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DERIVATIVES OF ALL ORDERS OF SMOOTH FUNCTIONS WITH GIVEN ZEROS

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Abstract. Let $f : B^n \to \mathbb{R}$ be an infinitely differentiable function on the unit ball B^n, with $\max_{z \in B^n} |f(z)| = 1$. A well-known fact is that if f vanishes on a set $Z \subset B^n$ with a non-empty interior, then, for each $d = 1, 2, \ldots$, the norm of the d-th derivative $||f^{(d)}||$ is at least $\frac{(d+1)!}{2^{d+1}}$. A natural question to ask is: What happens for other sets Z? For finite, but sufficiently dense sets Z? This question was partially answered by the second author of this paper. This study can be naturally related to a certain special settings of the classical Whitney smooth extension problem. However, in our results, as well as in most of publications on the Whitney problem, the differentiability degree d was assumed to be fixed. Our main goal in the present paper is to allow d to run to ∞ in our results above, and to present the corresponding lower bounds for the asymptotic behavior of the high-order derivatives of f in terms of simple geometric characteristics of the zero set Z of f.

Keywords. High order derivatives; Remez constant; Test curves.

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1. INTRODUCTION

Let $Z \subset B^n \subset \mathbb{R}^n$ be a closed subset of the unit ball $B^n \subset \mathbb{R}^n$. In [4]-[6] and [27]-[29], we look for the class $U(Z)$ of all C^{d+1}-smooth functions $f : B^n \to \mathbb{R}$, vanishing on Z. Such C^{d+1}-smooth (and even C^∞) functions f always exist, since any closed set Z is the set of zeroes of a certain C^∞-smooth function.

Remark 1.1. It is important to notice that the requirement for Z to be closed simplifies some issues, but it is not really essential in our main Definition 1.2 below. In particular, it is convenient to allow non-closed Z in our result in Section 2.2 below.

We normalize the functions $f \in U(Z)$ requiring $\max_{B^n} |f| = 1$. Put the point-wise norm $||f^{(d+1)}(x)||$ to be the sum of the absolute values of $d + 1$-st order partial derivatives of f at $x \in B^n$, and let the global norm $||f^{(d+1)}||$ be the maximum of $||f^{(d+1)}(x)||$ for all $x \in B^n$.

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Definition 1.2. Let Z be as above, and let natural d be fixed. The d-rigidity $\mathcal{RG}_d(Z)$ of Z is the minimal possible norm of the last derivative $||f^{(d+1)}||$, for $f \in U(Z)$. In other words, for each normalized C^{d+1}-smooth function $f : B^n \to \mathbb{R}$, vanishing on Z, we have $||f^{(d+1)}|| \geq \mathcal{RG}_d(Z)$, and $\mathcal{RG}_d(Z)$ is the maximal number with this property.

Let us now state some classical facts, illustrating the d-rigidity notion. As the "model" example, we consider the case of dimension $n = 1$. Here we have the following important fact, which can be proved by the standard interpolation (or finite difference) formulas:

Proposition 1.3. *For any $Z \subset B^1$, we have $\mathcal{RG}_d(Z) \geq \frac{(d+1)!}{2^{d+1}}$, if Z consists of at least $d + 1$ different points, and $\mathcal{RG}_d(Z) = 0$ if Z consists of at most d different points.*

Another simple observation is the following:

Proposition 1.4. *For any $Z \subset B^n$ with a non-empty interior,*

$$\mathcal{RG}_d(Z) \geq \frac{(d+1)!}{2^{d+1}}.$$

This fact easily follows from Proposition 1.3. We just restrict any function $f \in U_d(Z)$ to a certain straight line ℓ, passing through z_0 with $|f(z_0)| = 1$, and through an interior point of Z.

Let us mention here also an old result of [23], related to smooth rigidity. Informally it can be stated as follows: if the set of zeros $Y(f)$ of a smooth function f on B^n does not look like a union of smooth hypersurfaces, of a total area bounded by a constant, depending only on n, d, then the norm of $f^{(d+1)}$ is not smaller than a certain positive constant, depending only on n, d. This result was recently strongly extended in [14].

Many of the "near-polynomiality" results of [24, 30] can be naturally interpreted in terms of smooth rigidity. We plan to present some new results in this direction separately.

