agent is a risk-averse expected utility maximizer with a separable utility function $\sum_{t=0}^{2} -\kappa_t \exp(-d_t/\gamma)$, where d_t denotes the yield at stage t and $\gamma > 0$ is the risk tolerance. The utility discounting factor is fixed to $\kappa_t = 0.9^t$. Hence, assumptions A1 and A3 are satisfied.

The real option values $\hat{V}$ of an FMS project may be determined using Theorem 2.2 or Lemma 3.1 and stochastic programming directly. However, for instance, with 21 nodes in the scenario tree, there are $2^{21} > 2$ million alternative stochastic cash flow streams in the choice set

	first period		second period		both periods			
Example	p_1	p_2	p_1	p_2	r_{11}	r_{12}	r_{21}	r_{22}
1	0.5	0.5	0.5	0.5	1.0	0.0	0.0	1.0
2	0.5	0.5	0.5	0.5	0.8	0.2	0.2	0.8
3	0.5	0.5	0.8/0.2	0.2/0.8	0.8	0.2	0.2	0.8

TABLE 1. Transition probabilities in Examples 1–3. In Example 1, the realization of the the return R_1 or R_2 predicts perfectly the outcome of the up/down factors u_h and d_h for profits under modes $h = A, B$. In Example 3, p_1 and p_2 for high and low return (R_1 and R_2) in the second period depend on the outcome of the first period realization of the up and down factors u_h and d_h.

F, and therefore, valuation of each $f \in F$ separately seems computationally inefficient. Indeed, as indicated by all examples below, computations can be vastly reduced employing Theorem 6.1 for valuation. For the choice history l at node k, all what matters is the mode prevailing while entering node k. Hence, the choice history is $l = A$ or $l = B$. The possible choices h at node k in Theorem 6.1 are $h = A$ or $h = B$, and if h is chosen, then the choice history at the successor node is h.

Example 1. First, we discuss the complete market case by Trigeorgis [27] For each time period, there is an equal probability $p_1 = p_2 = 0.5$ of high return R_1 and low return R_2 of the risky asset. Furthermore, the realization R_i perfectly predicts changes in profits of the FMS. If high return R_1, is observed in a given period, then the profits go up in the same period with certainty: $r_{11} = 1$ and $r_{12} = 0$. Similarly, if low return R_2 is observed, then $r_{21} = 0$ and $r_{22} = 1$ and profits go down with certainty. Consequently, all nodes with probability zero, as well as their descendants, are omitted from the scenario tree which reduces to a two period binary tree with seven nodes only. One can easily check the completeness assumption CM and the no-arbitrage assumption A2.

This example is easy to analyze; see Trigeorgis [27]. Theorem 6.1 in this case provides the standard risk neutral valuation formula, which recursively leads to $\hat{V} = v_{0A} = 305.3$. Note that $\hat{V}$ is independent of preferences, by Lemma 2.3.

Example 2. This case is the same as Example 1, except that we assume $r_{ij} = 0.8$, for $i = j$, and $r_{ij} = 0.2$, for $i \neq j$. The realization of the return of risky asset does not perfectly predict the change in profits of the FMS. The risk neutral probabilities over the subset J_{ik} sum up to q_i which is uniquely determined by (4.1) for each i. The realization of changes in profits has no predictive power on future development of return of the risky asset. Therefore, by definition of Smith and Nau [25] the market is partially complete. However, the market defined by the scenario tree is not complete because the risk neutral probabilities of nodes in the subset J_{ik} are not unique, only their sum is unique. Therefore Lemma 2.3 does not apply in this case. Hence, assumption PCM holds, but CM does not. For the partially complete market, (5.20) implies that the auxiliary probabilities are $r^*_{ij} = r_{ij}$, because for $i = 1, 2$, the values u^* are equal in the two nodes of J_{ik}. Theorem 6.1 reduces to the integrated roll back procedure by Smith and Nau [25]. With constant risk tolerance $\gamma_t = 100$ we obtain $\hat{V} = v_{0A} = 288.4$. The optimal strategy is to employ mode A, except at stage $t = 1$, if profits went down in the first period then switch to mode B, and thereafter switch back to mode A if profits go up during the second period.