In [4]-[6], [23], and [27]-[29] some lower bounds on the d-rigidity $\mathcal{RG}_d(Z)$ were produced, under different kinds of assumptions on Z. However, in these results the differentiability degree d was assumed to be fixed.

Our main goal in the present paper is to allow d to run to ∞ in the results above, and to present the corresponding lower bounds for the asymptotic behavior of the high-order derivatives of f in terms of simple geometric characteristics of the zero set Z of f. We also present some results in this direction, not immediately related to our recent publications, mentioned above: one is in terms of the "capacity" of Z (see, e.g. [3]), another is based on [25].

In spite of the fact that d-rigidity certainly can be treated as a very special case of the general Whitney problem (see [2, 7, 8, 18, 19, 20]), we do not discuss the possible connections here, referring the reader to [5].

The paper is organized as follows: in Section 2 we recall the result of [27], connecting the rigidity $\mathcal{RG}_d(Z)$ and the Remez constant $\mathcal{R}_d(Z)$. Then in Subsection 2.1 we produce our first new result, connecting the asymptotic behavior in d of $\mathcal{RG}_d(Z)$, via $\mathcal{R}_d(Z)$, with the classical notion of the capacity of Z. Next, in Subsection 2.2, we use the result of [28] in order to bound asymptotics of $\mathcal{RG}_d(Z)$, via $\mathcal{R}_d(Z)$, through the topology of Z. Finally, in Subsection 2.3 we use [25] in order to bound $\mathcal{R}_d(Z)$, and hence $\mathcal{RG}_d(Z)$, in terms of the box (entropy) dimension of Z.

In Section 3 we use the results of [29] and [6] to bound the asymptotic behavior in d of $\mathcal{RG}_d(Z)$, not referring to the Remez constant $\mathcal{R}_d(Z)$. Instead, in [29] and [6] the "test curves"

are used. In Section 3 we give a short explanation of the approach, and some consequences for the asymptotic behavior of $\mathcal{RG}_d(Z)$.

Finally, let us mention that some asymptotic rigidity results can be obtained by "an inverse reading" (not always straightforward) of the results of [5, 13, 14, 22, 30]. We consider these connections as important for a better understanding of the geometry of infinitely smooth functions, and we plan to present some results here separately.

The authors would like to thank the referee for instructive remarks, which allowed us to significantly improve the presentation, especially in Section 2.3.

2. RIGIDITY AND REMEZ CONSTANT OF Z

First we present the results of [27]. We need a definition and some properties of the Remez (or Lebesgue, or norming, ...) constant (see, e.g., [1, 3, 15, 25] and references therein).

Definition 2.1. For a set $Z \subset B^n \subset \mathbb{R}^n$, the Remez constant $\mathcal{R}_d(Z)$ is the minimal K for which the inequality $\sup_{B^n} |P| \leq K \sup_Z |P|$ is valid for any real polynomial $P(x) = P(x_1,\dots,x_n)$ of degree d.

Clearly, we always have $\mathcal{R}_d(Z) \geq 1$. For some Z, the Remez constant $\mathcal{R}_d(Z)$ may be equal to ∞. In fact, $\mathcal{R}_d(Z)$ is infinite if and only if Z is contained in the set of zeroes

$$Y_P = \{x \in \mathbb{R}^n, \mid P(x) = 0\}$$

of a certain polynomial P of degree d. Sometimes, it is convenient to use the inverse Remez constant $\hat{\mathcal{R}}_d(Z) := \frac{1}{\mathcal{R}_d(Z)}$

The classical bounds on the Remez constant $\mathcal{R}_d(Z)$ in terms of the Lebesgue measure are given in [15] in dimension one, and in [1] in higher dimensions. The bound of [1] is, essentially, reproduced in Theorem 2.8 below.

In [26, 27], we demonstrated that the rigidity $\mathcal{RG}_d(Z)$ and the Remez constant $\hat{\mathcal{R}}_d(Z)$ are closely connected:

Theorem 2.2. ([27]) *For any $Z \subset B^n$, $\frac{(d+1)!}{2}\hat{\mathcal{R}}_d(Z) \leq \mathcal{RG}_d(Z)$.*

Below, in this section, we provide some approaches, which allow us to estimate, for certain specific classes of the sets Z, the asymptotic behavior in d of the Remez constant $\hat{\mathcal{R}}_d(Z)$. Then Theorem 2.2 translates these estimates into the results on the asymptotic behavior in d of the rigidity $\mathcal{RG}_d(Z)$.