Example 3. The third case is the same as Example 2, except that during the second period succeeding node k at stage $t = 1$, the risky asset yields the high return R_1 with probability $p_1 = 0.8$, given that the FMS profits increased from the root node to node k. In the opposite case, if profits went down, then $p_1 = 0.2$. As before, for the first period $p_1 = 0.5$. The high or low realization of return R_i provides partial information on changes of profits up or down, and symmetrically, the up and down movements of profits predict the return R_i in the subsequent period. In this case assumption LCM holds. However, because changes in profits have some predictive power on future return of the risky asset, the market is not partially complete. Therefore, we need to evaluate the auxiliary probabilities r^*_{ij} using (5.20). Consequently, we need the utility levels u^*_k from (5.15) and (5.3). There is no need to evaluate ψ_{t-1} from (5.12), because it cancels out while computing the auxiliary probabilities r^*_{ij}. Again with risk tolerance $\gamma = 100$, Theorem 6.1 yields $\hat{V} = v_{0A} = 284.5$. The optimal strategy of choosing modes at each node, is the same as in Example 2.

Example 4. The fourth case is the same as Example 3, except that there are three realizations for the return of risky asset. Instead of the two possible returns 1.8 and 0.6 in Examples 1-3, we now have $R_1 = 1.8$ (high) , $R_2 = 1.0$ (medium) and $R_3 = 0.6$ (low), with respective probabilities 0.5, 0.2 and 0.3 at the root node. For the second period, succeeding node k at stage $t = 1$, given that the FMS profits increased from the root node to node k, then the risky asset yields the high return R_1 with probability $p_1 = 0.7$, medium return R_2 with $p_2 = 0.1$ and low return R_3 with $p_3 = 0.2$, In the opposite case, if profits went down, the probabilities are $p_1 = 0.2$, $p_2 = 0.1$ and $p_3 = 0.7$. The high, medium or low realization of return R_i provides partial information on changes of profits up or down, and the probabilities r_{ij} are given by $r_{11} = 0.8$, $r_{12} = 0.2$, $r_{21} = 0.5$, $r_{22} = 0.5$, and $r_{31} = 0.2$, $r_{32} = 0.8$. In this case assumption LCM no longer holds. Therefore, we need to evaluate the risk neutral probabilities q^*_i, $\hat{q}_{ih}$, and auxiliary probabilities r^*_{ij}. Consequently, we need the utility levels u^*_k and $\hat{u}_{kl}$, for each k and choice history l. Again with risk tolerance $\gamma = 100$, Theorem 6.1 yields $\hat{V} = v_{0A} = 289.0$. The optimal strategy of choosing modes is to choose A, except if the profits move down during the first period, then at stage $t = 1$ choose B, and thereafter at stage $t = 2$, choose A if profits go up and B otherwise.

Table 2 shows the valuation results with the risk tolerance in the range $[1, 10^6]$. In the complete market case of Example 1, the values are independent of preferences. In the other three cases, the real option value increases as the risk tolerance increases; i.e., with increasing risk aversion, agents are willing to pay less for the risky FMS projects. By Lemma 3.2, all these valuation results are independent of the utility discounting factors κ_t. Table 2 also shows the CPU time in μs for solving the real option valuation problem by dynamic programming (DP, μs). The figures are averages for problems with different risk tolerances γ. The bottom line of Table 2 shows the average CPU time in ms for valuation of a single cash flow stream using stochastic programming (SP, ms); for this purpose, we use the optimal cash flow stream obtained by dynamic programming. As expected, the CM case of Example 1 with seven nodes only, solves fast. The PCM case of Example 2 and the LCM case of Example 3 are reasonably close to each other in solution time. In Example 4 without market completeness assumptions, the extra evaluations of risk neutral probabilities show in the solution time of 186 μs, which is six to seven times bigger than in Examples 2 and 3. However, it is over 2000 times smaller than the evaluation time 470 ms of a single cash flow stream using stochastic programming, and there are $2^{43} > 10^{12}$ possible cash flow streams in Example 4.

γ	Example 1	2	3	4
1	305.3	204.2	204.1	203.8
10	305.3	222.6	221.6	221.9
100	305.3	288.4	284.5	289.0
1000	305.3	314.5	309.9	319.2
10^4	305.3	317.7	313.1	323.0
10^5	305.3	318.0	313.4	323.4
10^6	305.3	318.1	313.4	323.4
DP (μs)	2	25	31	186
SP (ms)	306	413	390	470

TABLE 2. The real option values $\hat{V}$ of the FMS projects in Examples 1-4 for different levels of risk tolerance γ. On the bottom, we show the average CPU time in μs=10^{-6}s for solving the entire real option valuation by dynamic programming (DP, μs), and the average CPU time in ms=10^{-3}s for valuation of a single cash flow stream using stochastic programming (SP, ms).

On the basis of the run times reported in Table 2, the calculation of the optimal expected utilities u_j can be done with little additional work. The explanation is simple: equations (5.3) and (5.15) provide a convenient recursion to evaluate the values u_j analytically. Furthermore, such utilities are independent of choices allowed by the real option. Besides, in the context of this paper an interesting issue is whether DP provides computational advantage over alternative approaches. In Table 2 illustrating simple cases, the run time under LCM (Example 3) increases by 24 percent in comparison with the PCM (Example 2), and by 640 percent in the case without completeness assumptions (Example 4). However, even in Example 4 the run time is vastly smaller than the run time for SP.

8. CONCLUSIONS

A firm considering investment in a real option project faces choices of exercising the real option. Each choice yields a stochastic cash flow stream, and an optimal choice depends on private preferences. From the firm's perspective, the value of the real option is the bid price at which the firm is indifferent between investing and not investing in the project, given competing investment opportunities, such as financial or real investments. Such valuation is consistent with arbitrage pricing theory. However, under incomplete markets, unlike arbitrage arguments alone, bid price valuation provides a unique real option value. Properties of exponential utility allow dynamic programming approaches for computing real option values, and such values are independent of utility discounting factors. The earlier results by Smith and Nau [25] employing a partially complete market assumption PCM, are generalized in this article omitting market completeness assumptions. However, a locally complete market assumption LCM is proposed, under which computations are simplified.

For practical applications, LCM often is an important relaxation of PCM. Both PCM and LCM distinguish between market uncertainties and private uncertainties. Under PCM, the realizations of private uncertainties convey no information about the future price processes of competing assets, for instance, the market incertainty of stock prices. Unlike PCM, LCM allows realizations of private events to reveal information influencing, with possible delays, the future price processes of the competing assets. The new product development process in a major pharmaceutical company is suggsted as an example of a real option where LCM holds. While the Efficient Market Hypothesis (EMH) is a widely accepted concept in finance, it is up to the managerial judgement to decide to what extent EMH is applicable. For LCM, it is essential to keep in mind that the scenarios depicting both market and private uncertainties express the views of the management responsible for the real option project. Thereby, the management is responsible for justified choices in modeling uncertainties for real option valuation.

With examples in flexible manufacturing systems, the approach is demonstrated under PCM, LCM and without any completeness assumption. The results demonstrate the superiority of dynamic programming in these examples over stochastic programming. The valuation principle is easy to explain to managers and spread sheet calculations suit for valuation.

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Appendix: Proofs

Proof of Lemma 2.1. First, for item (i), a feasible solution for the problem (2.4) is constructed as follows. Begin with any investment strategy $s = s^0$. Then define $d = d^0$ to satisfy (2.1). If $d_k^0 \geq 0$ for all k such that $t_k \notin \Phi$ in A1, then s^0 and d^0 is a feasible solution for the problem (2.4). Otherwise, for $t = 0, 1, \ldots, T-1$ such that $t \notin \Phi$, update s^0 by borrowing an amount of the risk free asset to achieve $d_k^0 \geq 0$, for all nodes k at stage t. Thereby we end up with a feasible solution s^0 and d^0 with an objective function value $u^0 > -\infty$.

By A1, the objective function in (2.4) is continuous. Hence, to show existence of an optimal solution, it suffices to show that the closed and convex set D of feasible vectors d for which the objective function value is at least u^0, is bounded. Assume to the contrary, that D is not bounded. Then, for some vector $\delta = (\delta_k) \neq 0$ and $\delta_k \geq 0$, for $t_k \notin \Phi$, the ray $d(\lambda) = d^0 + \lambda\delta \in D$, for all $\lambda \geq 0$. Furthermore, δ together with some vector η (for investments s) constitute a homogenous solution for (2.1)-(2.3). Hence, A2 implies $\delta_k < 0$, for some node k, and $t_k \in \Phi$. Denote the objective function value along the ray by $u(\lambda)$. Assumption A1 implies that $u(\lambda)$ is concave in λ. The derivative is $u'(\lambda) = \sum_k \pi_k \delta_k u'_{t_k}(d_k(\lambda))$. Letting $\lambda \to \infty$, u'_{t_k} remains bounded if $\delta_k \geq 0$, and u'_{t_s} approaches ∞, as $\delta_s < 0$. Hence, $\lim_{\lambda\to\infty} u'(\lambda) = -\infty$, and $u(\lambda)$ decreases without limit as λ increases so that for large enough λ, $d(\lambda) \notin D$. This is a contradiction, wherefore D is bounded. Hence, an optimal solution exists for (2.4), and we denote the optimal objective function value by $u(V, f)$.