2.1. **Rigidity, Remez Constant, and Capacity.** In this section, we present our first new result on the asymptotic behavior of the smooth rigidity of sets Z, in terms of the classical notion of the "capacity" $cap(Z)$. Let us recall (following [3]) that an important class of "massive" sets Z with a predictable asymptotic behavior of the Remez constant $\hat{\mathcal{R}}_d(Z)$ consists of the so-called nonpluripolar subsets Z of $\mathbb{R}^n$. Recall that a compact subset $Z \in \mathbb{R}^n$ is pluripolar if there exists a nonidentically $-\infty$ plurisubharmonic function u on $\mathbb{C}^n$ such that $u|Z \equiv -\infty$. It is known (see, for example, [10]) that a compact subset Z is nonpluripolar if and only if there exists a constant $C > 0$ depending on Z and n only such that the Remez constants $\mathcal{R}_d(Z)$ of Z satisfy

$$\mathcal{R}_d(Z) \leq C^d, \; d = 1, 2, \dots \; .$$

For instance, the results of [1] show that any compact subset Z of positive Lebesgue n-measure is nonpluripolar. However, nonpluripolar sets in $\mathbb{R}^n$ may be of arbitrary small positive Hausdorff dimension (for example, the n-fold direct product of Cantor sets in $[-1;1]$ of sufficiently small positive Hausdorff dimensions is nonpluripolar in $\mathbb{R}^n$; see [12]).

If Z is nonpluripolar, then the upper bound for the Remez constant of Z can be expressed also in terms of the capacity $cap(Z)$ of Z, a positive number defined in one of the following equivalent ways: in terms of the Monge - Ampere measure of Z, the Robin constant of Z, the Chebyshev constant of Z, or the transfinite diameter of Z; see [10, 16]. Then, for such Z and all $d \in \mathbb{N}$, one has

$$\ln \mathcal{R}_d(Z) \leq \frac{c \cdot d}{cap(Z)}, \ or \ \mathcal{R}_d(Z) \leq e^{\frac{c \cdot d}{cap(Z)}}, \tag{2.1}$$

where $c > 0$ depends on n only, and $cap(Z)$ is defined as above.

2.1.1. *New result.* Now we combine (2.1) with Theorem 2.2 above, and obtain our first new result on the asymptotic behavior of the smooth rigidity of sets Z:

Theorem 2.3. *For any nonpluripolar $Z \subset B^n$, and for any $d = 1, 2, \ldots$, it holds*

$$\mathcal{RG}_d(Z) \geq \frac{(d+1)!}{2} e^{-\frac{c \cdot d}{cap(Z)}},$$

where the constant c is defined in (2.1).

2.2. **Remez and Rigidity via the Topology of Z.** The following result was obtained in [28]: let B^n be the unit n-dimensional ball. For a given integer d let $Z \subset B^n$ be a smooth compact hypersurface with $N = (d-1)^n + 1$ connected components Z_j *with disjoint interiors* U_j. Let μ_j be the n-volume of the interior U_j of Z_j, and put $\mu = \min \mu_j$, $j = 1, \ldots, N$.

Theorem 2.4. *([28])*

Under the conditions above,

1. *(Remez-type inequality):* $\mathcal{R}_d(Z) \leq (\frac{4n}{\mu})^d$;

2. *(Smooth rigidity):* $\mathcal{RG}_d(Z) \geq \frac{(d+1)!}{2}(\frac{\mu}{4n})^d$.

This result was significantly extended in [5].

2.2.1. *New result.* In our new result, we assume Z to be a smooth (non-compact) hypersurface with *infinitely many* compact connected components Z_j *with disjoint interiors* U_j, inside the interior of the unit ball B^n (compare Remark 1.1 in Section 1). Let μ_j be the n-volume of the interior U_j of Z_j, and assume that Z_j are ordered in such a way that always $\mu_j \geq \mu_{j+1}$. We also assume that for each subsequence of Z_j, which accumulates to the boundary of B^n, or to any point inside B^n, the volumes μ_j tend to zero.