For the dual multipliers in item (i), given A1, existence of optimal multipliers y_k, for all k, follows from Theorem 28.2 by Rockafellar [24]. We show by induction that $y > 0$. At the terminal stage $T \in \Phi$, by A1, and for all terminal nodes k, optimality conditions for d_k imply $y_k = \pi_k u'_{t_k}(d_k) > 0$. In the inductive step, consider any node k, such that $y_j > 0$, for all successor nodes $j \in J_k$. Optimality conditions for the investment vector s_k, for all $j \in J_k$, imply

$$\sum_{j \in K_k} y_j S_j = y_k e. \tag{8.1}$$

By the inductive hypothesis, for the risk free asset, the left side of (8.1) is strictly positive. Hence, the right side of (8.1) implies $y_k > 0$.

Assertion (ii) follows directly from (i) and standard theory of convex optimization; see Rockafellar (1970).

For assertion (iii), it follows from A0 and (ii) that an optimal solution $\hat{f} \in F$ for (2.5) exists. Next, let $V^1 < V^2$, with optimal choices $f^1 = \hat{f}(V^1)$ and $f^2 = \hat{f}(V^2)$. Suppose $\hat{u}(V^1) \leq \hat{u}(V^2)$. Then $\hat{u}(V^1) \leq u(V^2, f^2) < u(V^1, f^2)$, which is a contradiction. Hence, $\hat{u}(V)$ is strictly decreasing in V.

We now show continuity of $\hat{u}$. First, the epigraph of $\hat{u}(V)$ is closed, because it is the intersection of epigraphs of $u(V, f)$ which are closed, for all $f \in F$. Hence, $\hat{u}$ is lower semicontinuous. Second, consider the hypograph H of $\hat{u}(V)$, and a convergent sequence $\{u^n, V^n\} \subset H$ with $(u^n, V^n) \to (\bar{u}, \bar{V})$. Define $f^n = \hat{f}(V^n) \in F$. Then $\hat{u}(V^n) = u(V^n, f^n) \geq u^n$, for all n. By A0, there is a convergent subsequence $f^{n_i} \to \bar{f} \in F$, and $\hat{u}(\bar{V}) \geq u(\bar{V}, \bar{f}) \geq \bar{u}$, by continuity of $u(V, f)$. Hence, H is closed, and $\hat{u}$ is also upper semicontinuous. Consequently, $\hat{u}(V)$ is continuous.

To show the left side inequality (2.8), define $f^* = (f_k^*)$, such that $f_k^* = \max_{f \in F} \max_k f_k$, for all k. Then (ii) implies $u(V, f^*) \geq \hat{u}(V)$, for all V, and $\lim_{V \to +\infty} u(V, f^*) = -\infty$. Hence, for some V_1 large enough, $\hat{u}(V_1) \leq u(0,0)$.

To prove the right inequality (2.8), pick any $f \in F$, and let d^* be an optimal yield for (2.5) with $V = 0$ and $f = 0$. We proceed by taking V small enough such that risk free investments alone with $f \in F$ provide a yield $d \geq d^*$. Define $\bar{d} = \max\{0, \max_k (d_k^* - f_k)\}$, and $\bar{R} = \max_t R_{0,t}$, the risk free return from stage 0 to stage t. Here, $\bar{d} \geq 0$ is the cash required from risk free savings at node k sufficient to meet or exceed the yield d_k^* at any node $k \neq 0$, and $\bar{R}$ provides a lower bound for the total return from risk free investments at $t = 0$ until any stage $t > 0$. By assumption, $\bar{R} > 0$. Consider an investment strategy s employing the risk free asset only. At the root node $k = 0$, initial positions are closed, and for all $t > 0$, a risk free position $\Delta_t = \bar{d}/\bar{R}$ is taken and held until stage t. To meet the yield d_0^* at the root node, we need a cash amount of $\Delta_0 = d_0^* - f_0$. Consequently, defining $V_2 = -\sum_{t=0}^{T} \Delta_t$, we have $\hat{u}(V_2) \geq u(V_2, f) \geq u(0,0)$. □