Now for each $d = 1, 2, \ldots$ we put $N_d = (d-1)^n + 1$, and put $\bar{\mu}_d := \mu_{N_d}$.

Theorem 2.5. *Under the conditions above, and for $d = 1, 2, \ldots$,*

1. *(Remez-type inequality):* $\mathcal{R}_d(Z) \leq (\frac{4n}{\bar{\mu}_d})^d$;

2. *(Smooth rigidity):* $\mathcal{RG}_d(Z) \geq \frac{(d+1)!}{2}(\frac{\bar{\mu}_d}{4n})^d$.

Proof: For each $d = 1, 2, \ldots$ we consider the compact subset $Z_d \subseteq Z$, consisting of the first $N_d = (d-1)^n + 1$ components Z_j. We apply Theorem 2.4 to Z_d. Clearly, μ in this theorem is $\bar{\mu}_d := \mu_{N_d}$, and we obtain the required inequalities of Theorem 2.5. $\square$

This result can be significantly extended in the lines of [5]. We plan to present this extension separately.

2.3. **Remez and Rigidity via Covering Complexity.** In this section we utilize the results of [25] in order to provide asymptotic in d bounds on the Remez constant of Z (and hence, on the rigidity), starting with the covering complexity of Z. We consider subsets $Z \subseteq B^n \subset \mathbb{R}^n$, where $\mathbb{R}^n$ is equipped with the l^∞ metric.

Definition 2.6. The covering number $M(\varepsilon, Z)$ is the minimal number of the ε-cubes Q_ε^n in $\mathbb{R}^n$, covering Z.

Next we introduce "Vitushkin polynomials"

$$M_d(\varepsilon) = \sum_{j=0}^{n-1} C_j(n,d) \left(\frac{1}{\varepsilon}\right)^j, \tag{2.2}$$

which provide the upper bound for the covering number $M(\varepsilon, V)$ for any polynomial level set $V = \{P = 0\}$ inside the unit ball B^n, P being a polynomial of degree d. The first example of the bound of this form appeared in Vitushkin's book [17]. We are not aware of any preferable specific form of (2.2), so below we use the following expression, obtained in [9]:

$$M_d(\varepsilon) = \sum_{j=0}^{n-1} 2^j \binom{n}{j} (d-1)^{n-j} \left(\frac{1}{\varepsilon}\right)^j. \tag{2.3}$$

Next, following [25], for a given $Z \subset \mathbb{R}^n$, we define "the metric d-span" $\omega_d(Z)$ as

$$\omega_d(Z) = \sup_{\varepsilon > 0} \varepsilon^n (M(\varepsilon, Z) - M_d(\varepsilon)). \tag{2.4}$$

We believe, the characterization of sets Z, provided by the "measurement" $\omega_d(Z)$, may be important in understanding the geometry of Z. Some comments and examples can be found in [25]. But, up to now, one of the most important properties of $\omega_d(Z)$ is that, being non-zero for many discrete and even finite sets, it, in many senses, behaves like the Lebesgue measure μ_n. In fact, we have the following result:

Proposition 2.7. *([25, Proposition 3.2]) For a measurable subset $Z \subset B^n$, the d-span $\omega_d(Z)$ satisfies $\omega_d(Z) \geq \mu_n(Z)$.*

Now, the main result of [25] is that the d-span $\omega_d(Z)$ can be substituted instead of $\mu_n(Z)$ in the classical Remez inequalities of [1, 15], thus providing their significantly stronger version. Now, in contrast with [1, 15], it works for sets Z of Lebesgue measure zero:

Theorem 2.8. *([25, Theorem 2.3]) If $\omega_d(Z) = \omega > 0$, then the Remez constant $\mathcal{R}_d(Z)$ is finite and it satisfies*

$$\mathcal{R}_d(Z) \leq T_d \left(\frac{1 + (1-\omega)^{1/n}}{1 - (1-\omega)^{1/n}} \right) \leq \left(\frac{4n}{\omega} \right)^d. \tag{2.5}$$

Some initial conclusions, and some specific examples of applications of Theorem 2.8 can be found in [25]. In the present paper we obtain some new asymptotic results on $\mathcal{R}_d(Z)$ (and hence on $\mathcal{RG}_d(Z)$), using (2.5), in terms of the covering complexity of Z. While we do not make this assumption explicitly, it may be instructive to consider the discrete sets Z.

2.3.1. *New results.* 1. Let's recall the definition of the box (or Minkovski, or entropy, ...) dimension $dim_e(Z)$ of Z. It is defined as the upper limit

$$dim_e(Z) = \overline{lim}_{\varepsilon \to 0} \frac{\log M(\varepsilon, Z)}{\log 1/\varepsilon}.$$

We assume that the box dimension $dim_e(Z)$ of Z is n. Moreover, we assume that $\overline{lim}_{\varepsilon \to 0} \varepsilon^n M(\varepsilon, Z) = \gamma > 0$. Under this assumption we get the following result:

Theorem 2.9. *For $d = 1, 2, \ldots$,*

$$\omega_d(Z) \geq \gamma, \ \ \mathcal{R}_d(Z) \leq \left(\frac{4n}{\gamma}\right)^d, \ \ \mathcal{RG}_d(Z) \geq \frac{(d+1)!}{2}\left(\frac{\gamma}{4n}\right)^d.$$

Proof: In expression (2.4), defining $\omega_d(Z)$, we let $\varepsilon \to 0$. Since for any d the Vitushkin polynomial is of order $n-1$ in $\frac{1}{\varepsilon}$, we get γ in the upper limit. We conclude that $\omega_d(Z)$, being the maximum of (2.4) with respect to ε, is at least γ. Now application of Theorem 2.8 and Theorem 2.2 completes the proof. $\square$

2. One of consequences of Theorem 2.9 is:

Corollary 2.10. *Any subset $Z \subset B^n$ with $dim_e(Z) = n$ is nonpluripolar.*

3. Now, we relax the previous setting, and assume only that $dim_e(Z)$ of Z is $n - \eta$, $0 < \eta < 1$. The assumption $\eta < 1$ is necessary in our new setting, if we want to produce nontrivial bounds on $\mathcal{RG}_d(Z)$ for any d. Indeed, algebraic hypersurfaces Z of any degree d in $\mathbb{R}^n$ have $dim_e(Z) = n - 1$.

Below we put $\nu = \frac{n\eta}{\eta - 1} < 0$. In order to simplify the "fractal calculations" below (which still remain not completely negligible), we moreover assume that for $0 < \varepsilon < 1$ we have

$$M(\varepsilon, Z) \geq \frac{\kappa}{\varepsilon^{n-\eta}}, \tag{2.6}$$

for a certain fixed $\kappa > 0$. Under this assumption, we get the following result.

Theorem 2.11. *If $dim_e(Z)$ of Z is $n - \eta$, $0 < \eta < 1$, and (2.6) is satisfied, then*

$$\omega_d(Z) \geq C(d-1)^\nu, \ \ \mathcal{R}_d(Z) \leq C_1^d (d-1)^{-\nu d}, \ \ \mathcal{RG}_d(Z) \geq C_1^{-d} \frac{(d+1)!}{2}(d-1)^{\nu d},$$

where C, C_1 are the constants, depending only on n, η, κ, and given explicitly below.

Proof: First of all, we simplify the expression of (2.3) for the Vitushkin polynomial:

Lemma 2.12. *The Vitushkin polynomial (2.3) satisfies, for $0 < \varepsilon < 1$,*

$$M_d(\varepsilon) = \sum_{j=0}^{n-1} 2^j \binom{n}{j} (d-1)^{n-j} \left(\frac{1}{\varepsilon}\right)^j \leq 4^n (d-1)^n \left(\frac{1}{\varepsilon}\right)^{n-1}. \tag{2.7}$$