Proof of Theorem 2.2. Item (i) follows from (2.8) of Lemma 2.1 (iii) and the mean value theorem. For (ii), Lemma 2.1 (iii) implies that the optimum in (2.5), for $V = \hat{V}$, is attained with some $\hat{f} \in F$. For $V = \hat{V}$ and $f = \hat{f}$, by Lemma 2.1 (i), optimal multipliers $\hat{y} > 0$, for (2.1) exist satisfying (2.6). Similarly, for $V = 0$ and $f = 0$, denote the optimal multipliers by $y^* > 0$. Then $\hat{u}(\hat{V}) = u^*$ and Lemma 2.1 (ii) imply $\hat{y}_0 \hat{V} - \sum_k \hat{y}_k \hat{f}_k \geq 0$ and $-y_0^* \hat{V} + \sum_k y_k^* \hat{f}_k \geq 0$. Combining these two inequalities yields $\sum_k (y_k^*/y_0^*) \hat{f}_k \geq \hat{V} \geq \sum_k (\hat{y}_k/\hat{y}_0) \hat{f}_k$. Because the set of prices y/y_0 satisfying (2.6) is convex, there is a convex combination y of y^*/y_0^* and $\hat{y}/\hat{y}_0$, which satisfies (2.6) and (2.9).

For (iii), because $\hat{f}$ solves (2.5) with $V = \hat{V}$, for all $f \in F$, we have $u(\hat{V}, f) \leq u(\hat{V}, \hat{f}) = u^*$. Hence, $u(\hat{V}, f) \leq u(V(f), f)$. Because $u(V, f)$ is strictly decreasing in V by Lemma 2.1 (ii), it follows that $V(f) \leq \hat{V}$. Hence, (2.10) holds. □

Proof of Lemma 3.1. Let $f \in F$, choose a real option charge V and denote the resulting problem (2.4) by $P(V, f)$. For $V' = V - \Delta$, $P(V', f)$ denotes the revised problem. For $P(V, f)$, let $s = (s_k)$ and $d = (d_k)$ denote the optimal solution and $y = (y_k)$ the vector optimal dual multipliers for (2.1). By Lemma 2.1 (i), we have $y > 0$.

For the revised problem $P(V', f)$, a feasible solution s' and d' is obtained by adjusting the solution s and d as follows. For all $t = 0, 1, \ldots, T$ and nodes k with $t_k = t$, increment the yield to $d'_k = d_k + \delta\gamma_t$ and finance these increments by investing at the root node 0 in the risk free asset amounts $\delta\gamma_t / R_{0,t}$ yielding $\delta\gamma_t$ at stage t. Choose δ such that these risk free investments at the root sum up to Δ; i.e., $\sum_{t\geq 0} \delta\gamma_t / R_{0,t} = \Delta$. Then using (3.1), we have $\delta = \Delta / \hat{\gamma}_0$. The resulting solution s' and d' is feasible for revised problem $P(V', f)$, and the objective function value is $\alpha u(V, f)$ with $\alpha = \exp(-\delta) > 0$. Multiplying by α the gradient of the objective function of $P(V, f)$ yields the gradient for $P(V', f)$. Hence, after scaling the optimal multipliers y by α, we end up with a Karush-Kuhn-Tucker point for $P(V', f)$. Therefore, $u(V', f) = \alpha u(V, f)$, and consequently, the optimal choice $\hat{f}(V) \in F$ in (2.5) is independent of V and (3.3) holds. For the bid price $\hat{V}$, using (3.3) with $V = 0$ and $\Delta = -\hat{V}$, we have $u^* = u(0,0) = \hat{u}(\hat{V}) = \hat{u}(0)\exp(\hat{V}/\hat{\gamma}_0)$. Solving for $\hat{V}$, yields (3.4). □

Proof of Lemma 3.2. Given the real option value $\hat{V}$ and an optimal choice $\hat{f} \in F$, let $\hat{s}$ and $\hat{d}$ denote an optimal solution of (2.4) with $V = \hat{V}$ and $f = \hat{f}$. Similarly, let s^* and d^* denote an optimal solution of (2.4) with $V = 0$ and $f = 0$.

Next, consider the revised problem (2.4), where the discounting factor κ_t is multiplied by some factor $\alpha_t > 0$, for all $t \in \Phi$, Referring to the revised utility function, we denote the optimal values for (2.4) by $\bar{u}(V, f)$. Revise the solution $\hat{s}$ and $\hat{d}$ as follows: for all $t \in \Phi$, invest in the risk free asset an amount Δ_t at stage $t = 0$ until stage t and increment $\hat{d}$ at stage t by the total return $R_{0,t}\Delta_t$. In particular choose Δ_t such that $\alpha_t \exp(-R_{0,t}\Delta_t/\gamma_t) = 1$. Then, denoting $\Delta = \sum_t \Delta_t$, we observe that the revised solution is optimal for (2.4) with $V = \hat{V} - \Delta$ and $f = \hat{f}$. Furthermore, the optimal objective value is unchanged so that $u^* = u(\hat{V}, \hat{f}) = \bar{u}(\hat{V} - \Delta, \hat{f})$. Apply the same revision to s^* and d^* as well to obtain an optimal solution for (2.4) with $V = -\Delta$ and $f = 0$, and $u^* = u(0,0) = \bar{u}(-\Delta, 0)$. Then, by Lemma 3.1, $\bar{u}(\hat{V}, \hat{f}) = \bar{u}(0,0)$ so that $\hat{V}$ is the bid price of $\hat{f}$. It remains to be shown that $\hat{f}$ is an optimal choice with $V = \hat{V}$. For any $f \in F$, $u(\hat{V}, f) = \bar{u}(\hat{V} - \Delta, f) \leq u(0,0) = \bar{u}(-\Delta, 0)$. Hence, by Lemma 3.1, $\bar{u}(\hat{V}, f) \leq \bar{u}(0,0)$; i.e., $\hat{f}$ is optimal. □

Proof of Lemma 5.1. Suppressing k, denote for short $\delta = f_k - d_k$ and $h_i = p_i\hat{u}_i < 0$. The auxiliary result of Lemma 8.1 below shows that the risk neutral probabilities $\hat{q}_i$ in (5.10), for $i \in I_k$, are unique and independent of d_k, the decision variable of the outer problem in (5.5). Furthermore, the optimal objective function value in (5.9) is given by (8.3). Applied to (5.5), with $h_i = p_i\hat{u}_i$ and $\delta = f_k - d_k$, yields

$$\hat{u}_k = \max_{d_k} \{u_{t-1}(d_k) - \exp(-R\delta/\hat{\gamma}_t)\Pi_{i\in I_k}(-p_i\hat{u}_i/\hat{q}_i)^{\hat{q}_i}\}. \tag{8.2}$$

Optimal multipliers of the cash balance equations of $\hat{P}_k$ are strictly positive so that $d_k = 0$ is optimal, if $t_k = t - 1 \notin \Phi$. Otherwise, if $t_k \in \Phi$, then optimization over d_k in (8.2) with (3.2), after lengthy but straightforward algebra, yields (5.11). Note for $t_k = t - 1 \notin \Phi$, $d_k = 0$ and $\hat{\gamma}_{t-1} = \hat{\gamma}_t / R$ by (3.2), so that (8.2) implies (5.11) with $\psi_{t-1} = 0$. □

Proof of Theorem 5.2. By indifference (5.16) holds for terminal nodes. For node $k \notin K$ at stage $t-1$, from (3.4) of Lemma 3.1, we obtain $v_k = \hat{\gamma}_{t-1}\log(u^*_k/\hat{u}_k)$. Hence, Using (5.11) and (5.15) we obtain (5.17). Under LCM, $\hat{q}_i = q^*_i$, for $i \in I_k$. Hence (5.17) yields $v_k = f_k - (1/R)\sum_{i\in I_k} q^*_i\hat{\gamma}_t\log(\hat{u}_i/u^*_i)$. Furthermore, using $\hat{u}_i = \sum_{j\in J_{ik}} r_{ij}\hat{u}_j$ in (5.6) and $r^*_{ij} = r_{ij}(u^*_j/u^*_i)$ in (5.20), we get $v_k = f_k - (1/R)\sum_{i\in I_k} q^*_i\hat{\gamma}_t\log[\sum_{j\in J_{ik}} r^*_{ij}(\hat{u}_j/u^*_j)]$. Thus, observing that $\hat{u}_j/u^*_j =$