Proof: We just replace each of the factors in the coefficients of (2.3) by its maximal possible value, and get

$$M_d(\varepsilon) = \sum_{j=0}^{n-1} 2^j \binom{n}{j} (d-1)^{n-j} (\frac{1}{\varepsilon})^j \leq \left(2^n (d-1)^n (\frac{1}{\varepsilon})^{n-1} \right) \sum_{j=0}^{n-1} 2^j \leq$$

$$\leq 4^n (d-1)^n (\frac{1}{\varepsilon})^{n-1}. \tag{2.8}$$

□

Next we get, via (2.4), (2.6), and via Lemma 2.12, for each ε_0, $0 < \varepsilon_0 < 1$,

$$\omega_d(Z) = \sup_{\varepsilon > 0} \varepsilon^n (M(\varepsilon, Z) - M_d(\varepsilon)) \geq \varepsilon_0^n [M(\varepsilon_0, Z) - M_d(\varepsilon_0)] \geq$$

$$\geq \varepsilon_0^n [\kappa (\frac{1}{\varepsilon_0})^{n-\eta} - 4^n (d-1)^n (\frac{1}{\varepsilon_0})^{n-1}] = \varepsilon_0 [\kappa (\frac{1}{\varepsilon_0})^{1-\eta} - 4^n (d-1)^n]. \tag{2.9}$$

Now, we fix ε_0 to satisfy the equation

$$\kappa (\frac{1}{\varepsilon_0})^{1-\eta} = 4^{n+1} (d-1)^n, \tag{2.10}$$

which gives

$$\varepsilon_0 = [\frac{4^{n+1}}{\kappa} (d-1)^n]^{\frac{1}{\eta-1}} = [\frac{4^{n+1}}{\kappa}]^{\frac{1}{\eta-1}} (d-1)^{\frac{n}{\eta-1}}. \tag{2.11}$$

In particular, for this choice of ε_0, we have

$$\kappa (\frac{1}{\varepsilon_0})^{1-\eta} - 4^n (d-1)^n = 3 \cdot 4^n (d-1)^n. \tag{2.12}$$

Notice that for n and Z fixed the only free parameter in (2.11) is d. As a function of d, ε_0 decreases as $(d-1)^{\frac{n}{\eta-1}}$. Substituting this value of ε_0 into (2.9), we obtain

$$\omega_d(Z) \geq 3 \cdot 4^n (d-1)^n \varepsilon_0 = 3 \cdot 4^n (d-1)^n [\frac{4^{n+1}}{\kappa}]^{\frac{1}{\eta-1}} (d-1)^{\frac{n}{\eta-1}} =$$

$$= 3 \cdot 4^n [\frac{4^{n+1}}{\kappa}]^{\frac{1}{\eta-1}} (d-1)^{n+\frac{n}{\eta-1}} = C(n, \eta, \kappa)(d-1)^\nu, \tag{2.13}$$

with $\nu = \frac{n\eta}{\eta-1}$ as above, and with $C(n, \eta, \kappa) = C := 3 \cdot 4^n [\frac{4^{n+1}}{\kappa}]^{\frac{1}{\eta-1}}$, depending only on n, η, κ. Notice that, since $0 < \eta < 1$, we get $\nu < 0$.

Thus we finally obtain $\omega_d(Z) \geq C(d-1)^\nu$, which is the first item in Theorem 2.11. Accordingly, by Theorem 2.8, we have

$$\mathcal{R}_d(Z) \leq (\frac{4n}{\omega})^d \leq (\frac{4n}{C(d-1)^\nu})^d = (\frac{4n}{C})^d (d-1)^{-\nu d} = C_1^d (d-1)^{-\nu d},$$

with $C_1 := \frac{4n}{C}$ depending only on n, η, κ. This is the second item in Theorem 2.11.

Finally, via Theorem 2.2, we obtain

$$\mathcal{RG}_d(Z) \geq \frac{(d+1)!}{2} \hat{\mathcal{R}}_d(Z) \geq C_1^{-d} \frac{(d+1)!}{2} (d-1)^{\nu d},$$

which is the last item in Theorem 2.11. This completes the proof of Theorem 2.11. □

3. RIGIDITY VIA TEST CURVES

As it was mentioned above, *for Z with a non-empty interior we always have $\mathcal{RG}_d(Z) \geq \frac{(d+1)!}{2^{d+1}}$, independently of the size and the geometry of Z.*

In [29], this result was extended to sets with a sufficiently large box dimension:

Theorem 3.1. *If the box dimension $dim_e(Z)$ is greater than $n - \frac{1}{d+1}$, then*

$$\mathcal{RG}_d(Z) \geq M = M(n,d) > 0,$$

where the positive constant M depends only on n and d.

In particular, the result of Theorem 3.1 provides examples of discrete, but sufficiently dense, sets Z, for which $\mathcal{R}_d(Z)$ behaves in the same way as for sets with a non-empty interior. In Theorem 3.1 neither the geometry nor the size of the set Z are important. Z may be contained in the ball of an arbitrarily small radius.

The proof of Theorem 3.1 in [29] does not involve the Remez constant of Z. Instead, it uses what we call "test curves". The approach is as follows. If we could find a straight line ℓ in $\mathbb{R}^n$, passing through the point z_0, where the absolute value $|f(z)|$ is equal to one, and through some $d+1$ distinct points in Z, we could immediately get the required lower bound $\mathcal{RG}_d(Z) \geq \frac{(d+1)!}{2^{d+1}}$ on the $d+1$-st derivative of f, via the basic properties of d-rigidity, mentioned above (Proposition 1.3).

However, for a generic finite set Z any straight line ℓ meets Z at one or two points, at most. Instead we replace ℓ by a smooth curve ω, and try to mimic the calculations for ℓ. This requires analysis of the high order chain-rule expressions, on one side, and construction of curves ω with small high-order derivatives, passing through some $d+1$ distinct points in Z, on the other side.

Thus, Theorem 3.1, in contrast with the results of Section 2 above, directly bounds the rigidity of Z, not touching its Remez constant.

3.1. **New result.** As a direct corollary of Theorem 3.1 we get the following asymptotic result:

Theorem 3.2. *If the box dimension $dim_e(Z)$ is equal to n, then*

$$\mathcal{RG}_d(Z) \geq M(n,d) > 0, \ d = 1, 2, \ldots$$

Proof: Since $dim_e(Z)$ is equal to n, the conditions of Theorem 3.1 are satisfied for any d. $\square$

We do not give the explicit expression for $M = M(n,d)$, refereing the reader to [29]. Let us stress once more that, as for the sets with a non-empty interior, the bounds in Theorem 3.2 depend only on the box dimension of Z, but not on its specific geometry or size. We believe that behind this fact may be hidden an important property of smooth functions, which is not completely understood yet.

Another important remark concerns a comparison between Theorem 2.9 above and Theorem 3.2. Both assume $dim_e(Z) = n$. Theorem 2.9 bounds the Remez constant of Z and, consequently, its rigidity, but the bound depends not only on $dim_e(Z) = n$, but also on γ, which, essentially, measures the "size" of Z, and this bound tends to zero as $\gamma \to 0$. In contrast, Theorem 3.2 bounds only the smooth rigidity of Z, and provides no information on its Remez constant. But its bound depends only on the assumption $dim_e(Z) = n$, and not on the specific geometry or size of Z. We believe that the current paper, as well as [6, 25, 29] can provide approaches to a better understanding of this problem.

4. POLYNOMIAL TEST CURVES OF HIGHER DEGREE

Another application of test curves is given in [6]. In this paper we consider the intersection of the set Z of zeroes of f with a polynomial curve ω of degree s, and provide certain specific smooth rigidity results. Here we quote just one of the results of [6], concerning derivatives of all orders of f.

Consider polynomial parametric curves ω of degree s in $\mathbb{R}^n$. The curves ω are given in the coordinate form by $\omega(t) = (\omega_1(t), \ldots, \omega_n(t))$, with $t \in [-1, 1]$, and with $\omega_i(t)$ being polynomials in t of the degree at most s.

Theorem 4.1. *Let $f : B^n \to \mathbb{R}$ be an infinitely differentiable function on B^n, and let ω be as above, with $\omega([-1,1]) \subset B^n$ not contained in the zero set $Z(f)$. Assume that $|Z \cap \omega([-1,1])| = \infty$, i.e. there are infinitely many zeros of f on the curve ω. Then there is an infinite number of the derivatives orders m for which*

$$||f^{(m)}|| \geq C(m)\frac{(m+1)!}{2^{m+1}},$$

with the positive constants $C(m)$, explicitly given in [6].

Authors' Note

This paper is dedicated to V.M. Tikhomirov, on his 90's anniversary. The second author strongly appreciates a deep influence on his work of the profound results of Tikhomirov, especially of those ([11] and plenty of what followed), related to the complexity of smooth, analytic, and other functional classes. In particular, this concerns the second author's works [21, 30], the present paper, and many others. Best wishes!

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