$\exp(-v_j/\hat{\gamma}_t)$ and using $C_i = -\hat{\gamma}_t \log[\ \sum_{j\in J_{ik}} r^*_{ij} \exp(-v_j/\hat{\gamma}_t)\ ]$ in (5.19), we notice that (5.17) simplifies to (5.18). Under PCM, LCM holds and $u^*_j = u^*_i$ in (5.13), for all $j \in J_{ik}$. Hence, $r^*_{ij} = r_{ij}$ in (5.20). Under CM, both PCM and LCM hold, and in each set J_{ik} there is a single node. Hence $C_i = v_i$ in (5.19). □

Lemma 8.1. *Assuming A2, optimality conditions for (5.9) imply risk neutral probabilities $\hat{q}_i$, for $i \in I_k$, satisfying (5.8), which are unique and independent of δ. Denoting $\delta = f_k - d_k$ and $h_i = p_i\hat{u}_i$, the optimal objective function value in (5.9) is*

$$\bar{u} = -\exp(-R\delta/\hat{\gamma}_t)\Pi_i(-h_i/\hat{q}_i)^{\hat{q}_i}. \tag{8.3}$$

Proof: Under A2, an optimal solution $\bar{x}$ exists for the convex optimization problem (5.9). Denote $u_i = h_i \exp(-E_i\bar{x}/\hat{\gamma}_t)$ and $u = \sum_{i\in I_k} u_i$. Then $u_i < 0$, for $i \in I_k$, and $\bar{u} = \exp(-R\delta/\hat{\gamma}_t)\ u$. Optimality conditions for (5.9) imply $\sum_i -u_iE_i = 0$. Denote $\hat{q}_i = u_i/u$. Then $\hat{q}_i > 0$, $\sum_i \hat{q}_i = 1$ and, $\sum_i \hat{q}_iE_i = 0$ satisfying (5.8). Hence, the optimality conditions yield risk neutral probabilities $\hat{q}_i$, which are independent of δ.

To show uniqueness of $\hat{q}_i$, denote $g_i = E_i\bar{x}/\hat{\gamma}_t$ so that $u = \sum_i u_i = \sum_i h_i \exp(-g_i)$. Because $u_i = h_i \exp(-g_i)$, the risk neutral probabilities are $\hat{q}_i = h_i \exp(-g_i)/u$. Hence, if $\bar{x}$ is a unique optimal solution, then $\hat{q}_i$ is unique. Otherwise, if $\hat{x} \neq \bar{x}$ is an alternative optimal solution, define $d_i = E_i(\hat{x}-\bar{x})/\hat{\gamma}_t$. Then, because the set of optimal solutions is convex, $f(\lambda) = \sum_i h_i \exp[-(g_i + \lambda d_i)] = u$ is a constant, for all $\lambda \in [0,1]$. Suppose that $d_i \neq 0$, for some i. Then $d^2 f/d\lambda^2 = \sum_i h_i d_i^2 \exp[-(g_i + \lambda d_i)] < 0$, and hence, $f(\lambda)$ is strictly concave. This is a contradiction, because $f(\lambda)$ is constant, for $\lambda \in [0,1]$. Therefore $d_i = 0$, for all i, and consequently, risk neutral probabilities $\hat{q}_i$ are unique.

To show (8.3), definitions of u_i, u and $\hat{q}_i = u_i/u$ yield $\log(-u_i) = \log(-h_i) - E_i\bar{x}/\hat{\gamma}_t = \log(-u\hat{q}_i)$. Multiplying the latter equality by $\hat{q}_i$ and summing over i yields $\sum_i \hat{q}_i[\log(-h_i) - E_i\bar{x}/\hat{\gamma}_t] = \sum_i q_i \log(-u\hat{q}_i)$, where the left side is $\sum_i \hat{q}_i \log(-h_i)$. Hence, we obtain $\log(-u) = \sum_i \hat{q}_i[\log(-h_i) - \log(\hat{q}_i)]$. But $\log(-\bar{u}) = -R\delta/\hat{\gamma}_t + \log(-u) = -R\delta/\hat{\gamma}_t + \log[\Pi_i(-h_i/\hat{q}_i)^{\hat{q}_i}]$, which yields (8.3). □